Thai Journal of Mathematics
Volume 20 Number 4 (2022)
Pages 1661–1668

http://thaijmath.in.cmu.ac.th

ISSN 1686-0209

The Molting of Butterfly Theorem

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Abstract In this paper, we generalize the butterfly theorem for quadrilaterals and apply this result to a beauty on a retraction of the butterfly wings, which we call the molting of butterfly theorem.

MSC: 51M05; 51M04

Keywords: butterfly theorem; butterfly molting; Euclidean geometry; quadrilateral

Submission date: 14.06.2021 / Acceptance date: 08.02.2022

1. Introduction

One of the most beautiful result in Euclidean geometry is the butterfly theorem which formally states as follows.

Theorem 1.1 (Butterfly Theorem)**.** *Let I be the midpoint of a chord AB of a circle. Through I, two other chords DF and EG are drawn such that D and G are on opposite sides of AB. If DG and EF intersect AB at M and N, respectively (see Figure 1), then I is also the midpoint of MN.*

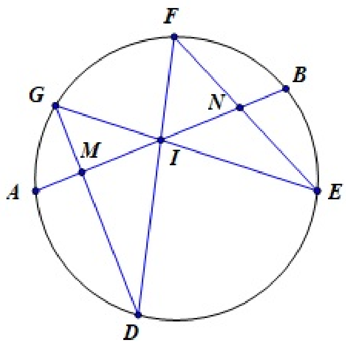

Figure 1. Butterfly Theorem

The most likely original proof was proposed by W. G. Horner in 1815 as a solution of a problem appeared in The Gentleman's Diary (see [1]). Since then, many different proofs were found using different tools which vary from the simplest to the most complicated (see [2], [3], [4], [5], [6]). The idea of the butterfly theorem was generalized in various aspects (see ,[7], [8], [9], [10], [11]). One of those remarkable results was reported by Bankoff (see [1]) who described how it would be when the common point of 2 wings is not at the middle of the chord. We paraphrase the theorem as follows.

Theorem 1.2 (Generalized a Butterfly Theorem)**.** *Let I be a point anywhere on a chord AB of a circle. Through I, two other chords DF and EG are drawn such that D and G are on opposite sides of AB. If DG and EF intersect AB at M and N, respectively (see Figure 2). Then* $\frac{AM}{MI} \cdot \frac{IN}{NB} = \frac{AI}{IB}$.

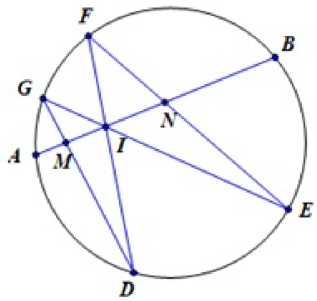

FIGURE 2. Generalize a Butterfly Theorem

From Theorem 1.2, if I is a midpoint of the chord, then $AI = IB$ and we finally have that $MI = IN$. It is obvious that we have created the butterfly with the same feature as in Theorem 1.1. Then we can say that Theorem 1.1 is an easy consequence of Theorem 1.2.

Theorem 1.3 (Butterfly Theorem for Quadrilaterals)**.** *Let I be the intersection of the diagonals AC, BD of a convex quadrilateral $ABCD$. Through I, draw two lines EF and HG that meet the sides of $ABCD$ at E, F, G, H such that E and G are on opposite sides of AC. If EG and HF intersect AC at M and N, respectively (see Figure 3). Then* $\frac{AM}{MI} \cdot \frac{IN}{NC} = \frac{AI}{IC}$.

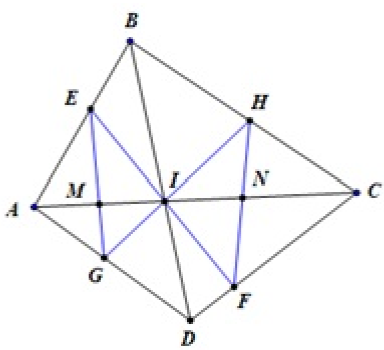

FIGURE 3. Butterfly Theorem for Quadrilaterals

We see that the feature of the butterfly Theorem 1.3 is quite specific. The 4 tips of the wings must lie on different sides of the quadrilateral. In this paper, we aim to figure out what features else of butterfly which is possible to maintain some interesting properties and apply this result to a beauty on a retraction of the butterfly wings, which we call the butterfly molting theorem. In this note, $[ABC]$ denotes the area of triangle ABC. Our proof depends primarily upon the following properties for areas of triangles:

Proposition 1. Given the triangles ABC and ABD. If E is the intersection of the lines CD and AB (see Figure 4), then $\frac{[ABC]}{[ABD]} = \frac{CE}{DE}$.

Proposition 2. Given the triangles ABC and DEF. IF $\angle CAB = \angle FDA$, then $\frac{[ABC]}{[DEF]} = \frac{CA \cdot AB}{FD \cdot DE}$.

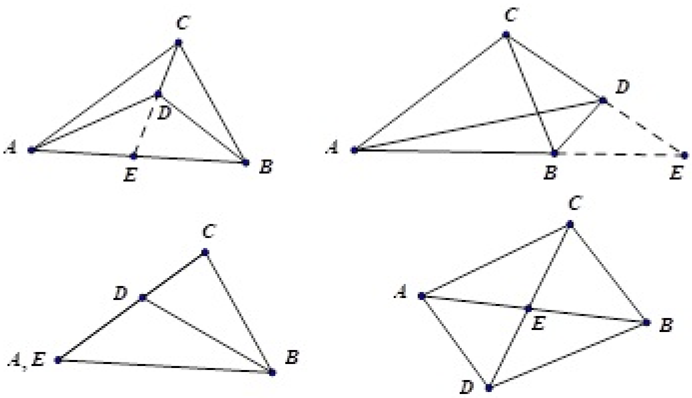

Figure 4. Properties for Areas of Triangles

2. Generalizations of the Butterfly Theorem for Quadrilaterals

In this section, we will generalize the butterfly theorem for quadrilaterals in two ways. The first one is an investigation under the position of the common point of 2 wings along a diagonal of the quadrilateral. The later is the exploration under the position of the wing tips which has never been discussed in Theorem 1.3.

Theorem 2.1. *Let X be any point on diagonal AC of a convex quadrilateral $ABCD$. Through X, draw two lines EF and HG that meet the sides of $ABCD$ such that E, H, F and G lie on AB, BC, CD and AD, respectively (see Figure 5). If EG and HF intersect AC at M and N, respectively. Then* $\frac{AM}{MX} \cdot \frac{XN}{NC} = \frac{[ABD]}{[CBD]}$.

Proof. Consider all sub-triangles in Figure 5 in pairs. Some pairs share a common edge, while some have one of their angle equal. By means of Proposition 1 and Proposition 2, we can deal with those ratios and obtain the following formula.

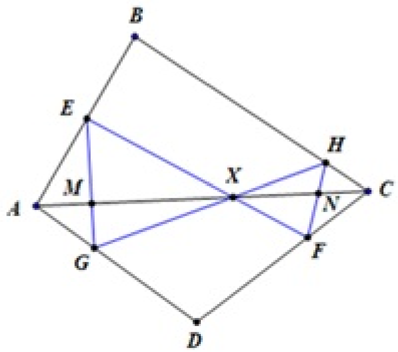

FIGURE 5. Generalize a Butterfly Theorem for Quadrilaterals Type I

$$\begin{aligned}
\frac{AM}{MX}\cdot\frac{XN}{NC} &= \frac{[AEG]}{[XEG]}\cdot\frac{[XHF]}{[CHF]}\\
&= \frac{[AEG]}{[XEG]}\cdot\frac{[XHF]}{[CHF]}\cdot\frac{[ABD]}{[ABD]}\cdot\frac{[CBD]}{[CBD]}\\
&= \frac{[AEG]}{[ABD]}\cdot\frac{[XHF]}{[XEG]}\cdot\frac{[CBD]}{[CHF]}\cdot\frac{[ABD]}{[CBD]}\\
&= \frac{AE{\cdot}AG}{AB{\cdot}AD}\cdot\frac{XH{\cdot}XF}{XE{\cdot}XG}\cdot\frac{CB{\cdot}CD}{CH{\cdot}CF}\cdot\frac{[ABD]}{[CBD]}\\
&= \frac{AE}{AB}\cdot\frac{AG}{AD}\cdot\frac{XH}{XG}\cdot\frac{XF}{XE}\cdot\frac{CB}{CH}\cdot\frac{CD}{CF}\cdot\frac{[ABD]}{[CBD]}\\
&= \frac{[AEC]}{[ABC]}\cdot\frac{[AGC]}{[ADC]}\cdot\frac{[AHC]}{[AGC]}\cdot\frac{[AFC]}{[AEC]}\cdot\frac{[ACB]}{[ACH])}\cdot\frac{[ACD]}{[ACF]}\cdot\frac{[ABD]}{[CBD]}\\
&= \frac{[ABD]}{[CBD]}
\end{aligned}$$

■

Now the value of $\frac{AM}{MX}\cdot\frac{XN}{NC}$ does not depend on a point X. Therefore, if I is the intersection of the diagonals AC, BD of a convex quadrilateral $ABCD$, then $\frac{AM}{MX}\cdot\frac{XN}{NC} = \frac{AI}{IC}$. Then we can conclude that Theorem 1.3 is a consequence of Theorem 2.1.

Theorem 2.2. *Let X be any point on diagonal AC of a convex quadrilateral $ABCD$. Through X, draw two lines EF and HG that meet the sides of $ABCD$ such that E and H lie on AB, F and G lie on CD (see Figure 6). If EG and HF intersect AC at M and N, respectively. Then $\frac{AM}{MX}\cdot\frac{XN}{NC} = \frac{AX}{XC}$.*

Proof. Now consider all pairs of sub-triangles in Figure 6. Again, we use Proposition 1 and Proposition 2 in determining the proper ratios from the corresponding pairs of triangles. Another formula can be derived as:

$$\begin{aligned}
\frac{AM}{MX}\cdot\frac{XN}{NC} &= \frac{[AEG]}{[XEG]}\cdot\frac{[XHF]}{[CHF]}\\
&= \frac{[AEG]}{[XEG]}\cdot\frac{[XHF]}{[CHF]}\cdot\frac{[AHG]}{[AHG]}\cdot\frac{[CHG]}{[CHG]}\\
&= \frac{[XHF]}{[XEG]}\cdot\frac{[CHG]}{[CHF]}\cdot\frac{[AEG]}{[AHG]}\cdot\frac{[AHG]}{[CHG]}\\
&= \frac{XH{\cdot}XF}{(XE{\cdot}XG}\cdot\frac{CH{\cdot}CG}{CH{\cdot}CF}\cdot\frac{AE{\cdot}AG}{AH{\cdot}AG}\cdot\frac{AX}{XC}\\
&= \frac{[AHC]}{[AGC]}\cdot\frac{[AFC]}{[AEC]}\cdot\frac{[CGA]}{[CFA]}\cdot\frac{[AEC]}{[AHC]}\cdot\frac{AX}{XC}\\
&= \frac{AX}{XC}.
\end{aligned}$$

We see that in this case, the value of $\frac{AM}{MX}\cdot\frac{XN}{NC}$ relies surprisingly on the common point X of 2 wings. ■

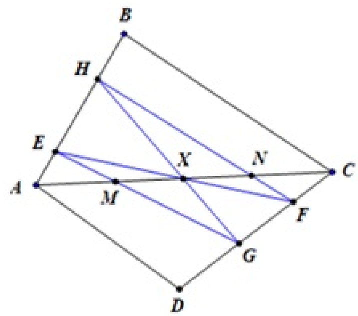

FIGURE 6. Generalize a Butterfly Theorem for Quadrilaterals Type II

From Theorem 2.1 and Theorem 2.2, it is amazing that the ratio $\frac{AM}{MX} \cdot \frac{XN}{NC}$ behaves differently depending on whether the pair of wing tips H and E lie on the same side or adjacent sides. This wonderful result can also be applied to another generalization of the butterfly theorem which we give the detail in the next section.

3. The Butterfly Molting

In this section, we discuss on a retraction of the butterfly wings (see Figure7) which defined as a butterfly molting. We also show a sufficient condition that makes the results on a butterfly molting still be true.

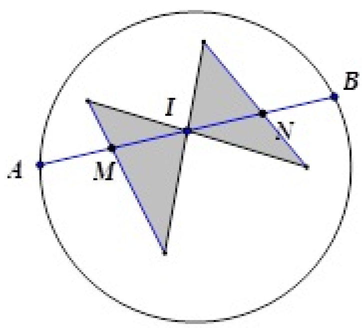

FIGURE 7. Retraction of a Butterfly in a Circle

Definition 3.1 (Butterfly Molting)**.** Let I be any point on a chord AB of a circle. Through I, two other chords DF and EG are drawn such that D and G are on opposite sides of AB. We will call a butterfly shaped $DEFG$ as $DEFG$-Butterfly over I, if the wings intersect AB. Let D', E', F' and G' be a point on ID, IE, IF and IG, respectively, the $D'E'F'G'$-Butterfly shaped is called as a butterfly molting of $DEFG$-Butterfly over I, or short, a butterfly molting of $DEFG$ (see Figure 8).

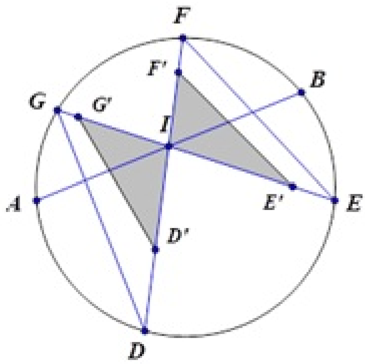

FIGURE 8. A Butterfly Molting of DEFG

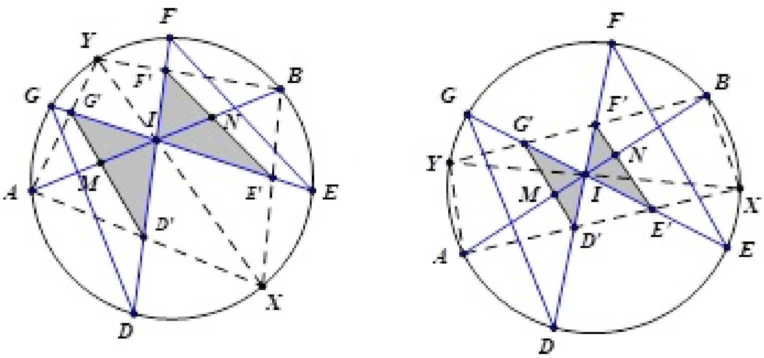

FIGURE 9. The Butterfly Molting Theorem

Theorem 3.2. *Let I be any point on a chord AB of a circle. Through I, three other chords DF, EG and XY are drawn such that D, E and X are on the same side of AB and $D'E'F'G'$ is a butterfly molting of $DEFG$, and $D'G'$ and $E'F'$ intersect AB at M and N, respectively. If D', E', F', and G' lie on a side of a quadrilateral $AXBY$ (see Figure 9), then* $\frac{AM}{MI} \cdot \frac{IN}{NB} = \frac{AI}{IB}$.

Proof. The result is clear by applying Theorem 2.1 and Theorem 2.2. ∎

3.1. Discussion

After the above investigation, we found that Theorem 2.1 is more comprehensive than Theorem 1.3. In Theorem 2.1, the common point of the wings does not have to be the intersection of the diagonals, but the result is still the same no matter where the common point is, along a diagonal. In addition, Theorem 2.2 reveals a new interesting property for a certain butterfly in a quadrilateral which is not mentioned in Theorem 1.3. However, Theorem 2.1 and Theorem 2.2 do not include the cases that the butterfly is imposed as in Figure 10. On the left hand side, the situation becomes a butterfly in a triangle, while on the right hand side, there is an empty side of the quadrilateral. In both cases, further verification is needed to identify the real behavior of the stated ratio.

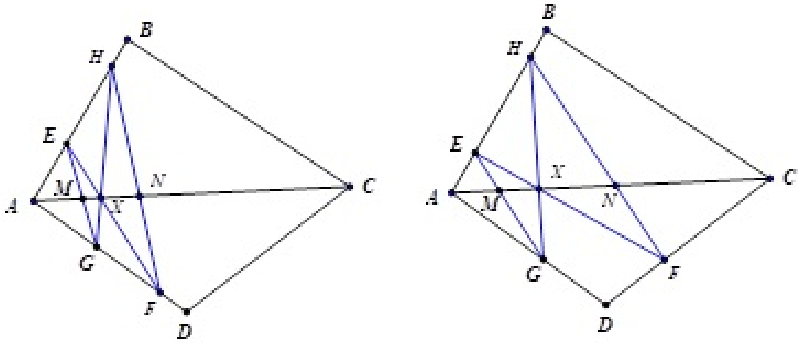

FIGURE 10. Other cases of the butterfly

ACKNOWLEDGEMENTS

The author would like to thank the referees for their careful reading of the article and insightful comments.

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