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Affine Rational Transformations of Copulas and Quasi-Copulas

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Abstract The transformation-based methods are indeed the convenient ways for constructing a new copula using known copulas. In this research, we characterize linear rational transformations for multivariate copulas and multivariate quasi-copulas. This is an extension to the already known results in the bivariate case. We found that this type of transformations extended naturally for the multivariate quasi-copulas. Yet, the only linear rational transformation of multivariate copulas is the identity function which is different from the bivariate case. This means that the set of linear rational functions that transform a multivariate copula varies depending on the copula itself. As an example, we also characterize such sets in the case of the trivariate product copula.

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1. Introduction

One of the important problems in statistics, especially in the probability theory, is to describe the intercorrelation between random variables. To tackle these problems, the concept of a copula was first introduced by Abe Sklar during his research in the 1950s, and this concept is known as Sklar's theorem. The theorem states that each joint distribution function can be represented by composing a multivariate copula with univariate marginal distribution functions, see for example [1]. In detail, a copula can be viewed as a multivariate distribution function for which each random variable has a uniform distribution on $[0, 1]$.

Since resources available for analyzing tremendous amounts of data are often limited, the concepts of copulas are widely studied in various fields such as finance, risk management [2], hydrology [3], and recently machine learning [4]. These came from the

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application of Sklar's theorem as it can conveniently perform the multivariate modeling by considering a copula and marginal distribution functions separately.

Later, more general concepts of a copula, a quasi-copula, were proposed in [5] in order to study the behavior of operations on distribution functions that are associated with operations on random variables [6]. Also, another concrete usage of quasi-copulas is their application in the areas of fuzzy logic and fuzzy preference modeling [7].

In addition, we can conclude from [8] that for a set of non-empty multivariate functions $\mathcal{S}$ that shares a common domain D, the pointwise infimum and supremum, which are given by

$$\underline{S}(u_1,...,u_k)=\inf\{S(u_1,...,u_k):S\in\mathcal{S}\}$$

and

$$\overline{S}(u_1,...,u_k)=\sup\{S(u_1,...,u_k):S\in\mathcal{S}\}$$

are the best possible point wise bounds for $\mathcal{S}$. Note that, neither $\underline{S}$ nor $\overline{S}$ is necessary in $\mathcal{S}$. Moreover, $\underline{S}$ and $\overline{S}$ of a set of copulas are quasi-copulas.

Unfortunately, finding the appropriate (quasi-)copulas to a given statistical model obtaining from experiments can sometimes be complicated. To deal with such a situation, several methods of constructing different (quasi-)copula families were presented, for instance, the inversion method, the geometric method, and the gluing method [1, 9].

The construction methods that we focus on in this research are the transformation-based methods that can yield a new (quasi-)copula by transforming known (quasi-)copulas. In 2013, Kolesrov et al. [10] proved that for any copula $C:[0,1]^2\to[0,1]$, the function $D_C:[0,1]^2\to[0,1]$, given by

$$D_C=C(x,y)(x+y-C(x,y))$$

is a copula. After that, [11] noted that the mentioned transformation can be considered a subcase of the transformation $K_P:[0,1]^2\to[0,1]$ under the quadratic polynomial P given by

$$K_{P,C}(x,y)=P(x,y,C(x,y)).$$

They also gave the conditions required of P that make $K_{P,C}$ still a copula. Later, [12] extended the results to the transformation $K'_{P,C_1,C_2}:[0,1]^2\to[0,1]$ by composing one more copula to the extended quadratic polynomial P, given by

$$K'_{P,C_1,C_2}(x,y)=P(x,y,C_1(x,y),C_2(x,y))$$

where C_1,C_2 are copulas. A short time later, [13] extended the result from the previous results to the transformation $\zeta_{P',C_1,...,C_k}:[0,1]^2\to[0,1]$ under any polynomial P', given by

$$\zeta_{P',C_1,...,C_k}(x,y)=P'(x,y,C_1(x,y),...,C_k(x,y))$$

for any bivariate copula $C_1,...,C_k$.

Quasi-copulas and copulas can also be seen as special types of aggregation functions. In [14], transformations for semi-copulas and quasi-copulas via quadratic functions P are studied as special types of transformations of aggregation functions. Formally, for a given $P:[0,1]^3\to\mathbb{R}$, they study a transformation τ_P, given by

$$\tau_P(A)(x,y)=P(x,y,A(x,y))$$

where $x, y \in [0,1]$ and for $A : [0,1]^2 \to \mathbb{R}$. Also, they gave necessary conditions of the coefficients of P so that τ_P transforms semi-copulas to semi-copulas and quasi-copulas to quasi-copulas, respectively.

We define $\mathcal{S}_k, \mathcal{Q}_k$ as a class of k-variate semi-copulas and k-variate quasi-copulas, respectively. In [15], they study a transformation τ_P, given by

$$\tau_P(A)(x_1, ..., x_n) = P(x_1, ..., x_n, A(x_1, ..., x_n))$$

which is a multivariate version of the study in [14]. They showed that τ_P transforms aggregation functions if and only if P itself is an aggregation function. (For the characterization of such quadratic polynomial P, see [16].) They also characterized quadratic polynomial P for which τ_P transforms semi-copulas and quasi-copulas. It follows from their characterization that for the multivariate case where $k > 2$, $\tau_P(\mathcal{S}_k) \subseteq \mathcal{S}_k$ if and only if $\tau_P(\mathcal{Q}_k) \subseteq \mathcal{Q}_k$. Note that this result is not true for the bivariate case. Later, [17] considered the case of multivariate transformations and characterized quadratic polynomial P such that

$$\tau_P(S_1, ..., S_k)(x_1, ..., x_n) = P(x_1, ..., x_n, S_1(x_1, ..., x_n), ..., S_k(x_1, ..., x_n)).$$

is a semi-copula transformation. However, no analogous result is given for quasi-copulas.

It can be seen that all of these transformations focus on polynomial functions, especially quadratic polynomials. Other types of functions can also be considered. Actually, [20] considered the affine rational transformation of copulas and quasi-copulas. They characterized affine rational transformations of both copulas and quasi-copulas, showing that transformations of both cases are the same. To date, no study has been done for the multivariate case. Thus, we are curious to find out whether or not these properties still hold true under similar rational transformations for copulas and quasi-copulas in higher dimensions.

2. Preliminaries

A function $Q : [0,1]^k \to \mathbb{R}$ is said to be *Lipschitz* if

$$|Q(x_1, ..., x_k) - Q(y_1, ..., y_k)| \leq \sum_{i=1}^{k} |x_i - y_i|$$

for all $x_i, y_i \in [0,1]$. Also, the *volume* of Q over a box $\prod_{i=1}^{k}[a_i, b_i] \subseteq [0,1]^k$ is defined via

$$V_Q\left(\prod_{i=1}^{k}[a_i, b_i]\right) = \sum_{\vec{x} \in \prod_{i=1}^{k}\{a_i, b_i\}} (-1)^{N((a_1,...,a_k),\vec{x})} Q(\vec{x})$$

where $N((a_1, ..., a_k), (x_1, ..., x_k))$ is the number of i such that $a_i = x_i$.

Definition 2.1. A function $Q : [0,1]^k \to [0,1]$ is said to be a *k-dimensional quasi-copula* if Q satisfies the following conditions:

(1) Q is *grounded*, that is, $Q(x_1, ..., x_{i-1}, 0, x_{i+1}, ..., x_k) = 0$ for all $x_i \in [0,1]$ and for every $i = 1, 2, ..., k$;
(2) Q has *uniform marginals*, that is, $C(1, ..., 1, x, 1, ..., 1) = x$ for all $x \in [0,1]$;
(3) Q is increasing in each place; and
(4) Q is Lipschitz.

A *k-dimensional copula* is a function $C : [0,1]^k \to [0,1]$, which is grounded, has uniform marginals and satisfies the *k-increasing* property, that is $V_C\left(\prod_{i=1}^k [a_i, b_i]\right) \geq 0$ for all $\prod_{i=1}^k [a_i, b_i] \subseteq [0,1]^k$.

Notice that for a continuously differentiable function Q, Q is increasing in each place and satisfies the Lipschitz condition if and only if $0 \leq \partial_i Q \leq 1$ for all $i = 1, \ldots, k$.

Consider the following functions defined on $[0,1]^k$ to the set of real number:

(1) $W_k(x_1, ..., x_k) = \max\{0, \sum_{i=1}^k x_i - k + 1\}$; and

(2) $M_k(x_1, ..., x_k) = \min\{x_1, ..., x_k\}$,

both W_k and M_k are quasi-copulas. Also,

$$W_k(x_1, ..., x_k) \leq Q(x_1, ..., x_k) \leq M_k(x_1, ..., x_k)$$

for all $x_1, \ldots, x_k \in [0,1]$ and all k-dimensional quasi-copulas Q. The function W_k is called the *Frechet-Hoeffding lower bounds* and the function M_k is called the *Frechet-Hoeffding upper bounds*, see for example [15] for more information. We note that the Frechet-Hoeffding lower bound W_k is not a copula if its dimension k is greater than 2.

Consider, a product function $\Pi : [0,1]^k \to \mathbb{R}$ is given by

$$\Pi(x_1, ..., x_k) = x_1 \cdots x_k \text{ is a copula.}$$

In fact, Π is known as the *(k-dimensional) product copula* for more details, see [18].

For an absolutely continuous function $C : [0,1]^k \to [0,1]$,

$$V_C\left(\prod_{i=1}^k [a_i, b_i]\right) = \int_{\prod_{i=1}^k [a_i, b_i]} \partial_1 \cdots \partial_k C(\vec{x}) d\vec{x} \tag{2.1}$$

for all $\prod_{i=1}^k [a_i, b_i] \subseteq [0,1]^k$. In particular, C is k-increasing if and only if its density $\partial_1 \cdots \partial_k C \geq 0$ a.e. We note here that the latter is usually easier to check than the former. This includes, for example, the Frechet-Hoeffding bounds. However, the density of the considered copula is not necessary to exist. Therefore, in order to show that all linear rational transformations of copulas are still satisfied the k-increasing property, we show that their k-mixed derivative (density) is non-negative whenever they exist. After that, we use the denseness property in the set of copulas to guarantee this must also holds for all copulas.

2.1. Bernstein Polynomials

Another key concept we used in this work came from [19]. The *m-th degree Bernstein polynomial* $b_{i,m} : [0,1] \to \mathbb{R}$, which is defined by

$$b_{i,m}(t) = \binom{m}{i} t^i (1-t)^{m-i}$$

where $i = 0, 1, ..., m$. We extend this to $B_{i,m}^k : [0,1]^k \to \mathbb{R}$ by taking i to be a multi-index, $i = (i_1, .., i_k)$, where each $i_j \in \{0, 1, ..., m\}$ and setting

$$B_{i,m}^k(x) = b_{i_1,m}(x_1) \cdots b_{i_k,m}(x_k)$$

where $x = (x_1, ..., x_k) \in [0,1]^k$. If $f : [0,1]^k \to \mathbb{R}$, we define the *Bernstein approximation* to f to be

$$\mathcal{B}_m^k(f) = \sum_i f\left(\frac{i}{m}\right) B_{i,m}^k$$

where i ranges over all multi-indices $i = (i_1, ..., i_k)$ such that each $i_j \in \{0, 1, ..., m\}$, and by $\frac{i}{m}$ we mean the vector $(\frac{i_1}{m}, ..., \frac{i_k}{m})$.

From now on, we denote $\mathcal{C}_k$, $\mathcal{Q}_k$ as the set of k-variate copulas and the set of k-variate quasi-copulas, respectively.

Theorem 2.2. [19] *If $f : [0,1]^k \to \mathbb{R}$ is continuous, then $\mathcal{B}_m^n(f) \to f$ uniformly on $[0,1]^k$ as $m \to \infty$. Moreover, if $C \in \mathcal{C}_k$, then $\mathcal{B}_m^k(C) \in \mathcal{C}_k$.*

Notice that $\partial_1 \cdots \partial_k \mathcal{B}_m^k(C)$ exists and is continuous for all copula C. We also prove the next Theorem by adjusting Theorem 2.2 for the purpose of applying in the quasi-copulas.

Theorem 2.3. *Let $Q \in \mathcal{Q}_k$ and $Q_m = \mathcal{B}_m^k(Q)$. Then Q_m is a quasi-copula. Moreover, $Q_m \to Q$ as $m \to \infty$ and each $\partial_l Q_m$ exists and is continuous.*

3. Results

In this work, we will characterize affine rational transformations $F(C)$ of multivariate (quasi-)copulas C given in the form

$$F(C)(x_1, \ldots, x_k) = \frac{a_0 + \sum_{i=1}^k a_i x_i + a_{k+1} C(x_1, \ldots, x_k)}{b_0 + \sum_{i=1}^k b_i x_i + b_{k+1} C(x_1, \ldots, x_k)}$$

where $x_1, \ldots, x_k \in [0,1]$. A special case of the above transformation is the transformation $F_\lambda(C)$ given by

$$F_\lambda(C)(x_1, \ldots, x_k) = \frac{C(x_1, \ldots, x_k)}{1 - \lambda + \lambda \sum_{i=1}^k (1 - x_i) + \lambda C(x_1, \ldots, x_k)}$$

where $x_1, \ldots, x_k \in [0,1]$. It has been proved in [20] for the case $k = 2$, that is, for the bivariate case, that F is a transformation of (quasi-)copulas if and only if it is equal to F_λ for some $\lambda \in [0,1]$. Thus, it is interesting to know whether this result still hold in the multivariate setting. First, we will show that at least the form of affine rational transformations remains unchanged when the dimension k varies.

3.1. The affine rational transformation of quasi-copulas

Theorem 3.1. *Assume that $F(C)$ is a (quasi-)copula for all copula $C : [0,1]^k \to [0,1]$. Then $F = F_\lambda$ for some $\lambda \in \mathbb{R}$.*

Proof. We first consider the grounded property, by plugging $x_i = 0$ to $F(C)$ separately for all $i = 1, ..., k$. We obtain that $a_1 x_1 = ... = a_k x_k$ for all $x_1, ..., x_k \in [0,1]$. That is $a_1 = ... = a_k = 0$. Next, substitute the coefficients $a_i = 0$ back to the equation yields $a_0 = 0$. Thus, $F(C)$ must be in the form

$$F(C)(x_1, ..., x_k) = \frac{a_{k+1} C(x_1, \ldots, x_k)}{b_0 + \sum_{i=1}^k b_i x_i + b_{k+1} C(x_1, \ldots, x_k)}$$

In order for $F(C)$ to be a (quasi-)copula, we must have $a_{k+1} > 0$. Thus, we can let $\hat{b}_i = \frac{b_i}{a_{k+1}}$ for all $i = 0, 1, ..., k+1$ and write

$$F(C)(x_1, ..., x_k) = \frac{C(x_1, \ldots, x_k)}{\hat{b}_0 + \sum_{i=1}^{k} \hat{b}_i x_i + \hat{b}_{k+1} C(x_1, \ldots, x_k)}$$

By the uniform marginals of $F(C)$, we substitute $x_i = 1$ for all $i = 1, ..., k$ to get

$$\hat{b}_0 + \hat{b}_1 + \cdots + \hat{b}_{k+1} = 1$$

and substitute $x_i = 1$ for all $i = 1, ..., k$ such that $i \neq l$ for every $l = 1, ..., k$ to get

$$\hat{b}_0 + \sum_{i \neq l} \hat{b}_i + \left(\hat{b}_l + \hat{b}_{k+1}\right) x_l = 1.$$

We get $\hat{b}_1 = ... = \hat{b}_k = -\hat{b}_{k+1}$. This would imply $\hat{b}_0 = 1 + (k-1)\hat{b}_{k+1}$. Finally, take $\lambda = \hat{b}_{k+1}$ and then substitute all these back to the $F(C)$ yields $F = F_\lambda$ for some $\lambda \in \mathbb{R}$ as desired. ∎

Next, we will investigate the possible range of λ. The idea is to recursively shown that the range of possible λ decrease as the dimension k increases. Again, $\mathcal{C}_k$ is the set of k-variate copulas and $\mathcal{Q}_k$ is the set of k-variate quasi-copulas.

Theorem 3.2. *For any λ, if $F_\lambda(\mathcal{C}_{k+1}) \subseteq \mathcal{Q}_{k+1}$, then $F_\lambda(\mathcal{C}_k) \subseteq \mathcal{Q}_k$. In particular, we must have $0 \leq \lambda \leq 1$.*

Proof. Let $C \in \mathcal{C}_k$. Define

$$\hat{\pi}_k(C)(x_1, \ldots, x_k, x_{k+1}) = x_{k+1} C(x_1, \ldots, x_k).$$

We see that $\hat{\pi}_k(C) \in \mathcal{C}_{k+1}$. Also, define

$$\pi_k(D)(x_1, \ldots, x_k) = D(x_1, \ldots, x_k, 1)$$

for all $D \in \mathcal{C}_{k+1}$. Again, $\pi_k(D)$ is always a copula. Moreover,

$$\pi_k F_\lambda \hat{\pi}_k(C)(x_1, ..., x_k) = \frac{1 \cdot C(x_1, ..., x_k)}{1 - \lambda + \lambda \sum_{i=1}^{k}(1 - x_i) + 0 + \lambda \cdot 1 \cdot C(x_1, ..., x_k)}.$$

Therefore, $F_\lambda(C) = \pi_k F_\lambda \hat{\pi}_k(C) \in \mathcal{Q}_k$. This concludes that $F_\lambda(C_{k+1}) \subseteq \mathcal{Q}_{k+1}$ implies $F_\lambda(C_k) \in \mathcal{Q}_k$.

By induction, $F_\lambda(C_k) \subseteq \mathcal{Q}_k$ would imply $F_\lambda(\mathcal{C}_2) \subseteq \mathcal{Q}_2$. The latter is only true when $0 \leq \lambda \leq 1$. ∎

Henceforth, we will denote $\gamma(x_1, \ldots, x_k) = \sum_{i=1}^{k}(1 - x_i)$ so that we may simply write

$$F_\lambda(C) = \frac{C}{1 + \lambda(\gamma + C - 1)}$$

for all quasi-copula C. Next, we will show that $0 \leq \lambda \leq 1$ is sufficient to guarantee that $F_\lambda(\mathcal{Q}_k) \subseteq \mathcal{Q}_k$ for all $k \geq 2$. The proof will be based on the following lemmas.

Lemma 3.3. *For any $0 \leq \lambda \leq 1$ and any quasi-copula(copula) C, the range of $F_\lambda(C)$ is in the unit interval $[0, 1]$.*

Proof. Since $1-\lambda$, $\lambda\gamma$ and λC are non-negative, $(1-\lambda)+\lambda\gamma+\lambda C \geq 0$. This implies $1+\lambda(\gamma+C-1)\geq 0$ for all $x_1, ..., x_k \in [0,1]$. Now, we assume that $1+\lambda(\gamma+C-1)=0$ for some $x_1, ..., x_k \in [0,1]$. Since $1-\lambda$, $\lambda\gamma$ and λC are non-negative, $1-\lambda=0$, $\lambda\gamma=0$, and $\lambda C=0$. Thus, $(1-\lambda)+\lambda\gamma+\lambda C=0$. That is, $\gamma=0$ and $C=0$ simultaneously, a contradiction. That is, $1+\lambda(\gamma+C-1)>0$ for all $x_1, ..., x_k \in [0,1]$.

Next, we will show that $C \leq 1+\lambda(\gamma+C-1)$ for any $\lambda \in [0,1]$. First, if $\lambda=1$, we automatically obtain the result. After that, we consider the case $0\leq\lambda<1$. We have that $(1-\lambda)C \leq 1-\lambda$ for all $x_1, ..., x_k \in [0,1]$. We have that

$$\begin{aligned}(1-\lambda)C &\leq 1-\lambda \\ &\leq 1-\lambda+\lambda\gamma \\ C &\leq 1+\lambda(\gamma+C-1)\end{aligned}$$

Therefore, $0 \leq \dfrac{C}{1+\lambda(\gamma+C-1)} \leq 1$ for all $x_1, ..., x_k \in [0,1]$. ■

Now, we are in a position to prove the first main result.

Theorem 3.4. *If $0\leq\lambda\leq 1$, then $F_\lambda(\mathcal{Q}_k)\subseteq\mathcal{Q}_k$ for all integer $k>1$.*

Proof. We note that

$$\partial_i F_\lambda(C) = F_\lambda(C)\left(\frac{\partial_i C}{C}+\frac{\lambda(1-\partial_i C)}{1+\lambda(\gamma+C-1)}\right).$$

Next, we denote $G_\lambda(C)=\frac{\partial_i C}{C}+\frac{\lambda(1-\partial_i C)}{1+\lambda(\gamma+C-1)}$. Then $G_\lambda(C)$ is non-decreasing in $\lambda\geq 0$ since both $(1-\partial_i C)$ and $(\gamma+C-1)$ are non-negative. Thus, $G_0(C)\leq G_\lambda(C)\leq G_1(C)$. Also, $F_\lambda(C)$ is non-increasing in $\lambda\geq 0$ so that $F_1(C)\leq F_\lambda(C)\leq F_0(C)$. It follows that

$$\partial_i F_\lambda(C) \geq F_1(C)\,G_0(C) = \frac{\partial_i C}{\gamma+C} \geq 0$$

and

$$\begin{aligned}\partial_i F_\lambda(C) &\leq F_0(C)\,G_1(C) \\ &= \partial_i C + \frac{(1-\partial_i C)\,C}{\gamma+C} \\ &\leq \partial_i C + (1-\partial_i C) \\ &= 1\end{aligned}$$

for all quasi-copula C such that $\partial_i C$ exists. Thus, $F_\lambda(C)$ must be a quasi-copula for such C. By Theorem 2.3, the result must holds for all quasi-copula and we are done. ■

Example 3.5. Johnson and Kott [21] denote a FGM copula by

$$C_\theta(x_1,x_2,x_3) = x_1x_2x_3(1+\theta(1-x_1)(1-x_2)(1-x_3))$$

where $\theta\in[-1,1]$. The transformation $F_\lambda C_\theta$ of the mentioned FGM copula is in the form

$$\frac{x_1x_2x_3(1+\theta(1-x_1)(1-x_2)(1-x_3))}{1+2\lambda-\lambda(x_1+x_2+x_3)+\lambda x_1x_2x_3(1+\theta(1-x_1)(1-x_2)(1-x_3))}.$$

It follows that $F_\lambda C_\theta$ is a quasi-copula for all $0\leq\lambda\leq 1$ for all $\theta\in[-1,1]$.

3.2. The iterations of quasi-copulas transformation

We define the iterations $F_\lambda^n(C)$, which is defined recursively by $F_\lambda^{n+1} := F_\lambda^n \circ F_\lambda$ where $F_\lambda^1 := F_\lambda$ for all $n \geq 1$. Also, for any k-variable continuous functions $f, g : [0,1]^k \to [0,1]$, we define $d_\infty(f,g)$ by

$$d_\infty(f,g) := \sup_{\vec{x}\in[0,1]^k} |f(\vec{x}) - g(\vec{x})|.$$

Theorem 3.6. *Let $0 < \lambda \leq 1$. Then for every $Q \in \mathcal{Q}_k$ where $k \geq 2$ we have that $\lim_{n\to\infty} d_\infty(F_\lambda^n Q, W) = 0$. Moreover, F_λ has W as the unique fixed point where W is the Frechet-Hoeffding lower bound of Q.*

Proof. We will first show that for $Q \in \mathcal{Q}_k$,

$$d_\infty(F_\lambda Q, W) \leq \frac{d_\infty(Q,W)}{1 + \lambda d_\infty(Q,W)}.$$

Before moving to the next step, it is followed from the definition of $d_\infty(F_\lambda^n Q, W)$ that

$$Q - W \leq d_\infty(F_\lambda^n Q, W).$$

Since $\dfrac{x}{1+\lambda x}$ is increasing over x for any $\lambda \in [0,1]$, we have

$$\frac{Q - W}{1 + \lambda(Q - W)} \leq \frac{d_\infty(Q,W)}{1 + \lambda d_\infty(Q,W)}.$$

By mathematical induction, we have

$$d_\infty(F_\lambda^n Q, W) \leq \frac{d_\infty(Q,W)}{1 + n\lambda d_\infty(Q,W)}.$$

for all $n \geq 2$.
Thus, $\lim\limits_{n\to\infty} d_\infty(F_\lambda^n Q_k, W) = 0$. Moreover, if C is any fixed point of F_λ, we get that $d_\infty(F_\lambda C, W) = d_\infty(C,W) \leq \dfrac{d_\infty(C,W)}{1+\lambda d_\infty(C,W)}$ which is only possible when $d_\infty(C,W) = 0$. That is $C = W$. ∎

Now, we are in a position to prove that F_λ is not a multivariate copula transformation unless $\lambda = 0$. Therefore, the class of affine linear transformations of multivariate quasi-copulas is different from the class of affine linear transformations of multivariate copulas. Thus, the situation is different from what happens in the bivariate case.

Theorem 3.7. *F_λ is not a multivariate copula transformation unless $\lambda = 0$.*

Proof. Assume that F_λ is a multivariate copula transformation. By Theorem 3.2, $\lambda \geq 0$. Suppose that $\lambda > 0$. Then by mathematical induction, $C_n = F_\lambda^n(\Pi)$ is a copula for all n. However, $C_n \to W$ by Theorem 3.6. Since the space of copula is complete, W must also be a multivariate copula. However, W is not a copula which leads to a contradiction. ∎

The above result states that $F_\lambda(C)$ is not necessary a copula when C is a copula but this is not a guarantee either. In general, the range of λ for which $F_\lambda(C)$ is a copula with depends on C and has to be analyzed case by case and it could also be very complicated. We demonstrate this for $C = \Pi$ in the next section.

3.3. The affine transformation of a product copula with the suitable range of λ

Theorem 3.8. *Let $\Pi(x,y,z) = xyz$ for all $x,y,z \in [0,1]$ and $\lambda \geq 0$. Then $F_\lambda(\Pi)$ is copula if and only if $\lambda \leq \frac{1}{4}$.*

Proof. Denote $q_\lambda = \Pi / F_\lambda(\Pi)$, that is,

$$q_\lambda(x,y,z) = 1 + \lambda(2 - x - y - z + xyz)$$

so that

$$q_\lambda^4 \partial_z \partial_y \partial_x F_\lambda(\Pi) = 1 + 2\beta(x,y,z)\lambda + \gamma(x,y,z)\lambda^2 + 2\alpha(x,y,z)\lambda^3$$

where

$$\begin{aligned}
\beta(x,y,z) &= 3 - x - y - z - 2xyz, \\
\gamma(x,y,z) &= 12 - 8x - 8y - 8z + x^2 + y^2 + z^2 + 4xy + 4xz + 4yz - 16xyz \\
&\quad + 4x^2yz + 4xy^2z + 4xyz^2 + x^2y^2z^2
\end{aligned}$$

and

$$\begin{aligned}
\alpha(x,y,z) &= 4 - 4x - 4y - 4z + x^2 + y^2 + z^2 + 4xy + 4xz + 4yz - x^2y - x^2z \\
&\quad - xy^2 - y^2z - xz^2 - yz^2 - 8xyz + 4x^2yz + 4xy^2z + 4xyz^2 - 2x^2y^2z \\
&\quad - 2x^2yz^2 - 2xy^2z^2 + x^2y^2z^2.
\end{aligned}$$

If $F_\lambda(\Pi)$ is a copula, then $\partial_z \partial_y \partial_x F_\lambda(\Pi) \geq 0$ on $[0,1]^3$. In particular,

$$1 - 4\lambda = \partial_z \partial_y \partial_x F_\lambda(\Pi)(1,1,1) \geq 0$$

which yields $\lambda \leq \frac{1}{4}$ as desired.

Now, we will prove the sufficiency. Notice that

$$\begin{aligned}
\alpha(x,y,z) &= (1-x)^2(1-y)^2(1-z)^2 + (1-x)(1-y)(1-xy) \\
&\quad + (1-x)(1-z)(1-xz) + (1-y)(1-z)(1-yz) \\
&\geq 0
\end{aligned}$$

for all $x,y,z \in [0,1]$. Thus,

$$q_\lambda^4 \partial_z \partial_y \partial_x F_\lambda(\Pi) \geq 1 + 2\beta(x,y,z)\lambda + \gamma(x,y,z)\lambda^2. \tag{3.1}$$

For $x,y,z \in [0,1]$ such that $\gamma(x,y,z) \geq 0$,

$$q_\lambda^4 \partial_z \partial_y \partial_x F_\lambda(\Pi) \geq 1 + 2\beta(x,y,z)\lambda.$$

Since the right side is a linear function of λ, its minimum occurs at the end points - either $\lambda = 0$ or $\lambda = \frac{1}{4}$. If the minimum occurs at $\lambda = 0$, then clearly, $\partial_z \partial_y \partial_x F_\lambda(\Pi) \geq 0$. Otherwise,

$$\begin{aligned}
q_\lambda^4 \partial_z \partial_y \partial_x F_\lambda(\Pi) &\geq 1 + \frac{1}{2}\beta(x,y,z) \\
&= 1 + \frac{1}{2}(3 - x - y - z - 2xyz) \\
&\geq 1 + \frac{1}{2}(-2) \\
&\geq 0
\end{aligned}$$

which also implies $\partial_z \partial_y \partial_x F_\lambda(\Pi) \geq 0$.

For $x, y, z \in [0,1]$ such that $\gamma(x,y,z) < 0$, the right side of the equation , considered as a function of λ, is a downward parabola. Thus, its minimum again occurs at the end points - either$\lambda = 0$ or $\lambda = \frac{1}{4}$. If the minimum occurs at $\lambda = 0$, then again, $\partial_z \partial_y \partial_x F_\lambda(\Pi) \geq 0$. Otherwise,

$$\begin{aligned}
16q_\lambda^4 \partial_z \partial_y \partial_x F_\lambda(\Pi) &\geq 16\left(1 + 2\beta(x,y,z)\left(\frac{1}{4}\right) + \gamma(x,y,z)\left(\frac{1}{4}\right)^2\right) \\
&= x^2y^2z^2 + 4x^2yz + 4xy^2z + 4xyz^2 - 32xyz + x^2 + y^2 + z^2 \\
&\quad + 4xy + 4yz + 4xz - 16x - 16y - 16z + 52 \\
&= (7 - xyz)(1 - xyz) + 4xy(1-z)^2 + 4xz(1-y)^2 + 4yz(1-x)^2 \\
&\quad + (15 - x)(1 - x) + (15 - y)(1 - y) + (15 - z)(1 - z) \\
&\geq 0
\end{aligned}$$

which again implies $\partial_z \partial_y \partial_x F_\lambda(\Pi) \geq 0$.

In any case, we have $\partial_z \partial_y \partial_x F_\lambda(\Pi) \geq 0$ as desired. ■

4. Conclusion and discussion

We are about to conclude our work throughout the report in this section. Durante et al. [20], characterized the linear rational transformation $T(C)$ of a bivariate (quasi-)copula $C : [0,1]^2 \to [0,1]$, given in the form

$$T(C)(x,y) = \frac{a_0 + a_1x + a_2y + a_3C(x,y)}{b_0 + b_1x + b_2y + b_3C(x,y)}$$

where $a_i, b_i \in \mathbb{R}$, for $i = 0,1,2,3$. For the set of all bivariate quasi-copula $\mathcal{Q}$ and all bivariate copula $\mathcal{C}$, they showed that $T(\mathcal{Q}) \subseteq \mathcal{Q}$ if and only if $T(\mathcal{C}) \subseteq \mathcal{C}$ if and only if $T = T_\lambda$, $\lambda \in [0,1]$, where

$$T_\lambda(C)(x,y) = \frac{C(x,y)}{1 - \lambda + \lambda(2 - x - y) + \lambda C(x,y)}$$

for any copula(quasi-copula) C. They also showed that the bivariate Frechet-Hoeffding lower bound is the unique fixed point of T_λ for all $0 < \lambda \leq 1$. Moreover, they showed that the iterations $T_\lambda^n(C)$ that is defined recursively by $T_\lambda^{n+1} := T_\lambda^n \circ T_\lambda$ where $T_\lambda^1 := T_\lambda$ for all $n \geq 1$ tends to the bivariate Frechet-Hoeffding lower bound as n tends to infinity. We note that the bivariate Frechet-Hoeffding lower bound is a copula.

In our works, we extend the previous result to a transformation for a multivariate (quasi-)copula C given in the form

$$F(C)(x_1, \ldots, x_k) = \frac{a_0 + \sum_{i=1}^k a_i x_i + a_{k+1} C(x_1, \ldots, x_k)}{b_0 + \sum_{i=1}^k b_i x_i + b_{k+1} C(x_1, \ldots, x_k)}$$

where $x_1, \ldots, x_k \in [0,1]$ and $a_i, b_i \in \mathbb{R}$ for $i = 0, ..., k$. We successfully show that all multivariate quasi-copula transformations must be in the form

$$F_\lambda(C)(x_1, \ldots, x_k) = \frac{C(x_1, \ldots, x_k)}{1 - \lambda + \lambda \sum_{i=1}^k (1 - x_i) + \lambda C(x_1, \ldots, x_k)}$$

where $x_1, \ldots, x_k \in [0,1]$ and $\lambda \in [0,1]$. This result is analogous to the bivariate case.

However, we found that the only way F is a multivariate copula transformation is that

$F = F_0$. In other words, the only multivariate copula linear rational transformation is the trivial identity transformation. This is different from what happen in the bivariate case. Actually, this follows from the fact that the multivariate Frechet-Hoeffding lower bound is not a copula and the fact that the multivariate Frechet-Hoeffding lower bound is the unique fixed point of F_λ for all $0 < \lambda \leq 1$.

The above fact implies that $F_\lambda(C)$ might not be a copula even though C is a copula. Nevertheless, $F_\lambda(C)$ might still be a copula for some specific copula C. We demonstrate this point by consider the transformation of the trivariate product copula Π. We show that $F_\lambda(\Pi)$ is a copula when $0 \leq \lambda \leq \frac{1}{4}$.

To go further, the next problem we are interested in is finding the suitable range of λ that makes $F_\lambda(C)$ still a copula for the specific copula C. However, this might be challenging due to several complicated tasks. One is to find the necessary and sufficient conditions of a given copula and how it interacts with the range of λ. The next step is to prove that the volume is still non-negative whenever each variable varies in $[0, 1]$, as you can see as the variable increases, the more difficult to prove that the volume is still non-negative. It might be interesting if we found a suitable family of copulas such that the range of λ for the transformation of this family does not decrease to zero when the dimension tends to infinity.

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