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A Practical Approach to Quasi-convex Optimization

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Abstract A new and simple method for quasi-convex optimization is introduced from which its various applications can be derived. Especially, a global optimum under constrains can be approximated for all continuous functions.

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1. Problems and Solutions

A classical form of optimization formulated as: To find

$$\begin{aligned} &\arg\min_x f(x) \\ \text{subject to } &g_i(x) = 0, \\ &h_j(x) \le 0, \quad (i = 1, \dots, m; j = 1, \dots, n). \end{aligned} \tag{1.1}$$

Following [1], a solution of (1.1) can be obtained by minimizing a deformed function

$$f_t(\cdot|K, M) := (1-t)(f(\cdot) - K) + tM\left[\sum_{i=1}^{m} |g(\cdot)| + \sum_{j=1}^{n} (h(\cdot) + |h(\cdot)|)\right], \quad t \in (0,1) \tag{1.2}$$

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t is very close to 1 and K, M are large. A problem, however, arrives when the function

$$F(\cdot) := \left[\sum_{i=1}^{m} |g(\cdot)| + \sum_{j=1}^{n} (h(\cdot) - |h(\cdot)|)\right],$$

is not convex. A minimizer may be too far away from the feasible set. To fix the problem, we replace f_t by $f_t - |f_t|$. The rest of the paper will now pay attention to look for a method for optimization of quasi-convex like functions.

Consider a function $g : C = \prod_{i=1}^{p}[a_i, b_i] \to (-\infty, 0]$. Suppose $g < 0$ on a small neighborhood containing $x^* = (x_1^*, \ldots, x_p^*)$. To find a point in this neighborhood we introduce a point $x' = (x_1', \ldots, x_p')$ which transforms a point $x = (x_1, \ldots, x_p) \in C$ under the rule:

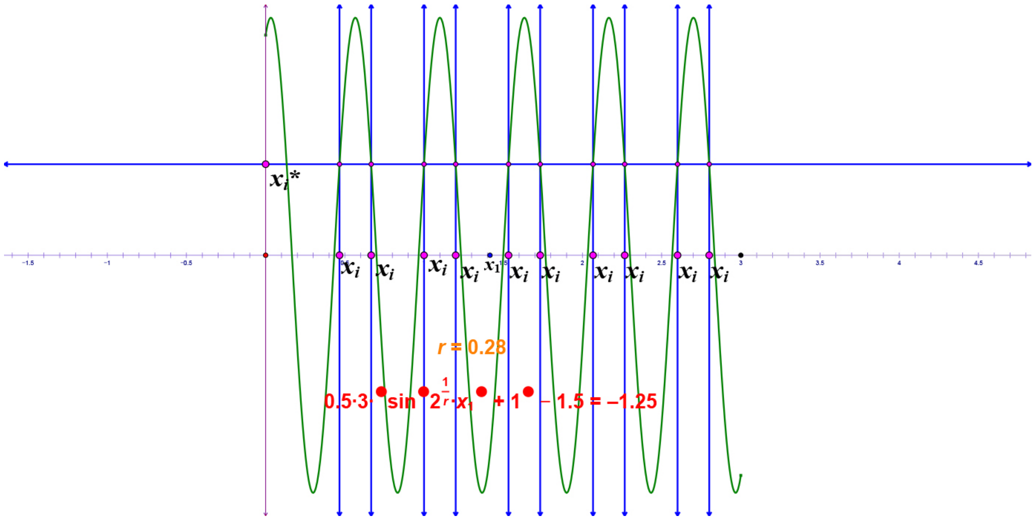
$$x_i' = .5(\sin(2^{\frac{1}{r_i}} x_i) + 1)(b_i - a_i) + a_i, \quad i = 1, \ldots, p, \tag{1.3}$$

where $r = (r_1, \ldots, r_p) \in (0, \infty)^p$.
It is seen that $x_i' = \gamma(\beta(x_i))$ where $\gamma(\beta)$ is the composition of $\beta : [a_i, b_i] \to [-1, 1], b \mapsto \sin(2^{\frac{1}{r_i}})b)$ followed by $\gamma : [-1, 1] \to [a_i, b_i], c \mapsto .5(b_i - a_i)(c + 1) + a_i$. Observe that $\gamma(\beta)$ sends $[a_i, b_i]$ onto itself many to one.
Put $u(x, r) = g(x')$. Clearly, if $x_i' = x_i^*$, then $u(x, r) < 0$ for all $x = (x_1, \ldots, x_p)$ for which the relation $x_i' = x_i^*$ holds for each i (see Figure 1). Figure 1a and 1b also reveal the fact that the number of points satisfying the relation $x_i' = x_i^*$, increases as r_i gets smaller.
This shows that, instead of working on f, we work on $u(x, r)$ to get more chances in finding a solution, i.e., a point in the neighborhood of x^*. Still, the problem remains, since we have no clue to move from any initial point $(x_0, r_0)= (x_{01}, \ldots, x_{0p}, r_{01}, \ldots, r_{0p})$ where $u(x_0, r_0) = 0$ to a new point (x_1, r_1) with $u(x_1, r_1) < 0$ (see Figure 4). Fortunately, we only add to $u(x, r)$ with a convex function

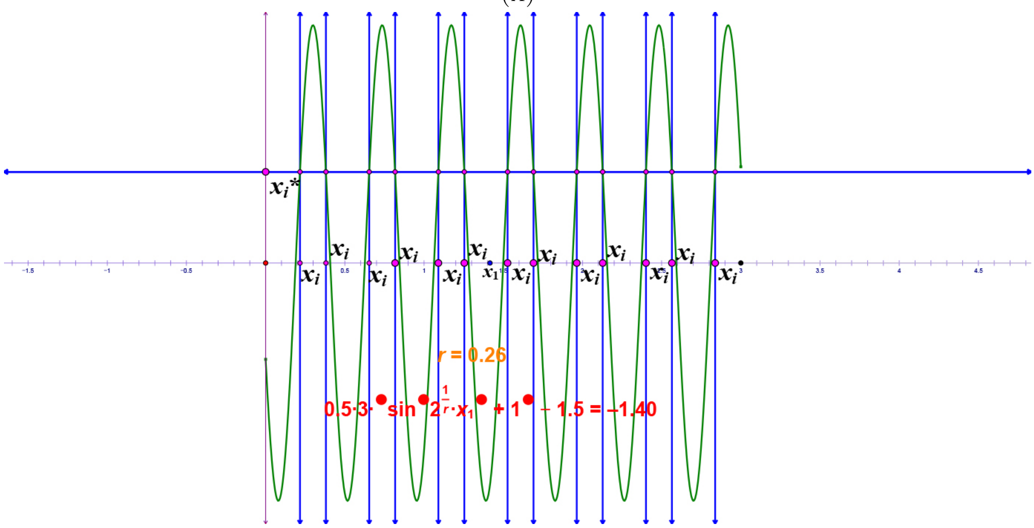
$$v(x, r) = (c_1 - x_1)^2 + \cdots + (c_p - x_p)^2 + (2 - r_1)^2 + \cdots + (2 - r_p)^2$$

and call $w = u + v$. Here c_i is the mid-point of $[a_i, b_i]$ for all i and 2 can be replaced by any positive number. Finally, we can apply any convex optimization algorithm to get any (local) minimizer of w over $(x_1, \ldots, x_p, r_1, \ldots, r_p)$. Note that any minimizer $x = (x_1, \ldots, x_p)$ solves the equation $x_i' = x_i^*$ at each i.
Now instead of finding $\arg\min_x f(x)$, we set to find x^* for which $f(x^*) < k$ for any given k. Then put $g = f - k - |f - k|$. So we shall find x^* satisfying $g(x^*) < 0$. This function g includes function f_t mentioned above.
In the following computation, we set $r_1 = \cdots = r_p$. Moreover, since $(c_1 - x_1)^2 + \cdots + (c_p - x_p)^2$ is convex as a function of x with its minimum value occurs at the point $c = (c_1, \ldots, c_p)$, we may choose to concentrate on a piece of the graph of w around $(c_1, \ldots, c_p)$.

Algorithm 1.1. (To find $x^*$for which $f(x^*) < k$)
Given $f : C = \prod_{i=1}^{p}[a_i, b_i] \to \mathbb{R}, k \in \mathbb{R}$. Set $g = f - k - |f - k|$.
Plug in x' by (1.3) to get $g(x')$. Recall that we fix all $r_1 = \ldots = r_p$ and set $v(x) = (c_1 - x_1)^2 + \cdots + (c_p - x_p)^2 + (2 - r_1)^2$, then compute $w(x, r_1) := g(x') + v(x, r_1)$.
Minimize w over (x, r_1) setting $(a_1, \ldots, a_p, r_1)$ as our initial point where r_1 is close to 0 and get any minimizer (x^*, r_1^*). We then get $g(x^{*'})) < 0$, i.e., $f(x^{*'}) < k$.



(A)



(B)

Figure 1

Obviously, a global minimum of f can be obtained by decreasing k as much as we can. Another way to minimize w is as follow: We are looking for cross sections of w at the point (c, r_1^*) where we find first a minimizer r_1^* of the function $r_1 \mapsto w(c, r_1)$. This means that the cross section along r_1 where the variables $x = (x_1, \ldots, x_p)$ is fixed at $c = (c_1, \ldots, c_p)$ has a local minimum at r_1^*. The point (c, r_1^*) will automatically a minimizer of w. Actually, we can set any point to start with instead of starting at c.

Example 1.2. Note, in Figure 2, 3, 4, 5a, 5b, that (x, y, z, u) stands for $(x_1, \ldots x_4)$ and set $r_1 = r_2 = r_3 = r_4 = r$ (dont be mixed up this r and the vector $r = (r_1, r_2, r_3, r_4)$ in the above context).
Consider the function
$f(x_1, \ldots, x_4) = (x_1 - 2)^2 + (x_2 + 2)^2 + (x_3 - 2)^2 + (x_4 + 2)^2 - .55 - |(x_1 - 2)^2 + (x_2 + 2)^2 + (x_3 - 2)^2 + (x_4 + 2)^2 - .55|, ((x_1, \ldots, x_4)) \in [-1.5, 1.5]^4$ (see Figure 2 for its graph). Find $((x_1, \ldots, x_4)) \in [-1.5, 1.5]^4$ satisfying $f((x_1, \ldots, x_4)) < 0$.

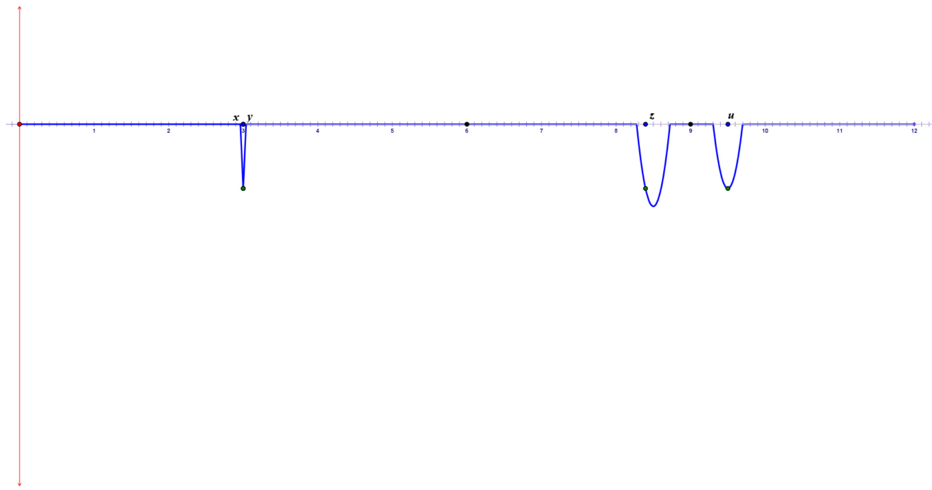


Figure 2

It is easy to guess that $(1.5, -1.5, 1, -1)$ is one such an answer. Most of the time, all cross sections at arbitrary $(x_1, \ldots, x_4)$ of f will be zero as in Figure 3.

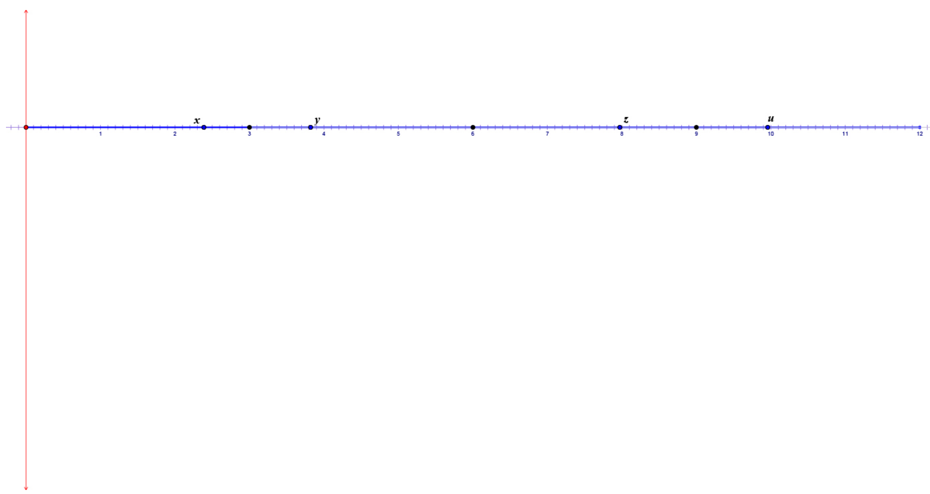


Figure 3

Plug in $(x_1^{'}, \ldots, x_4^{'})$ by (1.1) into f and get its graph as in Figure 4:

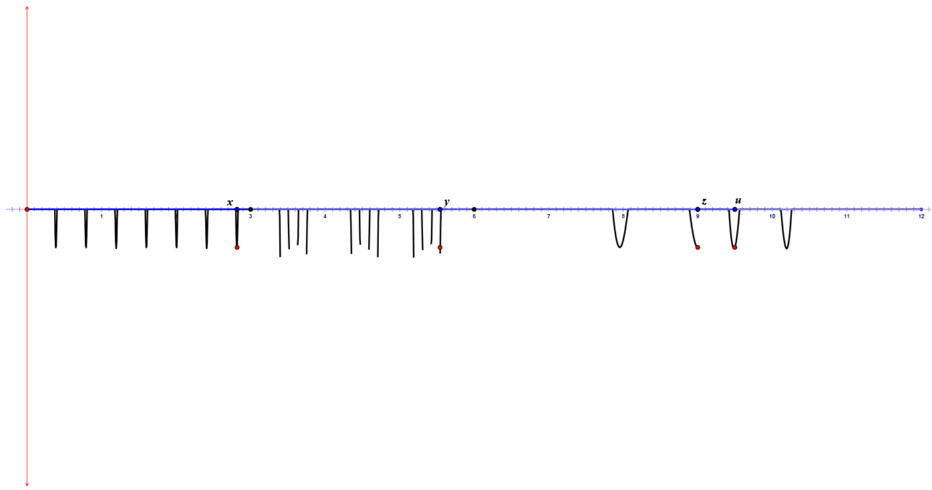
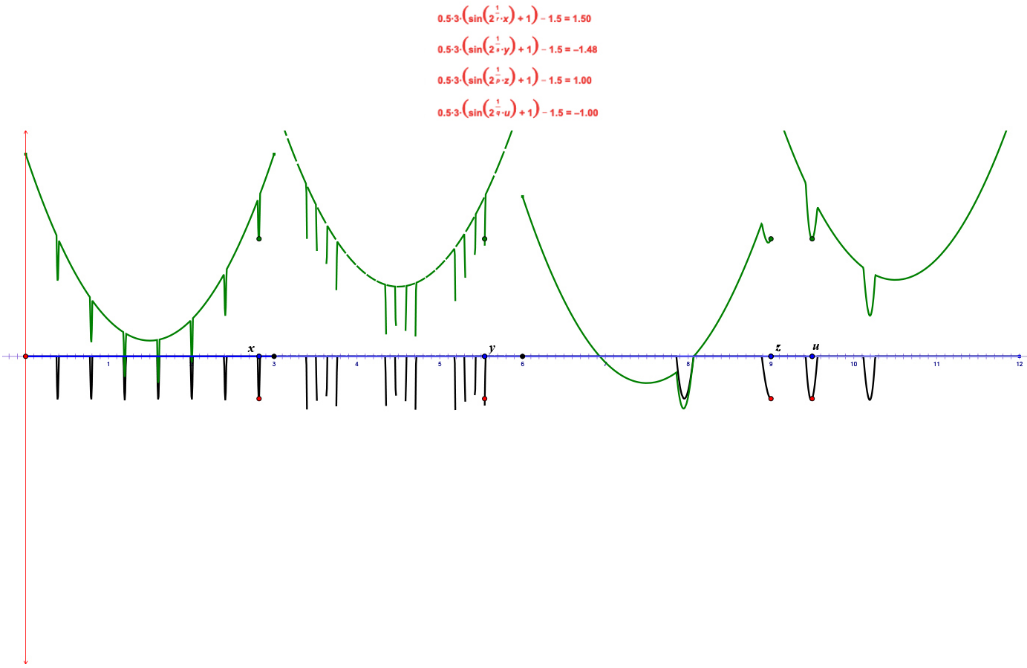
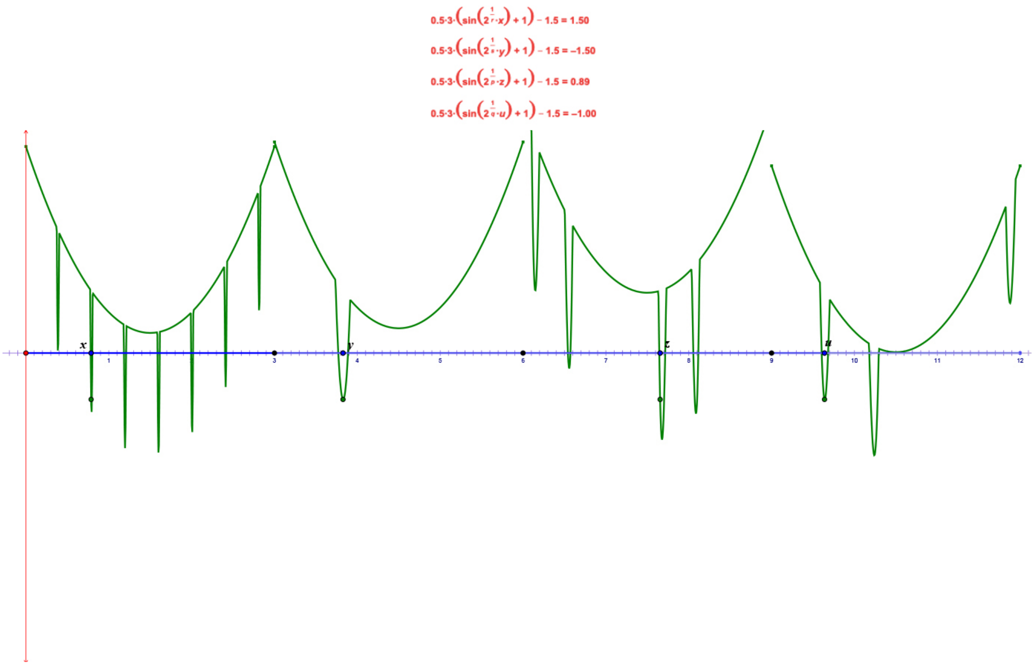


FIGURE 4

Now consider the graphs of w in Figure 5a and Figure 5b for different minimizers r_1^*. Each Figure shows a point in a neighborhood of x^* where $f(x^*) < 0$, namely, the point $(1.5, -1.48, 1, -1)$ and the point $(1.6, -1.5, .89. -1)$ respectively.


0.5·3·(sin(2^(1/r)·x) + 1) − 1.5 = 1.50
0.5·3·(sin(2^(1/s)·y) + 1) − 1.5 = −1.48
0.5·3·(sin(2^(1/p)·z) + 1) − 1.5 = 1.00
0.5·3·(sin(2^(1/q)·u) + 1) − 1.5 = −1.00
x
y
z
u

(A)


0.5·3·(sin(2^(1/r)·x) + 1) − 1.5 = 1.50
0.5·3·(sin(2^(1/s)·y) + 1) − 1.5 = −1.50
0.5·3·(sin(2^(1/p)·z) + 1) − 1.5 = 0.89
0.5·3·(sin(2^(1/q)·u) + 1) − 1.5 = −1.00
x
y
z
u

(B)

FIGURE 5

Acknowledgments

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