Thai Journal of Mathematics
Volume 20 Number 4 (2022)
Pages 1609–1619

http://thaijmath.in.cmu.ac.th

ISSN 1686-0209

Connectivity Concepts in Bipolar Fuzzy Incidence Graphs

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Abstract Incidence graph can be used to model some non-deterministic intercommunication networks having extra vertex-edge relationship. For two opposite sided thinking (i.e. bipolarity), bipolar fuzzy graph (BFG) can be used for better solution in many problems of real life. In this article, the intuition of bipolar fuzzy incidence graph (BFIG) and its matrix form are proposed. Bipolar fuzzy incidence subgraph is defined with several properties. Incidence pairs, paths and connectivities between pairs in BFIGs are introduced. Different types of strong and cut pair in BFIGs are examined with their properties.

MSC: 05C72; 05C76

Keywords: bipolar fuzzy incidence graph; incidence paths; incidence connectivities; strong pair; cut pair

Submission date: 02.09.2020 / Acceptance date: 24.03.2021

1. Introduction

Graph theory with its several operations have huge number of applications to solve many problems which are connected to the real life system. Zadeh's fuzzy sets [1] gives the mathematics an extraordinary tool for explain the situation in which uncertainty present. Due to the existence of fuzziness or uncertainty of vertices and edges in graphs, Rosenfeld's fuzzy graph (FG) [2] gives us solutions in many decision making application fields including optimization, computer engineering, network routing, artificial intelligence, city planning, image segmentation, medical science, etc. The strong path between vertices in FG are formulated in [3]. Many operations with their properties in FG theory have been clearly explained in [4]. Ghorai and Pal [5, 6] established various types of FGs with their several properties. Some basic definitions of paths, circuit, strong and complete FG and their applications are briefly discussed in [7]. Strong arcs and paths of generalized FGs and their real applications are given in [8]. Dinesh [9] first introduced the notion of fuzzy incidence graphs. Different types of nodes and properties of fuzzy incidence graph have been discussed in [10, 11]. Concepts of many intuitionistic FGs and their tolerance properties are investigated in [12, 13].

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Published by The Mathematical Association of Thailand.

In 1994, Zhang [14] established the notion of bipolar fuzzy sets. BFG gives more accuracy in some connective network fields when there is a positive thinking side and negative thinking side like: forward and backward, effect and side effect, cooperation and competition, gain and loss etc. Yang et al. [15] introduced the generalized BFG. Connectivity in bipolar fuzzy subgraphs are defined in [16]. The important parts like path, strength of connectedness in BFG are initiated by Akram [17] in 2011. The categorical properties and their operations of BFGs are given in [18]. Ghorai and Pal [19] established different types of product of BFGs and their properties. The geodesic distances and different types of nodes in BFG are described in [20]. Poulik and Ghorai [21–24] initiated degrees of nodes and indices of BFGs with applications in real life systems. Interval-valued fuzzy graphs and Interval-valued neutrosophic graphs and their application in education system have been discussed in [25]. Diffrent types of vague graphs with their operations and applications are described in [26–29]. Some labeling of FGs and intuitionistic FGs are introduced in [30–32]. Samanta et al. [33–35] introduced different types fuzzy planar graphs and their isomorphic properties, applications. In this paper, the BFIG and its properties are described with several examples in section 3. Differenet types of pair and their connectivity in BFIG are explained by examples in section 4. Many strong pair, cut pair and their properties in BFIG are examined by examples in section 5. A conclusion of this work is given in section 6.

2. Preliminaries

Here some basic definitions and properties are given.

Definition 2.1. [2] A FG $G = (V, \mu, \sigma)$ of the graph $G^* = (V, E)$ is defined by $\mu : V \to [0, 1]$ and $\sigma : V \times V \to [0, 1]$ such that for all $a, b \in V$,
$\sigma(ab) \leq \mu(a) \wedge \mu(b)$,
where $\mu(a)$ and $\sigma(ab)$ represent the membership values of the vertex a and the edge ab in G respectively and '$\wedge$' denote minimum value.

Definition 2.2. [15] A BFG of a graph $G^* = (V, E)$ is a triplet $G = (V, A, B)$ where $V = (\mu_A^P, \mu_A^N)$ is a bipolar fuzzy sets in V and $B = (\mu_B^P, \mu_B^N)$ is a bipolar fuzzy sets in $\widetilde{V^2}$ such that $\mu_B^P(xy) \leq min\{\mu_A^P(x), \mu_A^P(y)\}, \forall xy \in \widetilde{V^2}$ and $\mu_B^N(xy) \geq max\{\mu_A^N(x), \mu_A^N(y)\}, \forall xy \in \widetilde{V^2}$ and $\mu_B^P(xy) = \mu_B^N(xy) = 0$, $\forall xy \in \widetilde{V^2} - E$.

Definition 2.3. [16] A BFG $G' = (V', A', B')$ is said to be a bipolar fuzzy subgraph of a BFG $G = (V, A, B)$ if $V' \subseteq V$, $E' \subseteq E$ such that $\mu_A^P(a) = \mu_{A'}^P(a)$, $\mu_A^N(a) = \mu_{A'}^N(a)$, $\forall a \in V'$ and $\mu_{B'}^P(ay) = \mu_B^P(ay)$, $\mu_{B'}^N(ay) = \mu_B^N(ay)$ for every edge ay of G'.

Definition 2.4. [17] Let $G = (V, A, B)$ be a bipolar fuzzy graph and $x, y \in V$.

- A path $P : x = x_0, x_1, \ldots, x_{k-1}, x_k = y$ in G is a sequence of distinct vertices such that $(\mu_B^P(x_{i-1}x_i) > 0, \mu_B^N(x_{i-1}x_i) < 0), i = 1, 2, \ldots, k$ and the length of the path is k.
- If $P : x = x_0, x_1, \ldots, x_{k-1}, x_k = y$ be a path of length k between x and y, then $(\mu_B^P(xy))^k$ and $(\mu_B^N(xy))^k$ are defined as $(\mu_B^P(xy))^k = \sup\{\mu_B^P(xx_1) \wedge \mu_B^P(x_1x_2) \wedge \ldots \wedge \mu_B^P(x_{k-1}y)\}$ and $(\mu_B^N(xy))^k = \inf\{\mu_B^N(xx_1) \vee \mu_B^N(x_1x_2) \vee \ldots \vee \mu_B^N(x_{k-1}y)\}$. $((\mu_B^P(xy))^\infty, (\mu_B^N(xy))^\infty)$ is said to be the strength of connectedness between two

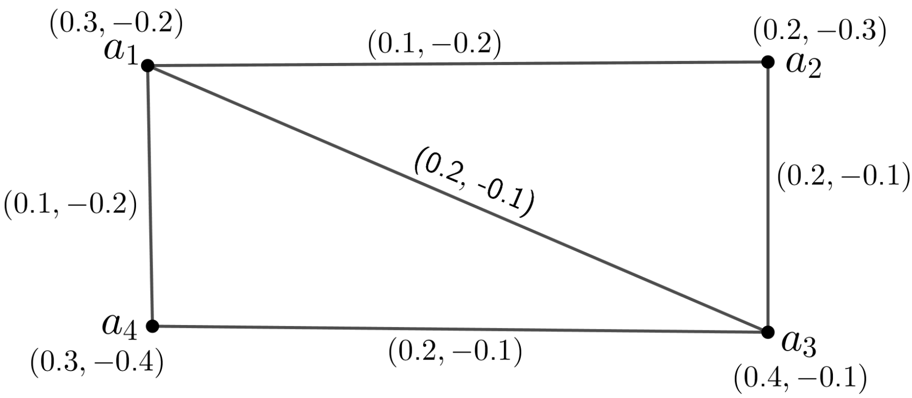

FIGURE 1. Example of a BFG G

vertices x and y in G, where $(\mu_B^P(xy))^\infty = \sup_{k\in\mathbb{N}}\{(\mu_B^P(xy))^k\}$ and $(\mu_B^N(xy))^\infty = \inf_{k\in\mathbb{N}}\{(\mu_B^N(xy))^k\}$.

- If $\mu_B^P(xy) \geq (\mu_B^P(xy))^\infty$ and $\mu_B^N(xy) \leq (\mu_B^N(xy))^\infty$, then the arc xy in G is said to be a strong arc. A path $x - y$ is strong path if all arcs on the path are strong.

Definition 2.5. [36] For a BFG G, if $\mu_B^P(ay) \geq \big(\mu_B^P(ay)\big)^\infty$ and $\mu_B^N(ay) \leq \big(\mu_B^N(ay)\big)^\infty$, then the edge ay is said to be a strong edge of G.

3. Bipolar Fuzzy Incidence Graph

In this section, we defined BFIG and its matrix form. Properties of bipolar fuzzy incidence subgraphs are given with examples.

Definition 3.1. Let $G = (V, A, B)$ be a BFG. Let $\psi = (\psi^P, \psi^N)$ be a mapping from $V \times E$ to $([0, 1], [-1, 0])$ such that $\psi^P(a, ab) \leq min\{\mu_A^P(a), \mu_B^P(ab)\}$ and $\psi^N(a, ab) \geq max\{\mu_A^N(a), \mu_B^N(ab)\}$, $\forall a \in V, \forall ab \in E$. Then ψ is called the bipolar fuzzy incidence of the BFG G and $\widetilde{G} = (V, A, B, \psi)$ is called a BFIG.

Example 3.2. In Fig. 1, G is a BFG. Here $V = \{a_1, a_2, a_3, a_4\}$ and $\widetilde{V^2} = \{a_1a_2, a_1a_3, a_1a_4, a_2a_3, a_3a_4\}$. $(\mu_A^P(a_1), \mu_A^N(a_1)) = (0.3, -0.2), (\mu_B^P(a_1a_2), \mu_B^N(a_1a_2)) = (0.1, -0.2)$. Now, from Fig. 2, we have $\psi^P(a_1, a_1a_2) = 0.1 \leq min\{\mu_A^P(a_1), \mu_B^P(a_1a_2)\}$ and $\psi^N(a_1, a_1a_2) = -0.1 \geq max\{\mu_A^N(a_1), \mu_B^N(a_1a_2)\}$. Similarly, $\psi^P(a_i, a_ia_j) \leq min\{\mu_A^P(a_i), \mu_B^P(a_ia_j)\}$ and $\psi^N(a_i, a_ia_j) \geq max\{\mu_A^N(a_i), \mu_B^N(a_ia_j)\}$, $\forall a_i \in V, \forall a_ia_j \in E$. Therefore $\widetilde{G}$ is a BFIG.

Definition 3.3. Let $\widetilde{G} = (V, A, B, \psi)$ be a BFIG an ψ be the bipolar fuzzy incidence $\widetilde{G}$. Consider $\widetilde{G}$ has n vertices $a_1, a_2, a_3, \ldots, a_n$ and m edges $e_1, e_2, e_3, \ldots, e_m$. The matrix form of the bipolar fuzzy incidence ψ of $\widetilde{G}$ is denoted by $[\psi_{ij}]_{n\times m}$ and is defined as

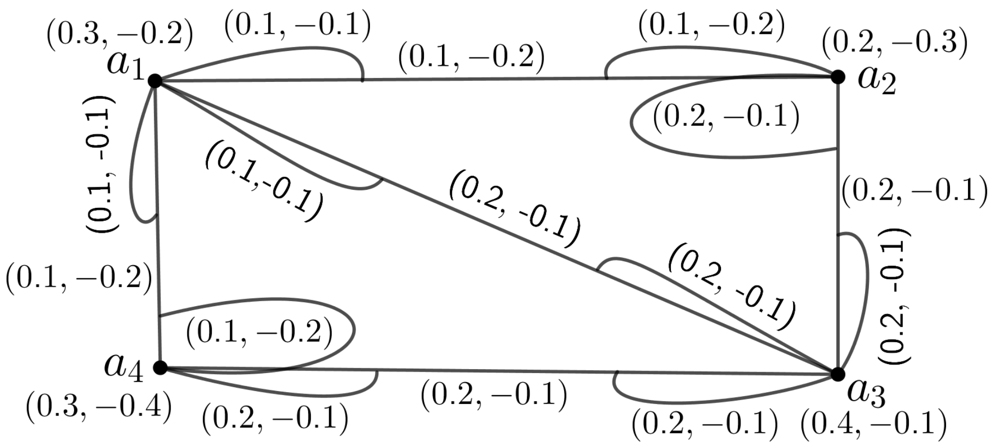

FIGURE 2. Example of a bipolar fuzzy incidence graph $\widetilde{G}$ of the BFG G of Fig. 1

$$[\psi_{ij}]_{n\times m} = \begin{array}{c} \\ a_1 \\ a_2 \\ . \\ . \\ a_n \end{array} \begin{array}{c} \begin{array}{ccccc} e_1 & e_2 & e_3 & \cdots & e_m \end{array} \\ \left[\begin{array}{ccccc} \psi_{11} & \psi_{12} & \psi_{13} & \cdots & \psi_{1m} \\ \psi_{21} & \psi_{22} & \psi_{23} & \cdots & \psi_{2m} \\ . & . & . & \cdots & . \\ . & . & . & \cdots & . \\ \psi_{n1} & \psi_{n2} & \psi_{n3} & \cdots & \psi_{nm} \end{array}\right] \end{array}$$

where $\psi_{ij} = \big(\psi^P(a_i, a_ia_j), \psi^N(a_i, a_ia_j)\big)$.

Example 3.4. Consider the BFIG $\widetilde{G}$ of Fig. 2. Here the number of vertices in $\widetilde{G}$ is 4 and the number of edges is 5. Therefore the matrix form of ψ of $\widetilde{G}$ is given below as:

$$[\psi_{ij}]_{4\times 5} = \begin{array}{c} \\ a_1 \\ a_2 \\ a_3 \\ a_4 \end{array} \begin{array}{c} \begin{array}{ccccc} a_1a_2 & a_1a_3 & a_1a_4 & a_2a_3 & a_3a_4 \end{array} \\ \left[\begin{array}{ccccc} (0.1,-0.1) & (0.1,-0.1) & (0.1,-0.1) & (0,0) & (0,0) \\ (0.1,-0.2) & (0.2,-0.1) & (0,0) & (0,0) & (0,0) \\ (0,0) & (0.2,-0.1) & (0,0) & (0.2,-0.1) & (0.2,-0.1) \\ (0,0) & (0,0) & (0.1,-0.2) & (0,0) & (0.2,-0.1) \end{array}\right] \end{array}$$

Definition 3.5. A BFIG $\widetilde{H} = (V', A', B', \psi')$ is said to be a bipolar fuzzy incidence partial subgraph of the BFIG $\widetilde{G} = (V, A, B, \psi)$ if $\mu^P_{A'}(a) \leq \mu^P_A(a)$, $\mu^N_{A'}(a) \geq \mu^N_A(a)$, $\mu^P_{B'}(ab) \leq \mu^P_B(ab)$, $\mu^N_{B'}(ab) \geq \mu^N_B(ab)$, ${\psi'}^P(a, ab) \leq \psi^P(a, ab)$ and ${\psi'}^N(a, ab) \geq \psi^N(a, ab)$, $\forall a \in V, \forall ab \in E$.

Also, $\widetilde{H}$ is said to be a bipolar fuzzy incidence subgraph of the BFIG $\widetilde{G}$ if $V' \subseteq V$, $E' \subseteq E$, $\psi' \subseteq \psi$, $\mu^P_{A'}(a) = \mu^P_A(a)$, $\mu^N_{A'}(a) = \mu^N_A(a)$, $\mu^P_{B'}(ab) = \mu^P_B(ab)$, $\mu^N_{B'}(ab) = \mu^N_B(ab)$, ${\psi'}^P(a, ab) = \psi^P(a, ab)$ and ${\psi'}^N(a, ab) = \psi^N(a, ab)$, $\forall a \in V, \forall ab \in E$.

If we delete a vertex or an edge from a BFIG, then the effects are initiated in next propositions.

Proposition 3.6. *A bipolar fuzzy incidence subgraph of a BFIG $\widetilde{G}$ must be a bipolar fuzzy incidence partial subgraph of $\widetilde{G}$.*

Proof. Let $\widetilde{H}$ be a bipolar fuzzy incidence subgraph of the BFIG $\widetilde{G}$. Then from the definitions, we have $\widetilde{H}$ satisfies all the conditions to be a bipolar fuzzy incidence partial subgraph of the BFIG $\widetilde{G}$. Thus, $\widetilde{H}$ is a bipolar fuzzy incidence subgraph of the BFIG $\widetilde{G}$. ∎

Proposition 3.7. *If $\widetilde{H}$ be a bipolar fuzzy incidence subgraph of the BFIG $\widetilde{G}$, then the BFG H is a bipolar fuzzy subgraph of the BFG G.*

Proof. $\widetilde{H}$ is a bipolar fuzzy incidence subgraph of the BFIG $\widetilde{G}$. Then from the definitions, we have the corresponding BFG H satisfies all the conditions to be a bipolar fuzzy subgraph of the BFG G. Therefore, H is a bipolar fuzzy subgraph of the BFG G. ∎

Definition 3.8. A BFIG $\widetilde{G} = (V, A, B, \psi)$ is said to be complete BFIG if $\psi^P(a, ab) = min\{\mu_A^P(a), \mu_B^P(ab)\}$ and $\psi^N(a, ab) = max\{\mu_A^N(a), \mu_B^N(ab)\}$, $\forall a \in V$ and $\forall ab \in E$.

A BFIG $\widetilde{G} = (V, A, B, \psi)$ is said to be strong if $\psi^P(a, ab) = min\{\mu_A^P(a), \mu_B^P(ab)\}$ and $\psi^N(a, ab) = max\{\mu_A^N(a), \mu_B^N(ab)\}$, $\forall$ *pairs* (a, ab) in $\widetilde{G}$.

If $\widetilde{G} = (V, A, B, \psi)$ is a complete BFIG and the vertices a, b are adjacnt to the edge ab, then $\psi^P(a, ab) = min\{\mu_A^P(a), \mu_B^P(ab)\} = \mu_B^P(ab) = min\{\mu_A^P(b), \mu_B^P(ab)\} = \psi^P(b, ab)$ and $\psi^N(a, ab) = max\{\mu_A^N(a), \mu_B^N(ab)\} = \mu_B^N(ab) = max\{\mu_A^N(b), \mu_B^N(ab)\} = \psi^N(b, ab)$.

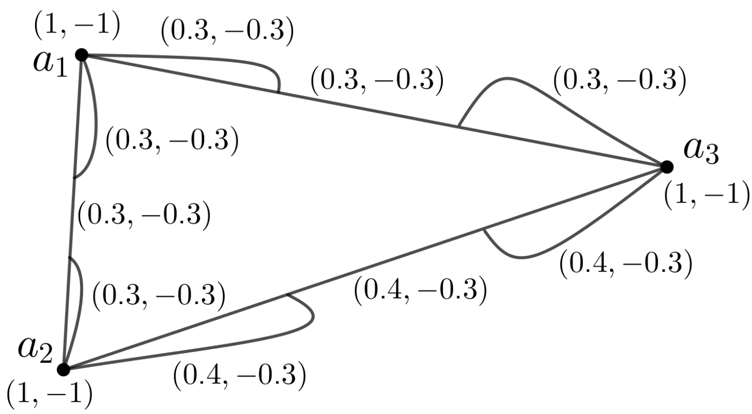

FIGURE 3. Example of a complete BFIG $\widetilde{G}$ which is also a strong BFIG

Theorem 3.9. *A complete BFIG is a strong BFIG.*

Proof. Let $\widetilde{G} = (V, A, B, \psi)$ be a complete BFIG and (a, ab) be a pair in $\widetilde{G}$. Then $\psi^P(a, ab) = min\{\mu_A^P(a), \mu_B^P(ab)\}$ and $\psi^N(a, ab) = max\{\mu_A^N(a), \mu_B^N(ab)\}$, $\forall a \in V$ and $\forall ab \in E$. So, $\psi^P(a, ab) = min\{\mu_A^P(a), \mu_B^P(ab)\}$ and $\psi^N(a, ab) = max\{\mu_A^N(a), \mu_B^N(ab)\}$, $\forall$ *pairs* (a, ab) in $\widetilde{G}$.
Therefore, $\widetilde{G}$ is a strong BFIG. ∎

4. Paths and Connectivities

Here we discussed many incidence path in BFIGs by examples. Some theorems are introduced.

Definition 4.1. Let $a = a_1, a_2, \ldots, a_{n-1} = b, a_n = c$ are the n vertices in a BFIG $\widetilde{G}$. Then $a_1, (a_1, a_1a_2), a_1a_2, (a_2, a_1a_2), a_2, \ldots, b, (b, bc), bc, (c, bc)$ is called an incidence path in $\widetilde{G}$. The incidence strength of this path is denoted by $(\psi^P_{a,bc}, \psi^N_{a,bc})$ and is defined as $\psi^P_{a,bc} = \psi^P(a_1, a_1a_2) \wedge \psi^P(a_2, a_1a_2) \wedge \ldots \wedge \psi^P(b, bc) \wedge \psi^P(c, bc)$ and $\psi^N_{a,bc} = \psi^N(a_1, a_1a_2) \vee \psi^N(a_2, a_1a_2) \vee \ldots \vee \psi^N(b, bc) \vee \psi^N(c, bc)$.
The incidence strength of connectedness between a and bc in $\widetilde{G}$ is $ICONN_G(a, bc)$ $= \big(ICONN^P_G(a, bc), ICONN^N_G(a, bc)\big)$, where $ICONN^P_G(a, bc) = \vee\{\psi^P_{a,bc}\}$ = maximum value of positive part of incidence strengths of all the paths between a and bc and $ICONN^N_G(a, bc) = \wedge\{\psi^N_{a,bc}\}$ = minimum value of negative part of incidence strengths of all the paths between a and bc.

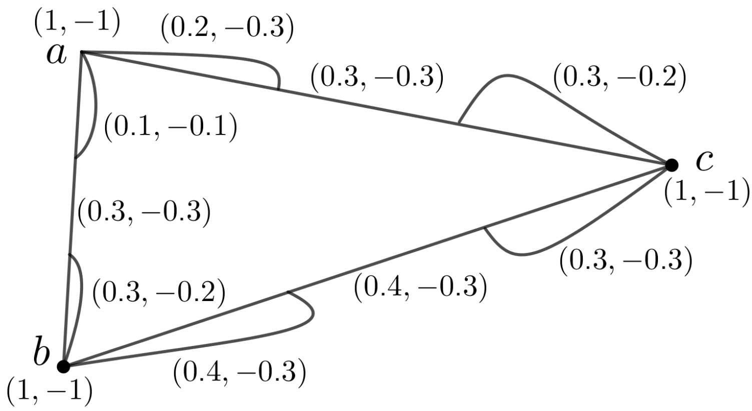

Figure 4. A BFIG $\widetilde{G}$

Example 4.2. Consider the connected BFIG $\widetilde{G}$ of Fig. 4. Here $\psi^P(a, ab) \wedge \psi^P(b, ab) \wedge \psi^P(b, bc) = 0.1$, $\psi^N(a, ab) \vee \psi^N(b, ab) \vee \psi^N(b, bc) = -0.1$, $\psi^P(a, ac) \wedge \psi^P(c, ac) \wedge \psi^P(c, bc) = 0.2$, $\psi^N(a, ac) \vee \psi^N(c, ac) \vee \psi^N(c, bc) = -0.2$. So, $ICONN^P_G(a, bc) = \vee\{0.2, 0.1\} = 0.2$ and $ICONN^N_G(a, bc) = \wedge\{-0.2, -0.1\} = -0.2$. Therefore the incidence strength of connectedness between a and bc is $ICONN_G(a, bc) = (0.2, -0.2)$.

Definition 4.3. Let ab be an edge of a BFIG $\widetilde{G} = (V, A, B, \psi)$. If $\psi^P(a, ab) > 0$, $\psi^P(b, ab) > 0$, $\psi^N(a, ab) < 0$ and $\psi^N(b, ab) < 0$, then (a, ab) and (b, ab) are called pairs. $\widetilde{G}$ is said to be connected if there exists an incidence path between every pair.

Theorem 4.4. *Let $\widetilde{G} = (V, A, B, \psi)$ be a BFIG and $\widetilde{H} = (V', A', B', \psi')$ be a bipolar fuzzy incidence subgraph of $\widetilde{G}$. Then for any pair (a, ab) in $\widetilde{G}$, $ICONN^P_G(a, ab) \geq ICONN^P_H(a, ab)$ and $ICONN^N_G(a, ab) \leq ICONN^N_H(a, ab)$.*

Proof. Consider $\widetilde{H}$ is a bipolar fuzzy incidence subgraph of $\widetilde{G}$. from definition of bipolar fuzzy incidence subgraph we have ${\psi'}^P(a, ab) = \psi^P(a, ab)$ and ${\psi'}^N(a, ab) = \psi^N(a, ab)$, for all pair (a, ab) in $\widetilde{H}$. But, $ICONN^P_G(a, ab)$, $ICONN^P_H(a, ab)$ and $ICONN^N_G(a, ab)$, $ICONN^N_H(a, ab)$ lies on same incidence pair of $\widetilde{H}$ and $\widetilde{H}$ or lies on defferent pairs of $\widetilde{H}$ and $\widetilde{H}$. Now there arises two cases.

Case 1. Suppose $ICONN^P_G(a, ab)$, $ICONN^P_H(a, ab)$ and $ICONN^N_G(a, ab)$, $ICONN^N_H(a, ab)$ lies on same pair (x, xy) of $\widetilde{H}$ and $\widetilde{G}$. Then from the definition of bipolar fuzzy incidence subgraph we have ${\psi'}^P((x,xy) = \psi^P((x,xy)$ and ${\psi'}^N((x,xy) = \psi^N((x,xy),$. Then $ICONN^P_G(a,ab) = ICONN^P_H(a,ab)$ and $ICONN^N_G(a,ab) = ICONN^N_H(a,ab)$.

Case 2. Suppose $ICONN^P_G(a,ab)$, $ICONN^P_H(a,ab)$ and $ICONN^N_G(a,ab)$, $ICONN^N_H(a,ab)$ lies on the pairs (x_1,x_1y_1) in $\widetilde{G}$ and (x_2,x_2y_2) in $\widetilde{H}$. This means both the pairs (x_1,x_1y_1) and (x_2,x_2y_2) are the pairs of $\widetilde{G}$. If $\psi^P(x_1,x_1y_1) = \psi^P(x_2,x_2y_2)$ and $\psi^N(x_1,x_1y_1) = \psi^N(x_2,x_2y_2)$, then $ICONN^P_G(a,ab) = ICONN^P_H(a,ab)$ and $ICONN^N_G(a,ab) = ICONN^N_H(a,ab)$. If $\psi^P(x_1,x_1y_1) \neq \psi^P(x_2,x_2y_2)$ or $\psi^N(x_1,x_1y_1 \neq \psi^N(x_2,x_2y_2)$ or both, then $ICONN^P_G(a,ab) > ICONN^P_H(a,ab)$ or $ICONN^N_G(a,ab) < ICONN^N_H(a,ab)$ or both.

Thus in any case $ICONN^P_G(a,ab) \geq ICONN^P_H(a,ab)$ and $ICONN^N_G(a,ab) \leq ICONN^N_H(a,ab)$. ■

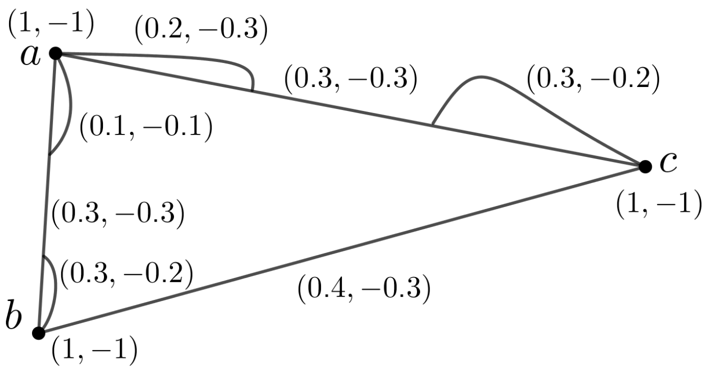

FIGURE 5. A bipolar fuzzy incidence subgraph $\widetilde{H}$ of the BFIG $\widetilde{G}$ of Fig. 4

Example 4.5. Consider the BFIGs $\widetilde{H}$ of Fig. 5 and $\widetilde{G}$ of Fig. 4. Here $\widetilde{H}$ is a bipolar fuzzy incidence subgraph of $\widetilde{G}$. Now, $ICONN^P_G(a,ab) = 0.2$, $ICONN^N_G(a,ab) = -0.2$, $ICONN^P_H(a,ab) = 0.1$, $ICONN^H_G(a,ab) = -0.1$. Therefore $ICONN^P_G(a,ab) > ICONN^P_H(a,ab)$ and $ICONN^N_G(a,ab) < ICONN^N_H(a,ab)$.

5. Strong Pair and Cut Pair in BFIG

Here Strong pair and cut pair in BFIG are defined with examples. Some theorems related to these are also given.

Definition 5.1. A pair (a,ab) in $\widetilde{G} = (V,A,B,\psi)$ is strong pair if $\psi^P(a,ab) \geq ICONN^P_G(a,ab)$ and $\psi^N(a,ab) \leq ICONN^N_G(a,ab)$.

Example 5.2. For the BFIG $\widetilde{G}$ of Fig. 4. Here $\psi^P(a,ab) = 0.1$, $\psi^N(a,ab) = -0.1$, $\psi^P(b,bc) = 0.4$, $\psi^N(b,bc) = -0.3$, $ICONN^P_G(a,ab) = 0.2$, $ICONN^N_G(a,ab) = -0.2$, $ICONN^P_G(b,bc) = 0.4$, $ICONN^N_G(b,bc) = -0.3$. So, $\psi^P(a,ab) < ICONN^P_G(a,ab)$, $\psi^N(a,ab) > ICONN^N_G(a,ab)$ and $\psi^P(b,bc) = ICONN^P_G(b,bc)$, $\psi^N(b,bc) = ICONN^N_G(b,bc)$. Therefore (a,ab) is not a strong pair but (b,bc) is a strong pair.

Theorem 5.3. *Let $\widetilde{G} = (V, A, B, \psi)$ be a BFIG and a pair (a, ab) in $\widetilde{G}$ such that $\big(\psi^P(a, ab), \psi^N(a, ab)\big) = \big(\vee\{\psi^P(x, xy)\}, \wedge\{\psi^N(x, xy)\}\big)$, for all pairs (x, xy) in $\widetilde{G}$. Then the pair (a, ab) is a strong pair.*

Proof. Let (a, ab) be a pair in the BFIG $\widetilde{G}$. If there is an unique pair (x, xy) in $\widetilde{G}$ such that $\psi^P(a, ab) = \vee\{\psi^P(x, xy)\}$ and $\psi^N(a, ab) = \wedge\{\psi^N(x, xy)\}$, for all pair (x, xy) in $\widetilde{G}$, then the positive part of the strength of all the paths without (a, ab) must be less than $\psi^P(a, ab)$ and the negative part of the strength of all the paths without (a, ab) must be greater than $\psi^N(a, ab)$. Then (a, ab) is a strong pair.
Again, if there are more than one pair (a, ab) in $\widetilde{G}$ such that $\psi^P(a, ab) = \vee\{\psi^P(x, xy)\}$ and $\psi^N(a, ab) = \wedge\{\psi^N(x, xy)\}$, for all pair (x, xy) in $\widetilde{G}$, then $\psi^P(a, ab) = \psi^P(x, xy)$ and $\psi^N(a, ab) = \psi^N(x, xy)$, for all pair (x, xy) in $\widetilde{G} - \{(a, ab)\}$. Now from the definition, we have $\psi^P(a, ab) \geq ICONN^P_G(x, xy)$, $\psi^N(a, ab) \leq ICONN^N_G(x, xy)$, for all pair (x, xy) in $\widetilde{G}$. Hence (a, ab) is a strong pair. ∎

Definition 5.4. Let $\widetilde{G} = (V, A, B, \psi)$ be a BFIG and $\widetilde{H} = (V', A', B', \psi')$ be a bipolar fuzzy incidence subgraph of $\widetilde{G}$ such that for a pair (a, ab) in $\widetilde{G}$, $\widetilde{H} = \widetilde{G} - \{(a, ab)\}$. If $ICONN^P_G(x, xy) > ICONN^P_H(x, xy)$ and $ICONN^N_G(x, xy) < ICONN^N_H(x, xy)$ for some pair (x, xy) in $\widetilde{G}$, then the pair (a, ab) is called a incidence cut pair of $\widetilde{G}$.

Theorem 5.5. *Let (a, ab) be a pair in a BFIG $\widetilde{G} = (V, A, B, \psi)$ such that $\psi^P(a, ab) = \wedge\{\mu^P_A(a), \mu^P_B(ab)\}$ and $\psi^N(a, ab) = \vee\{\mu^N_A(a), \mu^N_B(ab)\}$. Then the pair (a, ab) is a strong pair of $\widetilde{G}$.*

Proof. Let $\widetilde{H} = (V', A', B', \psi')$ be a bipolar fuzzy incidence subgraph of a BFIG $\widetilde{G} = (V, A, B, \psi)$ such that $\widetilde{H} = \widetilde{G} - \{(a, ab)\}$, where (a, ab) is a pair of $\widetilde{G}$. If $\widetilde{H}$ is disconnected, then (a, ab) must be a cut pair of $\widetilde{G}$. Then $ICONN^P_G(a, ab) > ICONN^P_H(a, ab)$ and $ICONN^N_G(a, ab) < ICONN^N_H(a, ab)$. Then by definition (a, ab) is a strong pair.
If $\widetilde{H}$ is connected, then there exists pairs (a, ac) for some $a \neq c$, such that (a, ac) and (b, ab) lies in a incidence path from a to ab in $\widetilde{H}$. Then $ICONN^P_H(a, ab) \leq \wedge\{\psi^P(a, ac), \psi^P(b, ab)\}$ $\leq \wedge\{\mu^P_A(a), \mu^P_B(ac), \mu^P_A(b), \mu^P_B(ab)\} \leq \wedge\{\mu^P_A(a), \mu^P_B(ab)\} = \psi^P(a, ab)$ and $ICONN^N_H(a, ab)$ $\geq \wedge\{\psi^N(a, ac), \psi^N(b, ab)\} \geq \wedge\{\mu^N_A(a), \mu^N_B(ac), \mu^N_A(b), \mu^N_B(ab)\} \geq \wedge\{\mu^N_A(a), \mu^N_B(ab)\} =$ $\psi^N(a, ab)$. Thus $\psi^P(a, ab) \geq ICONN^P_H(a, ab)$ and $\psi^N(a, ab) \geq ICONN^N_H(a, ab)$. Therefore, (a, ab) is a strong pair of $\widetilde{G}$. ∎

Theorem 5.6. *Every pair of a complete BFIG is a strong pair.*

Proof. Let (a, ab) be a pair in a complete BFIG $\widetilde{G} = (V, A, B, \psi)$. Then $\psi^P(a, ab) = \wedge\{\mu^P_A(a), \mu^P_B(ab)\}$ and $\psi^N(a, ab) = \vee\{\mu^N_A(a), \mu^N_B(ab)\}$. Now, by Theorem 5.5, it can be proved that (a, ab) is a strong pair of $\widetilde{G}$. Since the pair (a, ab) is arbitrary in the complete BFIG $\widetilde{G}$. then all the pairs of $\widetilde{G}$ are strong pair. ∎

6. Conclision

Incidence connectivity between vertices and edges had been used in BFG. First, BFIG and bipolar fuzzy incidence subgraphs are defined and explained their properties by examples. Second, different incidence paths and their sterngths, connectedness and properties are explained. The strength of connectedness of betwen pairs in BFIG and bipolar fuzzy incidence subgraph are investigated. Third, using srtength of connectedness of pairs and incidence paths the concept of strong and cut pairs with thier properties had been discussed. There are many future works of the concept incidence graphs in different uncertain fields like intuitionistic fuzzy incidence graphs, m-polar fuzzy incidence graphs, pythagorean fuzzy incidence graphs, etc.

Acknowledgements

We would like to thank the referees for their comments and suggestions on the manuscript. This work was supported by the Department of Higher Education, Science and Technology and Biotechnology, Government of West Bengal, India, Swami Vivekananda merit-cum-means scholarship (Award No. 52-Edn (B)/5B-15/2017 dated 07/06/2017).

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