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Infinite Families of Congruences Modulo 5 for Ramanujan's General Partition Function

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Abstract For any non-negative integer n and non-zero integer r, let $p_r(n)$ denote Ramanujan's general partition function. By employing q-identities, we prove some new Ramanujan-type congruences modulo 5 for $p_r(n)$ for $r = -(5\lambda+1), -(5\lambda+3), -(5\lambda+4), -(25\lambda+1)$, $-(25\lambda+2)$, and any integer λ.

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1. Introduction

A partition of a positive integer n is a non-increasing sequence of positive integers, called parts, whose sum equals n. For example, $n = 3$ has three partitions, namely,

$$3, \quad 2+1, \quad 1+1+1.$$

If $p(n)$ denote the number of partitions of n, then $p(3) = 3$. The generating function for $p(n)$ is given by

$$\sum_{n=0}^{\infty} p(n)q^n = \frac{1}{(q;q)_\infty}, \quad p(0) = 1,$$

where, here and throughout this paper

$$(a;q)_\infty = \prod_{k=1}^{\infty}(1-aq^{k-1}), \qquad |q| \leq 1.$$

Ramanujan [1, 2] proved following beautiful congruences for $p(n)$:

$$p(5n+4) \equiv 0 \pmod{5}, \; p(7n+5) \equiv 0 \pmod{7}, \text{ and } p(13n+6) \equiv 0 \pmod{11}.$$

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In a letter to Hardy written from Fitzroy House late in 1918 [3, p. 192-193], Ramanujan introduced the general partition function $p_r(n)$ for integers $n \geq 0$ and $r \neq 0$ as

$$\sum_{n=0}^{\infty} p_r(n)q^n = \frac{1}{(q;q)_\infty^r}, \qquad |q| < 1. \tag{1.1}$$

In order to describe the partition function $p_r(n)$, we first give the notion of colour partition. A part in a partition of n is said to have r colours if there are r copies of each part available and all of them are viewed as distinct objects.

Now, for $r > 1$, $p_r(n)$ denotes the number of partitions of n where each part may have r distinct colours. For example, if each part in the partitions of 3 have *two* colours, say red and green, then the number of two colour partitions of 3 is 10, namely

$$3_r, \quad 3_g, \quad 2_r + 1_r, \quad 2_r + 1_g, \quad 2_g + 1_g, \quad 2_g + 1_r,$$
$$1_r + 1_r + 1_r, \quad 1_g + 1_g + 1_g, \quad 1_r + 1_g + 1_g, \quad 1_r + 1_r + 1_g.$$

Thus, $p_2(3) = 10$. For $r = 1$, $p_1(n)$ is the usual partition function $p(n)$ which counts the number of unrestricted partitions of a non-negative integer n. Gandhi [4] studied the colour partition function $p_r(n)$ and found Ramanujan-type congruences for certain values of r. For example, he proved that

$$p_2(5n + 3) \equiv 0 \pmod 5$$

and

$$p_8(11n + 4) \equiv 0 \pmod{11}.$$

Newman [5] also found some congruences for colour partition. Recently, Hirschhorm [6] found congruences for $p_3(n)$ modulo higher powers of 3.

If $r < 0$, then

$$p_r(n) = (p_r(n, e) - p_r(n, o)),$$

where $p_r(n, e)$ (resp. $p_r(n, o)$) is the number of partitions of n with even (resp. odd) number of distinct parts and each part have r colours. For example, if $n = 5$ and $r = -1$ then $p_{-1}(5, e) = 2$ with relevant partitions $4 + 1$ and $3 + 2$, and $p_{-1}(5, o) = 1$ with the relevant partition 5. Thus, $p_{-1}(5) = 2 - 1 = 1$. Similarly, we see that $p_{-2}(3) = 4 - 2 = 2$. The case $r = -1$ is the famous Euler's pentagonal number theorem.

Ramanujan [3] showed that, if λ is a positive integer and $\overline{w}$ is a prime of the form $6\lambda - 1$, then

$$p_{-4}\left(n\overline{w} - \frac{(\overline{w} + 1)}{6}\right) \equiv 0 \pmod{\overline{w}}.$$

Ramanathan [7], Atkin [8], and Ono [9]. Baruah and Ojah [10] also proved some congruences for $p_{-3}(n)$. Recently, Baruah and Sharma [11] proved some arithmetic identities and congruences of $p_r(n)$ for some particular negative values of r. Saikia and Chetry [12] also proved infinite families of congruences modulo 7 of $p_r(n)$ for negative values of r.

In this paper, we prove some new Ramanujan-type congruences modulo 5 for $p_r(n)$ for $r = -(5\lambda + 1), -(5\lambda + 3), -(5\lambda + 4), -(25\lambda + 1)$, and $-(25\lambda + 2)$, where λ is any integer, by employing some q-identities. In particular, we prove the following infinite families of congruences modulo 5 for the $p_r(n)$:

Theorem 1.1. *For any integer λ and $\ell = 3, 4$, we have*

$$p_{-(5\lambda+1)}(5n + \ell) \equiv 0 \pmod 5.$$

Theorem 1.2. *For any integer λ and $\ell = 2, 3, 4$, we have*

$$p_{-(5\lambda+3)}(5n+\ell) \equiv 0 \pmod 5.$$

Theorem 1.3. *For any integer λ, we have*

$$p_{-(5\lambda+4)}(5n+4) \equiv 0 \pmod 5.$$

Theorem 1.4. *For any integer λ and $\ell = 1, 2, 3, 4$, we have*

$$p_{-(25\lambda+1)}(25n+5\ell+1) \equiv 0 \pmod 5.$$

Theorem 1.5. *For any integer λ and $\ell = 1, 2, 3, 4$, we have*

$$p_{-(25\lambda+2)}(25n+5\ell+2) \equiv 0 \pmod 5.$$

2. Preliminaries

To prove Theorems 1.1-1.5 we will employ some q-identities. Ramanujan [13] stated that, if

$$R(q) = \frac{(q^2;q^5)_\infty (q^3;q^5)_\infty}{(q;q^5)_\infty (q^4;q^5)_\infty}$$

then

$$(q;q)_\infty = \left(q^{25};q^{25}\right)_\infty \left(R(q^5) - q - \frac{q^2}{R(q^5)}\right). \tag{2.1}$$

Hirschhorn and Hunt [14, Lemma 2.2] proved that, if R is a series in powers of q^5, then

$$\eta = q^{-1}R - 1 - qR^{-1}, \tag{2.2}$$

where

$$\eta = \frac{(q;q)_\infty}{q\left(q^{25};q^{25}\right)_\infty}. \tag{2.3}$$

From Hirschhorn and Hunt [14], we have

$$H_5(\eta) = -1, \quad H_5(\eta^2) = -1, \quad H_5(\eta^3) = 5, \quad \text{and} \quad H_5(\eta^4) = -5, \tag{2.4}$$

where H_5 is an operator which acts on series of positive and negative powers of a single variable, and simply picks out the term in which the power is congruent to 0 modulo 5.

Note: Here we replaced the symbol H in [14] by H_5.

Lemma 2.1. *We have*

$$(q;q)_\infty^5 \equiv \left(q^5;q^5\right)_\infty \pmod 5.$$

Proof. This follows easily from the binomial theorem. ∎

Lemma 2.2. [15, p. 53] *We have*

$$(q;q)_\infty^3 = \sum_{k=0}^{\infty} (-1)^k (2k+1) q^{k(k+1)/2}.$$

3. Proof of Theorems 1.1-1.5

In this section, all congruences are to the modulus 5.

Proof of Theorem 1.1: Setting $r=-(5\lambda+1)$ in (1.1), we obtain

$$\sum_{n=0}^{\infty} p_{-(5\lambda+1)}(n)q^n=(q;q)_\infty^{5\lambda+1}=(q;q)_\infty^{5\lambda}(q;q)_\infty. \tag{3.1}$$

Employing Lemma 2.1 and (2.1) in (3.1), we obtain

$$\sum_{n=0}^{\infty} p_{-(5\lambda+1)}(n)q^n\equiv(q^5;q^5)_\infty^{\lambda}\left(q^{25};q^{25}\right)_\infty\left(R(q^5)-q-\frac{q^2}{R(q^5)}\right). \tag{3.2}$$

Extracting the terms involving $q^{5n+\ell}$, where $\ell=3,4$, in both sides of (3.2), we arrive at the desired result.

Proof of Theorem 1.2: Setting $r=-(5\lambda+3)$ in (1.1), we obtain

$$\sum_{n=0}^{\infty} p_{-(5\lambda+3)}(n)q^n=(q;q)_\infty^{5\lambda+3}=(q;q)_\infty^{5\lambda}(q;q)_\infty^{3}. \tag{3.3}$$

Employing Lemma 2.1 in (3.3), we obtain

$$\sum_{n=0}^{\infty} p_{-(5\lambda+3)}(n)q^n\equiv(q^5;q^5)_\infty^{\lambda}(q;q)_\infty^{3}. \tag{3.4}$$

Using Lemma 2.2 in (3.4), we obtain

$$\sum_{n=0}^{\infty} p_{-(5\lambda+3)}(n)q^n\equiv(q^5;q^5)_\infty^{\lambda}\times\sum_{k=0}^{\infty}(-1)^k(2k+1)q^{k(k+1)/2}. \tag{3.5}$$

Since there exist no non-negative integer k such that $k(k+1)/2$ is congruent to 2 or 4 modulo 5, extracting the terms involving q^{5n+k}, where $k=2,4$, in both sides of (3.5), we prove the cases $\ell=2$ and 4, respectively.

Again, extracting the terms involving q^{5n+3} in both sides of (3.5) and noting $2k+1\equiv 0 \pmod 5$ for $k=2$, we arrive at the case $\ell=3$.

Proof of Theorem 1.3: Setting $r=-(5\lambda+4)$ in (1.1), we obtain

$$\sum_{n=0}^{\infty} p_{-(5\lambda+4)}(n)q^n=(q;q)_\infty^{5\lambda+4}=(q;q)_\infty^{5\lambda}(q;q)_\infty^{4}. \tag{3.6}$$

Employing Lemma 2.1 and (2.3) in (3.6), we obtain

$$\sum_{n=0}^{\infty} p_{-(5\lambda+4)}(n)q^n\equiv(q^5;q^5)_\infty^{\lambda}\eta^4q^4(q^{25};q^{25})_\infty^{4}. \tag{3.7}$$

Extracting the terms involving q^{5n+4} and using the operator H_5 in (3.7), we obtain

$$\sum_{n=0}^{\infty} p_{-(5\lambda+4)}(5n+4)q^{5n+4}\equiv(q^5;q^5)_\infty^{\lambda}H_5\left(\eta^4\right)q^4(q^{25};q^{25})_\infty^{4}. \tag{3.8}$$

Using (2.4) in (3.8), we obtain

$$\sum_{n=0}^{\infty} p_{-(5\lambda+4)}(5n+4)q^{5n+4} \equiv (-5)(q^5;q^5)_\infty^{\lambda} q^4 (q^{25};q^{25})_\infty^4. \tag{3.9}$$

Now the desired result follows from (3.9) and the fact that $5 \equiv 0 \pmod 5$.

Proof of Theorem 1.4: Setting $r = -(25\lambda+1)$ in (1.1), we obtain

$$\sum_{n=0}^{\infty} p_{-(25\lambda+1)}(n)q^{n} = (q;q)_\infty^{25\lambda+1} = (q;q)_\infty^{25\lambda}(q;q)_\infty. \tag{3.10}$$

Employing Lemma 2.1 and (2.3) in (3.10), we obtain

$$\sum_{n=0}^{\infty} p_{-(25\lambda+1)}(n)q^{n} \equiv (q^{25};q^{25})_\infty^{\lambda} \eta q (q^{25};q^{25})_\infty. \tag{3.11}$$

Extracting the terms involving q^{5n+1} and using the operator H_5 in (3.11), we obtain

$$\sum_{n=0}^{\infty} p_{-(25\lambda+1)}(5n+1)q^{5n+1} \equiv q(q^{25};q^{25})_\infty^{\lambda+1} H_5(\eta). \tag{3.12}$$

Using (2.4) in (3.12), dividing by q, and replacing q^5 by q , we obtain

$$\sum_{n=0}^{\infty} p_{-(25\lambda+1)}(5n+1)q^{n} \equiv (-1)(q^5;q^5)_\infty^{\lambda+1}. \tag{3.13}$$

Extracting the terms involving $q^{5n+\ell}$, where $\ell = 1,2,3,4$, in both sides of (3.13), we arrive at the desired result.

Proof of Theorem 1.5: Setting $r = -(25\lambda+2)$ in (1.1), we obtain

$$\sum_{n=0}^{\infty} p_{-(25\lambda+2)}(n)q^{n} = (q;q)_\infty^{25\lambda+2} = (q;q)_\infty^{25\lambda}(q;q)_\infty^2. \tag{3.14}$$

Employing Lemma 2.1 and (2.3) in (3.14), we obtain

$$\sum_{n=0}^{\infty} p_{-(25\lambda+2)}(n)q^{n} \equiv (q^{25};q^{25})_\infty^{\lambda} \eta^2 q^2 (q^{25};q^{25})_\infty^2. \tag{3.15}$$

Extracting the terms involving q^{5n+2} and using the operator H_5 in (3.15), we obtain

$$\sum_{n=0}^{\infty} p_{-(25\lambda+2)}(5n+2)q^{5n+2} \equiv q^2(q^{25};q^{25})_\infty^{\lambda+2} H_5\left(\eta^2\right). \tag{3.16}$$

Using (2.4) in (3.16), dividing by q^2, and replacing q^5 by q , we obtain

$$\sum_{n=0}^{\infty} p_{-(25\lambda+2)}(5n+2)q^{n} \equiv 1 \times (q^5;q^5)_\infty^{\lambda+2}. \tag{3.17}$$

Extracting the terms involving $q^{5n+\ell}$, where $\ell = 1,2,3,4$, in both sides of (3.17), we arrive at the desired result.

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References

[1] S. Ramanujan, Some properties of $p(n)$, the number of partition of n, Proceedings of the Cambridge Philosophical Society, 19 (1919) 207–210.

[2] S. Ramanujan, Congruence properties of partitions, Mathematische Zeitschrift, 9 (1921) 147–153.

[3] B.C. Berndt and R.A. Rankin, Ramanujan: Letters and Commentary, Providence, Rhode Island, 1995.

[4] J.M. Gandhi, Congruences for $p_r(n)$ and Ramanujan's τ function, The American Mathematical Monthly, 70 (1963) 265–274.

[5] M. Newman, Congruence for the coefficients of modular forms and some new congruences for the partition function, Canadian Journal of Mathematics, 9 (1957) 549–552.

[6] M.D. Hirschhorn, Partitions in 3 colours, The Ramanujan Journal, 45 (2018) 399–411

[7] K.G. Ramanathan, Identities and congruences of the Ramanujan type, Canadian Journal of Mathematics, 2 (1950) 168–178.

[8] A.O.L. Atkim, Ramanujan congruence for $p_k(n)$, Canadian Journal of Mathematics, 20 (1968) 67–78.

[9] K. Ono, Distribution of the partition function modulo m, Annals of Mathematics, 151 (1) (2000) 293–307.

[10] N.D. Baruah and K.K. Ojah, Some congruences deducible from Ramanujan's cubic continued fraction, International Journal of Number Theory, 7 (2011) 1331–1343.

[11] N.D. Baruah and B.K. Sarmah, Identities and congruences for the general partition and Ramanujan *tau* functions, Indian Journal of Pure and Applied Mathematics, 44 (5) (2013) 643–671.

[12] N. Saikia and J. Chetry, Infinite families of congruences modulo 7 for Ramanujans general partition function, Annales mathmatiques du Québec, 42 (2018) 127–132

[13] S. Ramanujan, Collected Papers, Chelsea, New York,1962.

[14] M.D. Hirschhorn and D.C. Hunt, A simple proof of Ramanujan conjecture for power of 5, Journal fur die Reine und Angewandte Mathematik, 326 (1981) 1–17.

[15] B.C. Berndt, Ramanujan's Notebooks, Part III. Springer-Verlag, New York, 1991.