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# A Fixed Point Approach to Stabilities of Functional Equations in Intuitionistic Fuzzy Banach Spaces

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**Abstract** In this paper we consider some functional equations for studying their Hyers-Ulam-Rassias stability. This stability has been studied for a variety of mathematical structures. Our framework of discussion is intuitionistic fuzzy normed linear space. We consider both Archimedean and non-Archimedean varieties of such spaces. The approach to the present problem is a fixed point approach, that is, we obtain our results by applications of an extension of the Banach contraction mapping principle on generalized metric spaces where infinite distances are allowable.

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**Keywords:** Hyers-Ulam stability; non-Archimedean intuitionistic fuzzy normed spaces; intuitionistic fuzzy normed spaces; generalized metric space.

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## 1. Introduction

The purpose of this paper is to establish some stability results for functional equations in non-Archimedean and Archimedean intuitionistic fuzzy Banach spaces. The kind of stability for which the results are established here is Hyers-Ulam-Rassias stability. Such stabilities have originated from the works of Hyers [1], Ulam [2] and Rassias [3] wherein Ulam formulated this problem for group homomorphisms which was solved by Hyers for Cauchy functional equations and thereafter it was extended by Rassias to the case of linear mappings. This concept of stability has very wide applications which includes problems from differential equations [4–6], problems relating to isometrics [7] and the like. It is well known that fuzzy concepts are new tenets of modern mathematics which have made inroads in almost all branches of mathematical studies. Intuitionistic fuzzy sets [8, 9] are further extensions of fuzzy sets [10] which characterize the uncertain situations

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from two different but complementary angles. Our work is in the domain of intuitionistic fuzzy Banach spaces. Just like many other branches of mathematics, the linear space was generalized through the incorporation of fuzzy concepts while intuitionistic fuzzy linear spaces are obtained by a process of further generalization. We consider some functional equations in these spaces for the purpose of studying their Hyers-Ulam-Rassias stability properties. We consider some of our problems in the framework of intuitionistic extensions of non-Archimedean normed linear space which is a linear space defined on a non-Archimedean field having valuations associated with each of its members. Other are worked out in intuitionistic fuzzy normed linear spaces on Archimedean fields.

Amongst several approaches to the problems of the Hyers-Ulam-Rassias stability for functional equations, a fixed point approach has been adopted in a good number of papers of which some instances are [11–15] . Here we adopt this approach [16]. In essence, our main results are obtained by applications of a fixed point theorem on generalized metric spaces. It may be further mentioned that the study of Hyers-Ulam-Rassias stability encompasses different domains of mathematics. The literature on this topic is very extensive. Some instances of these works are noted in references [17–28].

## 2. Mathematical Preliminaries:

**Definition 2.1.** [29] Let $K$ be a field. A non-Archimedean absolute value on $K$ is a function $|.| : K \to R$, such that for any $a, b \in K$ we have

(i) $|a| \geq 0$ and equality holds if and only if $a = 0$;
(ii) $|a\,b| = |a|\,|b|$;
(iii) $|a+b| \leq \max\{\,|a|, |b|\}$.

It is note that $|n| \leq 1$ for each integer $n$. We assume that $|.|$ is non-trivial, that is, there exists an $a_0 \in K$ such that $|a_0| \neq 0, 1$.

**Definition 2.2.** [30] Let $X$ be a vector space over a field $K$ with a non-Archimedean valuation $|.|$. A function $||.|| : X \to [0, \infty)$ is said to be a non-Archimedean norm if it satisfies the following conditions :

(i) $||x|| = 0$ if and only if $x = 0$;
(ii) $||rx|| = |r|\,||x||\,(r \in K, x \in X)$;
(iii) the strong triangle inequality $||x+y|| \leq \max\{\,||x||, ||y||\,\}\ \forall x, y \in X$ holds.

Then $(X, ||.||)$ is called a non-Archimedean normed space.

**Definition 2.3.** [31] A binary operation $* : [0,1] \times [0,1] \longrightarrow [0,1]$ is continuous $t$ - norm if $*$ satisfies the following conditions :

(i) $*$ is commutative and associative;
(ii) $*$ is continuous;
(iii) $a * 1 = a\ \forall\ a \in [0,1]$;
(iv) $a * b \leq c * d$ whenever $a \leq c, b \leq d$ and $a, b, c, d \in [0,1]$.

**Definition 2.4.** [31] A binary operation $\diamond : [0,1] \times [0,1] \longrightarrow [0,1]$ is continuous $t$ co- norm if $\diamond$ satisfies the following conditions:

(i) $\diamond$ is commutative and associative;
(ii) $\diamond$ is continuous;
(iii) $a \diamond 0 = a \quad \forall\ a \in [0,1]$;
(iv) $a \diamond b \leq c \diamond d$ whenever $a \leq c, b \leq d$ and $a, b, c, d \in [0,1]$.

**Definition 2.5.** [10] A fuzzy set $A$ of a non-empty set $X$ is characterized by a membership function $\mu_A$ which associates each point of $X$ to a real number in the interval [0, 1]. With the value of $\mu_A(x)$ at $x$ representing the grade of membership of $x$ in $A$.

**Definition 2.6.** [9] Let E be any nonempty set. An intuitionistic fuzzy set A of E is an object of the form $A = \{(x, \mu_A(x), \nu_A(x)) : x \in E\}$, where the functions $\mu_A : E \to [0, 1]$ and $\nu_A : E \to [0, 1]$ denote the degree of membership and the degree of non-membership of the element $x \in E$ respectively and for every $x \in E$, $0 \leq \mu_A(x) + \nu_A(x) \leq 1$.

**Definition 2.7.** [32–34] The five-tuple $(X, \mu, \nu, *, \diamond)$ is said to be an intuitionistic fuzzy normed space, ( in short, IFN space ) if $X$ is a vector space over a field $R$, $*$ is a continuous t-norm, $\diamond$ is a continuous t-conorm, and $\mu, \nu$ are functions from $X \times R \to [0, 1]$ satisfying the following conditions.
For every $x, y \in X$ and $s, t \in R$

(i) $\mu(x, t) = 0, \forall t \leq 0$;
(ii) $\mu(x, t) = 1$ if and only if $x = 0, t > 0$ ;
(iii) $\mu(cx, t) = \mu\left(x, \frac{t}{|c|}\right)$ if $c \neq 0, t > 0$ ;
(iv) $\mu(x, s) * \mu(y, t) \leqslant \mu(x + y, s + t), \forall s, t \in R$;
(v) $\lim_{t \to \infty} \mu(x, t) = 1$ ;

and

(vi) $\nu(x, t) = 1, \forall t \leq 0$;
(vii) $\nu(x, t) = 0$ if and only if $x = 0, t > 0$ ;
(viii) $\nu(cx, t) = \nu\left(x, \frac{t}{|c|}\right)$ if $c \neq 0, t > 0$ ;
(ix) $\nu(x, s) \diamond \nu(y, t) \geqslant \nu(x + y, s + t), \forall s, t \in R$;
(x) $\lim_{t \to \infty} \nu(x, t) = 0$.

If we change the triangular inequalities $(iv)$ and $(ix)$ in the above as

(iv) $\mu(x, s) * \mu(y, t) \leqslant \mu(x + y, \max\{s, t\}), \forall s, t \in R$; and
(ix) $\nu(x, s) \diamond \nu(y, t) \geqslant \nu(x + y, \max\{s, t\}), \forall s, t \in R$

then $(X, \mu, \nu, *, \diamond)$ is said to non-Archimedean intuitionistic fuzzy normed space ( in short, non-Archimedean IFN space ).

**Remark 2.8.** From $(ii)$ and $(iv)$, it follows that $\mu(x, t)$ is a non-decreasing function of $R$ and from $(vii)$ and $(ix)$, it follows that $\nu(x, t)$ is a non-increasing function of $R$.

**Example 2.9.** Let $(X, \|\cdot\|)$ be a non-Archimedean normed space, and let $a * b = ab$ and $a \diamond b = \min\{a, b\}$ for all $a, b \in [0, 1]$. Let $\mu(x, t) = \frac{t}{t + \|x\|}$ and $\nu(x, t) = \frac{\|x\|}{t + \|x\|}$ for all $x \in X$ and $t > 0$. Then $(X, \mu, \nu, *, \diamond)$ is a non-Archimedean fuzzy normed space.

**Definition 2.10.** [32, 34, 35] Let $(X, \mu, \nu, *, \diamond)$ be a non-Archimeden intuitionistic fuzzy normed space. Then, a sequence $\{x_n\}$ in $X$ is said to be convergent or converge if there exists an $x \in X$ for all $t > 0$, such that $\lim_{n \to \infty} \mu(x_n - x, t) = 1$ and $\lim_{n \to \infty} \nu(x_n - x, t) = 0$. In this case, $x$ is called the limit of the sequence $\{x_n\}$ and we denote it by $(\mu, \nu) - \lim_{n \to \infty} x_n = x$.

**Definition 2.11.** [32, 34, 35] Let $(X, \mu, \nu, *, \diamond)$ be a non-Archimeden intuitionistic fuzzy normed space. A sequence $\{x_n\}$ in $X$ is called Cauchy sequence if for each $\varepsilon > 0$ and $t > 0$ there exists an $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ and all $p > 0$, we have $\mu(x_{n+p} - x_n, t) > 1 - \varepsilon$ and $\nu(x_{n+p} - x_n, t) < \varepsilon$.

**Definition 2.12.** [32, 34, 35] Let $(X, \mu, \nu, *, \diamond)$ be a non-Archimeden intuitionistic fuzzy normed space. Then $(X, \mu, \nu, *, \diamond)$ is said to be complete if every Cauchy sequence is convergent. In this case $(X, \mu, \nu, *, \diamond)$ is called a non-Archimedean intuitionistic fuzzy Banach space.

We require the following generalized metric space to establish our result of stability in this paper.

**Definition 2.13.** Let $X$ be a nonempty set. A function $d : X \times X \to [0, \infty]$ is called a generalized metric on $X$ if $d$ satisfies

(i) $d(x, y) = 0$ if and only if $x = y$;
(ii) $d(x, y) = d(y, x)$ for all $x, y \in X$;
(iii) $d(x, y) \leq d(x, z) + d(z, y)$ for all $x, y, z \in X$.

Then $(X, d)$ is called a generalized metric space or a g. m. s..

**Definition 2.14.** [36]) Let $(X, d)$ be a g. m. s., $\{x_n\}$ be a sequence in $X$, and $x \in X$. We say that $\{x_n\}$ is g. m. s. convergent to $x$ if and only if $d(x_n, x) \to 0$ as $n \to \infty$. We denote this by $x_n \to x$.

**Definition 2.15.** [36]) Let $(X, d)$ be a g. m. s. and $\{x_n\}$ be a sequence in $X$. We say that $\{x_n\}$ is Cauchy sequence if and only if for each $\varepsilon > 0$, there exists a natural number $N$ such that $d(x_n, x_m) < \varepsilon$ for all $n > m > N$.

**Definition 2.16.** [36]) Let $(X, d)$ be a g. m. s. Then $(X, d)$ is called a complete g. m. s. if every g. m. s. Cauchy seuqence is g. m. s. convergent in $X$.

The following theorem is pivotal to the proof of our main result in the present paper.

**Theorem 2.17.** [37, 38] *Let $(X, d)$ be a complete generalized metric space and let $J : X \to X$ be a strictly contractive mapping with Lipschitz constant $0 < L < 1$, that is,*

$$d(Jx, Jy) \leq L\, d(x, y),$$

*for all $x, y \in X$.*
*Then for each $x \in X$, either*

$$d(J^n x, J^{n+1} x) = \infty, \ \forall\, n \geq 0$$

*or,*

$$d(J^n x, J^{n+1} x) < \infty \ \ \forall\, n \geq n_o$$

*for some non-negative integers $n_0$. Moreover, if the second alternative holds then*

*(1) the sequence $\{J^n x\}$ converges to a fixed point $y^\star$ of $J$;*
*(2) $y^\star$ is the unique fixed point of $J$ in the set*

$$Y = \{y \in X : d(J^{n_0} x, y) < \infty\};$$

*(3) $d(y, y^\star) \leq (\frac{1}{1-L})\, d(y, Jy)$ for all $y \in Y$.*

**Lemma 2.18.** [39] *Let $X$ and $Y$ be vector spaces. If a mapping $f : X \to Y$ satisfies*

$$f(x+y) - f(x) - f(y) = \rho\left(2f\left(\frac{x+y}{2}\right) - f(x) - f(y)\right)$$

*for all $x, y \in X$, then $f : X \to Y$ is additive.*

**Lemma 2.19.** [39] *Let $X$ and $Y$ be vector spaces. If a mapping $f : X \to Y$ satisfies $f(0) = 0$ and*

$$2f\left(\frac{x+y}{2}\right) - f(x) - f(y) = \rho(f(x+y) - f(x) - f(y))$$

*for all $x, y \in X$, then $f : X \to Y$ is additive.*

For our notational purpose we consider the function $f : X \to Y$ and define

$$\begin{aligned} D_1 f(x,y) &= f(x+y) - f(x) - f(y)) \\ &\quad - \rho\left(2f\left(\frac{x+y}{2}\right) - f(x) - f(y)\right) \end{aligned} \tag{2.1}$$

and

$$\begin{aligned} D_2 f(x,y) &= \left(2f\left(\frac{x+y}{2}\right) - f(x) - f(y)\right) \\ &\quad - \rho(f(x+y) - f(x) - f(y)) \end{aligned} \tag{2.2}$$

## 3. Non-Archimedean intuitionistic fuzzy stability of additive $\rho$-functional equation (2.1).

Throughout this section $X$ is considered to be a linear space over the non-Archimedean field $K$, $(Y, \mu, \nu)$ a non-Archimedean IF-real Banach space, $(Z, \mu', \nu')$ a non-Archimedean IFN-space.

To establish our main theorems, we first prove the following Lemma and for our purpose we take $t$-norm and $t$ co-norm as Example 2.9.

**Lemma 3.1.** *Let $(Z, \mu', \nu')$ be a non-Archimedean IFN-space and $\phi : X \times X \to Z$ be a function. Let $E = \{g : X \to Y; g(0) = 0\}$ and define $d$ by*

$$d(g,h) = \inf\left\{k \in R^+ : \begin{cases} \mu(g(x) - h(x), kt) \geq \mu'(\phi(x,x), t) \\ \nu(g(x) - h(x), kt) \leq \nu'(\phi(x,x), t), \end{cases} \forall x \in X, t > o\right\}$$

*and $g, h \in E$. Then $(E, d)$ is a complete generalized metric space.*

*Proof.* Let $f, g, h \in E$ and $d(f,g) = k_1 < \infty$, $d(g,h) = k_2 < \infty$.

$$\text{Then, } \begin{cases} \mu(f(x) - g(x), k_1 t) \geq \mu'(\phi(x,x), t) \\ \nu(f(x) - g(x), k_1 t) \leq \nu'(\phi(x,x), t) \end{cases}$$

$$\text{and } \begin{cases} \mu(g(x) - h(x), k_2 t) \geq \mu'(\phi(x,x), t) \\ \nu(g(x) - h(x), k_2 t) \leq \nu'(\phi(x,x), t) \end{cases}$$

Therefore

$$\begin{cases} \mu(f(x) - h(x), \max\{k_1, k_2\}t) \geq \mu(f(x) - g(x), k_1 t) * \mu(g(x) - h(x), k_2 t) \\ \geq \mu'(\phi(x, x), t) * \mu'(\phi(x, x), t) \geq \mu'(\phi(x, x), t) \\ \text{and} \\ \nu(f(x) - h(x), \max\{k_1, k_2\}t) \leq \nu(f(x) - g(x), k_1 t) \diamond \nu(g(x) - h(x), k_2 t) \\ \leq \nu'(\phi(x, x), t) \diamond \nu'(\phi(x, x), t) = \nu'(\phi(x, x), t) \end{cases}$$

for all $x \in X, t > 0$.
Hence $d(f, h) \leq \max\{k_1, k_2\}$ so that
$d(f, h) \leq \max\{d(f, g), d(g, h)\}$.
This proves the triangle inequality for $(E, d)$. Rest of the proof is obvious and hence $(E, d)$ is a generalized metric space.
Now we prove that $(E, d)$ is complete.
Let $\{g_n\}$ be a Cauchy sequence in $(E, d)$.
Now for each fixed $x \in X$ and for every $\epsilon > 0$ there exists $\lambda > 0$ such that

$$\begin{cases} \mu'\left(\phi(x, x), \frac{t}{\lambda}\right) > 1 - \varepsilon \\ \nu'\left(\phi(x, x), \frac{t}{\lambda}\right) < \varepsilon \end{cases}$$

Since $\{g_n\}$ is a Cauchy sequence in $(E, d)$ corresponding to $\lambda > 0$, there exists $n_0 \in N$ such that
$d(g_n, g_m) < \lambda$ for all $m, n \geq n_0$.
Since

$$d(g_n, g_m) = \inf\left\{ k \in R^+ : \begin{cases} \mu(g_n(x) - g_m(x), kt) \geq \mu'(\phi(x, x), t) \\ \nu(g_n(x) - g_m(x), kt) \leq \nu'(\phi(x, x), t), \end{cases} \right.$$

$$\forall x \in X, t > o\},$$

that is,

$$d(g_n, g_m) = \inf\left\{ k \in R^+ : \begin{cases} \mu(g_n(x) - g_m(x), t) \geq \mu'(\phi(x, x), , \frac{t}{k}) \\ \nu(g_n(x) - g_m(x), t) \leq \nu'(\phi(x, x), , \frac{t}{k}), \end{cases} \right.$$

$$\forall x \in X, t > o\}$$

then there exists $k_3 \in [0, \infty)$ such that
$d(g_n, g_m) \leq k_3 < \lambda$ for all $m, n \geq n_0$;

$$\begin{cases} \mu(g_n(x) - g_m(x), t) \geq \mu'(\phi(x, x), \frac{t}{k_3}) \geq \mu'(\phi(x, x), \dfrac{t}{\lambda}) > 1 - \varepsilon \\ \textit{and} \\ \nu(g_n(x) - g_m(x), t) \leq \nu'(\phi(x, x), \frac{t}{k_3}) \leq \nu'(\phi(x, x), \frac{t}{\lambda}) < \varepsilon \end{cases}$$

since $\mu(x, t)$ is non-decreasing and $\nu(x, t)$ is non-increasing w.r.t. $t$ for all $m, n \geq n_0$.
This show that for fixed $x \in X$, $\{g_n(x)\}$ is a Cauchy sequence in $Y$. Also since $Y$ is Banach space, for each fixed $x \in X$, there exists $g(x) \in Y$ such that $g(x) := (\mu, \nu) - \lim_{n\to\infty} g_n(x)$. So we have a mapping $g : X \to Y$ such that $g(x) := (\mu, \nu) - \lim_{n\to\infty} g_n(x)$ for all $x \in X$.

Again $\{ g_n \}$ is a Cauchy sequence in $( E, d )$ so $\epsilon > 0$ and $t > 0$ there exists $n_0 \in N$ such that $d( g_n, g_m ) \leq k' < \epsilon \; \forall m, n \geq n_0$,
that is,

$$\begin{cases} \mu ( g_m (x) - g_n (x) , t ) \geq \mu' ( \phi ( x, x ), \frac{t}{k'} ) \geq \mu' ( \phi ( x, x ), \frac{t}{\epsilon} ) \\ \nu ( g_m (x) - g_n (x) , t ) \leq \nu' ( \phi ( x, x ), \frac{t}{k'} ) \leq \nu' ( \phi ( x, x ), \frac{t}{\epsilon} ) \end{cases},$$

that is,

$$\begin{cases} \mu ( g_m (x) - g_n (x) , \epsilon t ) \geq \mu' ( \phi ( x, x ), t ) \\ \nu ( g_m (x) - g_n (x) , \epsilon t ) \leq \nu' ( \phi ( x, x ), t ), \; \forall n , m \geq n_0. \end{cases}$$

Now let $\epsilon , \delta > 0$ be given and $m, n > n_0$ , $t > 0$, then

$$\begin{cases} \mu ( g_n (x) - g (x) , \max\{\epsilon , \delta \} t ) \\ \geq \mu ( g_n (x) - g_m (x) , \epsilon t ) * \mu ( g_m (x) - g (x) , \delta t ) \\ \geq \mu' ( \phi ( x, x ), t ) * \mu ( g_m (x) - g (x) , \delta t ) \\ \geq \mu'( \phi ( x, x ), t ) * 1, [ \textit{by taking limit } m \to \infty ] \\ = \mu' ( \phi ( x, x ), t ) ) ( \textit{ by boundary condition } ) \\ \text{and} \\ \nu ( g_n (x) - g (x) , \max\{\epsilon , \delta \} t ) \\ \leq \nu ( g_n (x) - g_m (x) , \epsilon t ) \diamond \nu ( g_m (x) - g (x) , \delta t ) \\ \leq \nu' ( \phi ( x, x ), t ) \diamond \nu ( g_m (x) - g (x) , \delta t ) \\ \leq \nu' ( \phi ( x, x ), t ) \diamond 0, [ \textit{by taking limit } m \to \infty ] \\ = \nu' ( \phi ( x, x ), t ) ) ( \textit{ by boundary condition } ). \end{cases}$$

That is., $d ( g_n, g ) \leq \max \{\epsilon , \delta \}$ for all $x \in X$ and $m, n \geq n_0$ .
Taking $\delta \to 0$ we have a mapping $g : X \to Y$ such that
$g ( x ) = ( \mu, \nu ) - \lim_{n \to \infty} g_n ( x ) \in E$ .
Clearly $g ( 0 ) = \lim_{n \to \infty} g_n ( 0 ) = 0$ .
Therefore $( E , d )$ is a complete generalized metric space.

∎

**Theorem 3.2.** *Let* $\phi : X \times X \to Z$ *be a function such that*

$$\begin{cases} \mu \left( \phi ( \frac{x}{2} , \frac{y}{2} ), t \right) \geq \mu' \left( \frac{\alpha}{|2|} \phi ( x , y ), t \right) \\ \textit{and} \\ \nu \left( \phi ( \frac{x}{2} , \frac{y}{2} ), t \right) \leq \nu' \left( \frac{\alpha}{|2|} \phi ( x , y ), t \right) \end{cases} \tag{3.1}$$

*for some real* $\alpha$ *with* $0 < \alpha < 1, \forall x, y \in X, t > 0$. *If* $f : X \to Y$ *be a mapping satisfying*

$$\begin{cases} \mu ( D_1 f ( x , y ), t ) \geq \mu'( \phi ( x , y ) , t ) \\ \textit{and} \\ \nu ( D_1 f ( x , y ), t ) \leq \nu'( \phi ( x , y ) , t ) \end{cases} \tag{3.2}$$

$(x, y \in X, t > 0)$, *where* $D_1 f(x, y)$ *is given by* (2.1).
*Then there exists a unique additive mapping* $A : X \to Y$ *define by* $A(x) := (\mu, \nu) - \lim_{n\to\infty} 2^n f\left(\frac{x}{2^n}\right)$ *for all* $x \in X, t > 0$ *satisfying*

$$\begin{cases} \mu(A(x) - f(x), t) \geq \mu'\left(\phi(x, x), \frac{|2|(1-\alpha)}{\alpha} t\right) \\ \textit{and} \\ \nu(A(x) - f(x), t) \leq \nu'\left(\phi(x, x), \frac{|2|(1-\alpha)}{\alpha} t\right) \end{cases} \tag{3.3}$$

*Proof.* Putting $y = x$ in (3.2) we get

$$\begin{cases} \mu(f(2x) - 2f(x), t) \geq \mu'(\phi(x, x), t) \\ \text{and} \\ \nu(f(2x) - 2f(x), t) \leq \nu'(\phi(x, x), t) \end{cases} \tag{3.4}$$

Now consider the set $E := \{g : X \to Y, g(0) = 0\}$ and introduce a complete generalized metric on E as per Lemma 3.1.
Define a mapping $J : E \to E$ by
$Jg(x) := 2g\left(\frac{x}{2}\right)$ for all $g \in E$ and $x \in X$. We now prove that J is a strictly contracting mapping of E with the Lipschitz constant $\alpha$.
Let $g, h \in E$ and $\epsilon > 0$. Then there exists $k' \in R^+$ satisfying

$$\begin{cases} \mu(g(x) - h(x), k't) \geq \mu'(x, t) \\ \text{and} \\ \nu(g(x) - h(x), k't) \leq \nu'(x, t) \end{cases}$$

such that $d(g, h) \leq k' < d(g, h) + \epsilon$
Then,

$$\inf\left\{k \in R^+ : \begin{cases} \mu(g(x) - h(x), kt) \geq \mu'(\phi(x,x), t) \\ \nu(g(x) - h(x), kt) \leq \nu'(\phi(x,x), t), \end{cases} \forall x \in X, t > o\right\} \leq k' < d(g,h) + \epsilon,$$

that is,

$$\inf\left\{k \in R^+ : \begin{cases} \mu(2g(\frac{x}{2}) - 2h(\frac{x}{2}), |2|kt) \geq \mu'(\phi(\frac{x}{2}, \frac{x}{2}), t) \\ \nu(2g(\frac{x}{2}) - 2h(\frac{x}{2}), |2|kt) \leq \nu'(\phi(\frac{x}{2}, \frac{x}{2}), t), \end{cases} \forall x \in X, t > o\right\}$$
$$\leq k' < d(g,h) + \epsilon,$$

that is,

$$\inf\left\{k \in R^+ : \begin{cases} \mu(Jg(x) - Jh(x), |2|kt) \geq \mu'\left(\frac{\alpha}{|2|}\phi(x, x), t\right) \\ \nu(Jg(x) - Jh(x), |2|kt) \leq \nu'\left(\frac{\alpha}{|2|}\phi(x, x), t\right), \end{cases} \forall x \in X, t > o\right\}$$
$$< d(g, h) + \epsilon,$$

that is,

$$\inf\left\{k \in R^+ : \begin{cases} \mu(Jg(x) - Jh(x), \alpha kt) \geq \mu'(\phi(x, x), t) \\ \nu(Jg(x) - Jh(x), \alpha kt) \leq \nu'(\phi(x, x), t), \end{cases} \forall x \in X, t > o\right\} <$$
$$d(g, h) + \epsilon$$

or, $d\left\{\frac{1}{\alpha}(Jg, Jh)\right\} < d(g, h) + \epsilon$
or,, $d\{(Jg, Jh)\} < \alpha\{d(g, h) + \epsilon\}$
Taking $\epsilon \to 0$ we get $d\{(Jg, Jh)\} \leq \alpha\{d(g, h)\}$

Therefore J is strictly contractive mapping with Lipschitz constant $\alpha < 1$.
Also from (3.4),

$$\begin{cases} \mu\left( f(x) - 2f(\frac{x}{2}), \frac{\alpha}{|2|}t \right) \geq \mu'(\phi(x, x), t) \\ \text{and} \\ \nu\left( f(x) - 2f(\frac{x}{2}), \frac{\alpha}{|2|}t \right) \leq \nu'(\phi(x, x), t) \end{cases}$$

Therefore

$$d(f, Jf) \leq \frac{\alpha}{|2|}$$

Again replacing x by $2^{-(n+1)}x$ in (3.4) we get

$$\begin{cases} \mu\left(2^n f\left(\frac{x}{2^n}\right) - 2^{n+1} f\left(\frac{x}{2^{n+1}}\right), |2|^n t\right) \\ \geq \mu'\left(\phi\left(\frac{x}{2^{n+1}}, , \frac{x}{2^{n+1}}\right), t\right) \geq \mu'\left(\left(\frac{\alpha}{|2|}\right)^{n+1} \phi(x, x), t\right) \\ \text{and} \\ \nu\left(2^n f\left(\frac{x}{2^n}\right) - 2^{n+1} f\left(\frac{x}{2^{n+1}}\right), |2|^n t\right) \\ \leq \nu'\left(\phi\left(\frac{x}{2^{n+1}}, , \frac{x}{2^{n+1}}\right), t\right) \leq \nu'\left(\left(\frac{\alpha}{|2|}\right)^{n+1} \phi(x, x), t\right) \end{cases}$$

$$\text{or, } \begin{cases} \mu\left( J^n f(x) - J^{n+1} f(x), t\frac{\alpha^{n+1}}{|2|} \right) \geq \mu'(\phi(x, x), t) \\ \text{and} \\ \nu\left( J^n f(x) - J^{n+1} f(x), t\frac{\alpha^{n+1}}{|2|} \right) \leq \nu'(\phi(x, x), t) \end{cases}$$

Hence $d(J^{n+1} f, J^n f) \leq \frac{\alpha^{n+1}}{|2|} < \infty$ as Lipschitz constant $\alpha < 1$ for $n \geq n_0 = 1$.
Therefore by Theorem 2.17 there exists a mapping $A : X \to Y$ satisfying the following:
1. A is a fixed point of J, that is, $A(x) = 2A(\frac{x}{2})$ for all $x \in X$.
The mapping A is a unique fixed point of J in the set
$E_1 = \{g \in E : d(J^{n_0} f, g) = d(Jf, g) < \infty\}$
Therefore $d(Jf, A) < \infty$.
Also from (3.4), $d(Jf, f) \leq \frac{\alpha}{|2|} < \infty$
Thus $f \in E_1$
Now, $d(f, A) \leq \max\{d(f, Jf), d(Jf, A)\} < \infty$.
Thus there exists $k \in (0, \infty)$ satisfying

$$\begin{cases} \mu(f(x) - A(x), kt) \geq \mu'(\phi(x, x), t) \\ \text{and} \\ \nu(f(x) - A(x), kt) \leq \nu'(\phi(x, x), t) \end{cases} \tag{3.5}$$

for all $x \in X$, $t > 0$;
Also from (3.5) we have

$$\begin{cases} \mu\left(f\left(\frac{x}{2^n}\right) - A\left(\frac{x}{2^n}\right), kt\right) \geq \mu'\left(\phi\left(\frac{x}{2^n}, \frac{x}{2^n}\right), t\right) \\ \text{and} \\ \nu\left(f\left(\frac{x}{2^n}\right) - A\left(\frac{x}{2^n}\right), kt\right) \leq \nu'\left(\phi\left(\frac{x}{2^n}, \frac{x}{2^n}\right), t\right) \end{cases}$$

or

$$\begin{cases} \mu\left(2^n f\left(\frac{x}{2^n}\right) - 2^n A\left(\frac{x}{2^n}\right), |2|^n kt\right) \geq \mu'\left(\phi\left(\frac{x}{2^n}, \frac{x}{2^n}\right), t\right) \geq \mu'\left(\frac{\alpha^n}{|2|^n}\phi(x, x), t\right) \\ \text{and} \\ \nu\left(2^n f\left(\frac{x}{2^n}\right) - 2^n A\left(\frac{x}{2^n}\right), |2|^n kt\right) \leq \nu'\left(\phi\left(\frac{x}{2^n}, \frac{x}{2^n}\right), t\right) \leq \nu'\left(\frac{\alpha^n}{|2|^n}\phi(x, x), t\right). \end{cases}$$

$$\text{That is, } \begin{cases} \mu\left(J^n f(x) - A(x), \alpha^n kt\right) \geq \mu'\left(\phi(x, x), t\right) \\ \textit{and} \\ \nu\left(J^n f(x) - A(x), \alpha^n kt\right) \leq \nu'\left(\phi(x, x), t\right). \end{cases}$$

as $A(x) = 2A(\frac{x}{2}) = 2^2 A(\frac{x}{2^2}) = ... = 2^n A(\frac{x}{2^n})$.

2. $d(J^n f, A) = \inf \left\{ k \in R^+ : \begin{cases} \mu(J^n f(x) - A(x), \alpha^n kt) \geq \mu'(\phi(x, x), t) \\ \nu(J^n f(x) - A(x), \alpha^n kt) \leq \nu'(\phi(x, x), t), \end{cases} \right.$

$\forall x \in X, t > o\}$

Therefore $d(J^n f, A) \leq \alpha^n k \to 0$ as $n \to \infty$ and $\alpha < 1$.
This implies the equality

$$A(x) := (\mu, \nu) - \lim_{n\to\infty} J^n f(x) = (\mu, \nu) - \lim_{n\to\infty} 2^n f\left(\frac{x}{2^n}\right) \tag{3.6}$$

for all $x \in X$.

3. $d(f, A) \leqslant \frac{1}{1-L} d(f, Jf)$ with $f \in E_1$ which implies the inequality

$$d(f, A) \leq \frac{1}{1-\alpha} \times \frac{\alpha}{|2|} = \frac{\alpha}{|2|(1-\alpha)}$$

it follows the results (3.3).
Now replacing $x$ and $y$ by $2^{-n}x$ and $2^{-n}y$ in (3.2) we have

$$\begin{cases} \mu\left(2^n f\left(\frac{x+y}{2^n}\right) - 2^n f\left(\frac{x}{2^n}\right) - 2^n f\left(\frac{y}{2^n}\right)\right. \\ \left.-\rho\left(2^{n+1} f\left(\frac{x+y}{2^{n+1}}\right) - 2^n f\left(\frac{x}{2^n}\right) - 2^n f\left(\frac{y}{2^n}\right)\right), t\right) \geq \mu'\left(\phi(x, x), \frac{t}{\alpha^n}\right) \\ \textit{and} \\ \nu\left(2^n f\left(\frac{x+y}{2^n}\right) - 2^n f\left(\frac{x}{2^n}\right) - 2^n f\left(\frac{y}{2^n}\right)\right. \\ \left.-\rho\left(2^{n+1} f\left(\frac{x+y}{2^{n+1}}\right) - 2^n f\left(\frac{x}{2^n}\right) - 2^n f\left(\frac{y}{2^n}\right)\right), t\right) \leq \nu'\left(\phi(x, x), \frac{t}{\alpha^n}\right). \end{cases} \tag{3.7}$$

Taking limit as $n \to \infty$ in (3.7) and using $\alpha < 1$, $\mu(x, t) = 1$ if and only if $x = 0, t > 0$, $\nu(x, t) = 0$ if and only if $x = 0, t > 0$,
( using Definition 2.7 )
we obtain,

$$A(x + y) - A(x) - A(y) = \rho\left(2A\left(\frac{x+y}{2}\right) - A(x) - A(y)\right) \tag{3.8}$$

Therefore by the Lemma1, we can say that $A : X \to Y$ is additive.

The uniqueness of A follows from the fact that A is the unique fixed point of J by the Theorem 2.17. This completes the proof of the theorem. ∎

**Corollary 3.3.** *Let $X$ be a non-Archimedean normed space with norm $||.||$ over the non-Archimedean field $K$ and $p < 1$ be a non-negative real number, $z_0 \in Z$ and let $f : X \to Y$ be a mapping such that*

$$\begin{cases} \mu(D_1 f(x, y), t) \geq \mu'(z_0(||x||^p + ||y||^p), t) \\ \textit{and} \\ \nu(D_1 f(x, y), t) \leq \nu'(z_0(||x||^p + ||y||^p), t) \\ . \end{cases} \tag{3.9}$$

$(x, y \in X, t > 0)$, *where $D_1 f(x, y)$ is given by (2.1).*
*Then there exists a unique additive mapping $A : X \to Y$ for all $x \in X, t > 0$ satisfying*

$$\begin{cases} \mu(A(x) - f(x), t) \geq \mu'\left(z_0 ||x||^p, \frac{(|2|^p - |2|)}{2} t\right) \\ \textit{and} \\ \nu(A(x) - f(x), t) \leq \nu'\left(z_0 ||x||^p, \frac{(|2|^p - |2|)}{2} t\right). \end{cases} \tag{3.10}$$

*Proof.* Define $\phi(x, y) = z_0(||x||^p + ||y||^p)$ and the proof follows from Theorem 3.2 by taking $\alpha = |2|^{1-p}$. ∎

**Theorem 3.4.** *Let $\phi : X \times X \to Z$ be a function such that*

$$\begin{cases} \mu(\phi(x, y), t) \geq \mu'\left(|2| \alpha \phi\left(\frac{x}{2}, \frac{y}{2}\right) t\right) \\ \textit{and} \\ \nu(\phi(x, y), t) \leq \nu'\left(|2| \alpha \phi\left(\frac{x}{2}, \frac{y}{2}\right) t\right). \end{cases} \tag{3.11}$$

*for some real $\alpha$ with $0 < \alpha < 1, \forall x, y \in X, t > 0$. If $f : X \to Y$ be a mapping satisfying (3.2) then there exists a unique additive mapping $A : X \to Y$ define by $A(x) := (\mu, \nu) - \lim\limits_{n\to\infty} \frac{1}{2^n} f(2^n x)$ for all $x \in X, t > 0$ satisfying*

$$\begin{cases} \mu(A(x) - f(x), t) \geq \mu'(\phi(x, x), |2|(1 - \alpha), t) \\ \textit{and} \\ \nu(A(x) - f(x), t) \leq \nu'(\phi(x, x), |2|(1 - \alpha), t). \end{cases} \tag{3.12}$$

*Proof.* Putting $y = x$ in (3.2) we get

$$\begin{cases} \mu\left(f(x) - \frac{1}{2} f(2x), \frac{t}{|2|}\right) \geq \mu'(\phi(x, x), t) \\ \text{and} \\ \nu\left(f(x) - \frac{1}{2} f(2x), \frac{t}{|2|}\right) \leq \nu'(\phi(x, x), t). \end{cases} \tag{3.13}$$

As before consider the set $E := \{g : X \to Y, g(0) = 0\}$ and introduce a complete generalized metric on $E$ such that $J : E \to E$ by
$J g(x) := \frac{1}{2} g(2x)$ for all $g \in E$ and $x \in X$.

Also from (3.13) we see that $d(f, Jf) \leq \frac{1}{|2|}$.
As before we conclude that the result (3.12). ∎

**Corollary 3.5.** *Let $p > 1$ be a non-negative real number, $X$ be a non-Archimedean normed space with norm $||.||$ over the non-Archimedean field $K$ $z_0 \in Z$ and let $f : X \to Y$ be a mapping satisfying (3.9) Then there exists a unique additive mapping $A : X \to Y$ for all $x \in X$, $t > 0$ satisfying*

$$\begin{cases} \mu(A(x) - f(x), t) \geq \mu'\left(z_0 ||x||^p, \frac{(|2|-|2|^p)}{2} t\right) \\ \text{and} \\ \nu(A(x) - f(x), t) \leq \nu'\left(z_0 ||x||^p, \frac{(|2|-|2|^p)}{2} t\right). \end{cases} \tag{3.14}$$

*Proof.* Define $\phi(x, y) = z_0(||x||^p + ||y||^p)$ and the proof follows from Theorem 3.4 by taking $\alpha = |2|^{p-1}$. ∎

## 4. Non-Archimedean Intuitionistic Fuzzy stability of Additive $\rho$-functional Equation (2.2)

Throughout this section also we consider $X$ to be a linear space over the non-Archimedean field $K$, $(Y, \mu, \nu)$ a non-Archimedean IF-real Banach space, $(Z, \mu', \nu')$ a non-Archimedean IFN-space.

**Theorem 4.1.** *Let $\phi : X \times X \to Z$ be a function satisfying (3.1).
If $f : X \to Y$ be a mapping with $f(0) = 0$ satisfying*

$$\begin{cases} \mu(D_2 f(x, y), t) \geq \mu'(\phi(x, y), t) \\ \text{and} \\ \nu(D_2 f(x, y), t) \leq \nu'(\phi(x, y), t) \end{cases} \tag{4.1}$$

*$(x, y \in X, t > 0)$, where $D_2$ is given by (2.2).
Then there exists a unique additive mapping $A : X \to Y$ define by $A(x) := (\mu, \nu) - \lim_{n\to\infty} 2^n f\left(\frac{x}{2^n}\right)$ for all $x \in X$, $t > 0$ satisfying*

$$\begin{cases} \mu(A(x) - f(x), t) \geq \mu'(\phi(x, 0), (1-\alpha)t) \\ \text{and} \\ \nu(A(x) - f(x), t) \leq \nu'(\phi(x, 0), (1-\alpha)t) \end{cases} \tag{4.2}$$

*Proof.* Putting $y = 0$ in (4.1) and the rest of the proof follows from the Theorem 3.2. ∎

**Corollary 4.2.** *Let $p < 1$ be a non-negative real number and $X$ be a non-Archimedean normed space with norm $||.||$ over the non-Archimedean field $K$, $z_0 \in Z$ and let $f : X \to Y$ be a mapping with $f(0) = 0$ such that*

$$\begin{cases} \mu(D_2 f(x, y), t) \geq \mu'(z_0(||x||^p + ||y||^p), t) \\ \text{and} \\ \nu(D_2 f(x, y), t) \leq \nu'(z_0(||x||^p + ||y||^p), t) \end{cases} \tag{4.3}$$

$(x, y \in X, t > 0)$, *where* $D_2$ *is given by* (2.2).
*Then there exists a unique additive mapping* $A : X \to Y$ *for all* $x \in X$, $t > 0$ *satisfying*

$$\begin{cases} \mu(A(x) - f(x), t) \geq \mu'\left(z_0\,||x||^p, \frac{(|2|^p - |2|)}{|2|^p}\, t\right) \\ \text{and} \\ \nu(A(x) - f(x), t) \leq \nu'\left(z_0\,||x||^p, \frac{(|2|^p - |2|)}{|2|^p}\, t\right). \end{cases} \tag{4.4}$$

*Proof.* Define $\phi(x, y) = z_0(||x||^p + ||y||^p)$ and the proof follows from Theorem 4.1 by taking $\alpha = |2|^{1-p}$. ∎

**Theorem 4.3.** *Let* $\phi : X \times X \to Z$ *be a function satisfying* (3.11).
*If* $f : X \to Y$ *be a mapping with* $f(0) = 0$ *satisfying* (4.1) *where* $D_2$ *is given by* (2.2).
*Then there exists a unique additive mapping* $A : X \to Y$ *for all* $x \in X$, $t > 0$ *satisfying*

$$\begin{cases} \mu(A(x) - f(x), t) \geq \mu'\left(\phi(x, 0), \frac{(1-\alpha)}{\alpha}\, t\right) \\ \text{and} \\ \nu(A(x) - f(x), t) \leq \nu'\left(\phi(x, 0), \frac{(1-\alpha)}{\alpha}\, t\right). \end{cases} \tag{4.5}$$

*Proof.* Putting $y = 0$ in (4.1) we get

$$\begin{cases} \mu(f(x) - \frac{1}{2} f(2x), \alpha t) \geq \mu'(\phi(x, 0), t) \\ \text{and} \\ \nu(f(x) - \frac{1}{2} f(2x), \alpha t) \leq \nu'(\phi(x, 0), t). \end{cases} \tag{4.6}$$

Hence $d(f, Jf) \leq \alpha$.
Similarly as before we conclude the result (4.5). ∎

**Corollary 4.4.** *Let* $p > 1$ *be a non-negative real number and* $X$ *be a non-Archimedean normed space with norm* $||.||$ *over the non-Archimedean field* $K$, $z_0 \in Z$ *and let* $f : X \to Y$ *be a mapping with* $f(0) = 0$ *such that* (4.3) *hold. Then there exists a unique additive mapping* $A : X \to Y$ *for all* $x \in X$, $t > 0$ *satisfying*

$$\begin{cases} \mu(A(x) - f(x), t) \geq \mu'\left(z_0\,||x||^p, \frac{(|2| - |2|^p)}{|2|^p}\, t\right) \\ \text{and} \\ \nu(A(x) - f(x), t) \leq \nu'\left(z_0\,||x||^p, \frac{(|2| - |2|^p)}{|2|^p}\, t\right). \end{cases} \tag{4.7}$$

*Proof.* Define $\phi(x, y) = z_0(||x||^p + ||y||^p)$ and the proof follows from Theorem 4.3 by taking $\alpha = |2|^{p-1}$. ∎

## 5. H-U-R stability of the Additive $\rho$-functional Equations in IF Banach Spaces.

**Theorem 5.1.** *Let $X$ be a linear space, $(Z, \mu', \nu')$ be an IFN-space, $\phi : X \times X \to Z$ be a function such that*

$$\begin{cases} \mu\left(\phi(\frac{x}{2}, \frac{y}{2}), t\right) \geq \mu'\left(\frac{\alpha}{2}\phi(x, y), t\right) \\ \text{and} \\ \nu\left(\phi(\frac{x}{2}, \frac{y}{2}), t\right) \leq \nu'\left(\frac{\alpha}{2}\phi(x, y), t\right). \end{cases} \tag{5.1}$$

*for some real $\alpha$ with $0 < \alpha < 1$, $x \in X$. Let $(Y, \mu, \nu)$ be a complete IFN-space. If $f : X \to Y$ be a mapping satisfying (3.2)*
*Then there exists a unique additive mapping $A : X \to Y$ define by $A(x) := (\mu, \nu) - \lim_{n\to\infty} 2^n f\left(\frac{x}{2^n}\right)$ for all $x \in X$, $t > 0$ satisfying*

$$\begin{cases} \mu(A(x) - f(x), t) \geq \mu'\left(\phi(x, x), \frac{2(1-\alpha)}{\alpha}t\right) \\ \text{and} \\ \nu(A(x) - f(x), t) \leq \nu'\left(\phi(x, x), \frac{2(1-\alpha)}{\alpha}t\right). \end{cases} \tag{5.2}$$

*Proof.* Putting $y = x$ in (3.2) we get

$$\begin{cases} \mu(f(2x) - 2f(x), t) \geq \mu'(\phi(x, x), t) \\ \text{and} \\ \nu(f(2x) - 2f(x), t) \leq \nu'(\phi(x, x), t). \end{cases} \tag{5.3}$$

Now consider the set $E := \{g : X \to Y, g(0) = 0\}$ and introduce a complete generalized metric on E as per Lemma 3.1.

Define a mapping $J : E \to E$ by
$Jg(x) := 2g\left(\frac{x}{2}\right)$ for all $g \in E$ and $x \in X$.

As before it is easy to prove that $J$ is strictly contractive mapping with Lipschitz constant $\alpha < 1$.
Also from (5.3),

$$\begin{cases} \mu(f(x) - 2f(\frac{x}{2}), t) \geq \mu'(\phi(\frac{x}{2}, \frac{x}{2}), t) \geq \mu'\left(\frac{\alpha}{2}\phi(x, x), t\right) \\ \text{and} \\ \nu(f(x) - 2f(\frac{x}{2}), t) \leq \nu'(\phi(\frac{x}{2}, \frac{x}{2}), t) \leq \nu'\left(\frac{\alpha}{2}\phi(x, x), t\right) \end{cases}$$

Therefore,

$$d(f, Jf) \leq \frac{\alpha}{2}$$

Again replacing x by $2^{-(n+1)}x$ in (5.3) we get $d(J^{n+1}f, J^n f) \leq \frac{\alpha^{n+1}}{2} < \infty$ as Lipschitz constant $\alpha < 1$ for $n \geq n_0 = 1$.

Therefore by Theorem 2.17, there exists a mapping $A : X \to Y$ satisfying the following:
1. A is a fixed point of J, that is, $A(x) = 2A(\frac{x}{2})$ for all $x \in X$
and, $d(f, A) \leq \{d(f, Jf) + d(Jf, A)\} < \infty$.
2. $d(J^n f, A)$

$= \inf\{ k \in R^{+} : \mu ( J^{n} f ( x ) - A ( x ), \alpha^{n} k t ) \geq \mu' ( \phi( x, x ), t ) \}$
Therefore $d ( J^{n} f , A ) \leq \alpha^{n} k \to 0$ as $n \to \infty$.
This implies the equality

$$A ( x ) := (\mu, \nu ) - \lim_{n \to \infty} J^{n} f ( x ) = (\mu, \nu ) - \lim_{n \to \infty} 2^{n} f \left( \frac{x}{2^{n}} \right) \tag{5.4}$$

for all $x \in X$.
3. $d( f , A ) \leq \frac{1}{1-\alpha} \times \frac{\alpha}{2} = \frac{\alpha}{2 ( 1-\alpha )}$ it follows the results ( 5.2 ).
This completes the proof of the theorem. ■

**Corollary 5.2.** *Let $X$ be normed space with norm $||.||$, $p < 1$ be a non-negative real number, $z_0 \in Z$ and let $f : X \to Y$ be a mapping such that (3.9) hold. Then there exists a unique additive mapping $A : X \to Y$ for all $x \in X$, $t > 0$ satisfying*

$$\begin{cases} \mu ( A ( x ) - f ( x ) , t ) \geq \mu' \left( z_0 ||x||^p, \frac{(2^p-2)}{2} t \right) \\ and \\ \nu ( A ( x ) - f ( x ) , t ) \leq \nu' \left( z_0 ||x||^p, \frac{(2^p-2)}{2} t \right). \end{cases} \tag{5.5}$$

*Proof.* Define $\phi ( x, y ) = z_0 ( ||x||^p + ||y||^p )$ and the proof follows from Theorem 5.1 taking $\alpha = 2^{1-p}$. ■

**Theorem 5.3.** *Let $X$ be a linear space, $( Z , \mu' , \nu' )$ be a IFN-space, $\phi : X \times X \to Z$ be a function such that*

$$\begin{cases} \mu ( \phi ( x , y ), t) \geq \mu \left(2 \alpha \phi \left( \frac{x}{2} , \frac{y}{2} \right) , t \right) \\ and \\ \nu ( \phi ( x , y ), t) \leq \nu \left(2 \alpha \phi \left( \frac{x}{2} , \frac{y}{2} \right) , t \right), \end{cases} \tag{5.6}$$

*for some real $\alpha$ with $0 < \alpha < 1$, for all $x \in X$, $t > 0$. Let $( Y, \mu, \nu )$ be a complete IFN-space. If $f : X \to Y$ be a mapping satisfying (3.2). Then there exists a unique additive mapping $A : X \to Y$ satisfying*

$$\begin{cases} \mu ( A ( x ) - f ( x ) , t ) \geq \mu' ( \phi ( x , x ), 2 (1 - \alpha ) t ) \\ and \\ \nu ( A ( x ) - f ( x ) , t ) \leq, \nu' ( \phi ( x , x ), 2 (1 - \alpha ) t ) \end{cases} \tag{5.7}$$

$( \forall x \in X, t > 0 )$.

*Proof.* Putting $y = x$ in (3.2) we get $d( f, Jf ) \leq \frac{1}{2}$.
As before we conclude the result ( 5.7 ). ■

**Corollary 5.4.** *Let $p > 1$ be a non-negative real number and $X$ be a normed space with norm $||.||$, $z_0 \in Z$ and let $f : X \to Y$ be a mapping such that (3.9) hold.
Then there exists a unique additive mapping $A : X \to Y$ for all $x \in X$, $t > 0$ satisfying*

$$\begin{cases} \mu ( A ( x ) - f ( x ) , t ) \geq \mu' \left( z_0 ||x||^p, \frac{(2-2^p)}{2} t \right) \\ and \\ \nu ( A ( x ) - f ( x ) , t ) \leq \nu' \left( z_0 ||x||^p, \frac{(2-2^p)}{2} t \right) \end{cases} \tag{5.8}$$

*Proof.* Define $\phi(x, y) = z_0(||x||^p + ||y||^p)$ and the proof follows from Theorem 5.3 taking $\alpha = 2^{p-1}$. ■

**Theorem 5.5.** *Let $X$ be a linear space, $(Z, \mu', \nu')$ be a IFN-space, $\phi : X \times X \to Z$ be a function such that*

$$\begin{cases} \mu\left(\phi(\frac{x}{2}, \frac{y}{2}), t\right) \geq \mu'\left(\frac{\alpha}{2}\phi(x, y), t\right) \\ and \\ \nu\left(\phi(\frac{x}{2}, \frac{y}{2}), t\right) \leq \nu'\left(\frac{\alpha}{2}\phi(x, y), t\right) \end{cases} \tag{5.9}$$

*for some real $\alpha$ with $0 < \alpha < 1$, for all $x, y \in X, t > 0$. Let $(Y, \mu, \nu)$ be a complete IFN-space. If $f : X \to Y$ be a mapping with $f(0) = 0$ satisfying (4.1). Then there exists a unique additive mapping $A : X \to Y$ satisfying*

$$\begin{cases} \mu(A(x) - f(x), t) \geq \mu'(\phi(x, 0), (1-\alpha)t) \\ and \\ \nu(A(x) - f(x), t) \leq \nu'(\phi(x, 0), (1-\alpha)t) \end{cases} \tag{5.10}$$

$(\forall x \in X, t > 0)$.

*Proof.* Putting $y = 0$ in (4.1) and the rest of the proof of the theorem similar as before. ■

**Corollary 5.6.** *Let $p < 1$ be a non-negative real number and $X$ be a normed space with norm $||.||$, $z_0 \in Z$ and let $f : X \to Y$ be a mapping with $f(0) = 0$ such that (4.3) hold. Then there exists a unique additive mapping $A : X \to Y$ for all $x \in X, t > 0$ satisfying*

$$\begin{cases} \mu(A(x) - f(x), t) \geq \mu'\left(z_0||x||^p, \frac{(2^p-2)}{2^p}t\right) \\ and \\ \nu(A(x) - f(x), t) \leq \nu'\left(z_0||x||^p, \frac{(2^p-2)}{2^p}t\right). \end{cases} \tag{5.11}$$

*Proof.* Define $\phi(x, y) = z_0(||x||^p + ||y||^p)$ and the proof follows from Theorem 5.5 taking $\alpha = 2^{1-p}$. ■

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## References

[1] D.H. Hyers, On the stability of the linear functional equation Proceedings of the National Academy of Sciences of the United States of America, 27 (1941) 222–224.

[2] S. M. Ulam, Problems in Modern Mathematics, Wiley, New York, 1964.

[3] Th.M. Rassias, On the stability of the linear mapping in Banach spaces, Proceedings of the American Mathematical Society, 72 (1978) 297–300.

[4] S. M. Jung, Hyers-Ulam stability of linear differential equations of first order, II, Applied Mathematics Letters, 19 (2006) 854–858.

[5] S.-M. Jung, HyersUlam stability of a system of first order linear differential equations with constant coefficients, Journal of Mathematical Analysis and Applications, 320 (2) (2006) 549–561.

[6] T. Miura, S. Miyajima and S.E. Takahasi, A characterization of HyersUlam stability of first order linear differential operators, Journal of Mathematical Analysis and Applications, 286 (2003) 136–146.

[7] Y. Dong, On approximate isometries and application to stability of a function, Journal of Mathematical Analysis and Applications, 426 (2015) 125–137.

[8] G. Deschrijiver and E.E. Kerre, On the relationship between some extensions of fuzzy set theory, Fuzzy Sets and Systems, 23 (2003) 227–235.

[9] K.T. Atanassov, Intuitionistic fuzzy sets, Fuzzy Sets and Systems, 20 (1986) 87–96.

[10] L.A. Zadeh, Fuzzy sets, Information and Control, 8 (1965) 338–353.

[11] P. Mondal, N.C. Kayal and T.K. Samanta, The stability of Pexider type functional equation in intuitionistic fuzzy banach spaces via fixed point technique, Journal of Hyperstructures, 4 (2015) 37–49.

[12] P. Saha, T.K. Samanta, P. Mondal and B.S. Choudhuary, Stability of additive-quadratic $\rho-$ functional equations in non-Archimedean intuitionistic fuzzy Banach spaces, Matematicki Vesnik, 72 (2) (2020) 154–164.

[13] P. Saha, T. K. Samanta, P. Mondal, B. S. Choudhury and M.D.L. Sen, Applying fixed point techniques to stability problems in intuitionistic fuzzy Banach spaces, Mathematics, 8 (6) (2020), 15 pages.

[14] T.Z. Xu, M.J. Rassias, W.X. Xu and J.M. Rassias, A fixed point approach to the intuitionistic fuzzy stability of quintic and sextic functional equations, Iranian Journal Fuzzy Systems, 9 (2012) 21–40.

[15] V. Radu, The fixed point alternative and the stability of functional equations, Fixed point theory, 4 (2003) 91–96.

[16] J.B. Diaz and B. Margolisi, A fixed point theorem of the alternative for contractions on a generalized complete metric space, Bulletin of the American Mathematical Society 74 (1968) 305–309.

[17] D.H. Hyers, G. Isac and T.M. Rassias, Stability of Functional Equations in Several Variables, Birkhauser, Massachusetts,1998.

[18] E. Elqorachi and M.Th. Rassias , Generalized Hyers-Ulam Stability of Trigonometric Functional Equations, Mathematics, 6 (5) (2018) 11 pages.

[19] J. Aczel and J. Dhombres, Functional Equations in Several Variables, Cambridge University Press, 1989.

[20] J. Wang, Some further generalization of the Ulam-Hyers-Rassias stability of functional equations, Journal of Mathematical Analysis and Applications, 263 (2001), 406–423.

[21] P.K. Sahoo and P. Kannappan, Introduction to Functional Equations, CRC Press, Florida, 2011.

[22] Pl. Kannappan, Functional Equations and Inequalities with Applications, Springer, New York, 2009.

[23] S. Czerwik, Functional Equations and Inequalities in Several Variables, World Scientific, New Jersey, 2002.

[24] S.-M. Jung, Hyers-Ulam-Rassias Stability of Functional Equations in Nonlinear Analysis, Springer, New York, 2011.

[25] S.-M. Jung, D. Popa and Th.M. Rassias, On the stability of the linear functional equation in a single variable on complete metric groups, Journal of Global Optimization, 59 (2014) 165–171.

[26] S.-M. Jung, S.K. Lee, Th.M. Rassias and S.M. Yang, Approximation properties of solutions of a mean value-type functional Inequality, II, Mathematics, 8 (2020) 8 pages.

[27] T. Trif, On the stability of a functional equation deriving from an inequality of Popoviciu for convex functions, Journal of Mathematical Analysis and Applications, 272 (2002) 604–616.

[28] Y.-H. Lee, S.M. Jung and Th.M. Rassias, On an n-dimensional mixed type additive and quadratic functional equation, Applied Mathematics and Computation, 228 (2014), 13–16.

[29] A.K. Mirmostafaee and M.S. Moslehian, Stability of additive mappings in non-Archimedean fuzzy normed spaces, Fuzzy Sets and Systems, 160 (2009) 1643–1652.

[30] M.S. Moslehian and G. Sadeghi, A Mazur-Ulam theorem in non-Archimedean normed spaces, Nonlinear Analysis, 69 (2008) 3405–3408.

[31] B. Schweizer and A. Sklar, Statistical metric space, Pacific Journal of Mathematics, 10 (1960) 314–334.

[32] S.A. Mohiuddine, A. Aiotaibi and M. Obaid, Stability of various functional equations in non-Archimedean intuitionistic fuzzy normed spaces, Discrete Dynamics in Nature and Society, 2012 (2012), Article ID 234727, 16 pages.

[33] T. Bag and S. K. Samanta, Finite dimensional intuitionistic fuzzy normed linear spaces, Annals of Fuzzy Mathematics and Informatics, 6 (2013) 45–57.

[34] T. K. Samanta and I.H. Jebril, Finite dimentional intuitionistic fuzzy normed linear space, International Journal of Open Problems in Computer Science and Mathematics, 2 (4) (2009) 574–591.

[35] P. Salimi, C. Vetro and P. Vetro, Some new fixed point results in non-Archimedean fuzzy metric spaces, Nonlinear Analysis: Modelling and Control, 18 (2013) 344–358.

[36] A. Branciari, A fixed point theorem of Banach-Caccioppoli type on a class of generalized metric spaces, Publicationes Mathematicae Debrecen, 57 (2000) 31–37.

[37] D. Mihet, The fixed point method for fuzzy stability of the Jensen functional equation, Fuzzy sets and systems, 160 (2009) 1663–1667.

[38] L. Cadariu and V. Radu, Fixed points and stability for functional equations in probabilistic metric and random normed spaces, Fixed Point Theory and Applications, (2009) 2009, Article ID 589143, 18 pages,

[39] C. Park, D. Y. Shin and J.R. Lee, Fixed points and additive $\rho$-functional equations, Journal of Fixed Point Theory Applications, 18 (2016), 569–586.