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Quasi–Multipliers on Banach Algebras Arising from Hypergroups

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Abstract In this paper, we obtain a characterization for $\mathfrak{QM}(L(\mathcal{X}))$, the quasi-multiplier algebra of $L(\mathcal{X})$, where $\mathcal{X}$ is a foundation hypergroup . It is shown that $\mathfrak{QM}(L(\mathcal{X}))$ can be identified by $M(\mathcal{X})$ for this class of hypergroups. We also investigate quasi-multipliers on the second dual Banach algebra $L_c(\mathcal{X})^{**}$. Indeed, we show that $\mathfrak{QM}(L_c(\mathcal{X})^{**})$ is isomorphic with $M(\mathcal{X})$. As an application, we prove that $\mathfrak{QM}(L_c(\mathcal{X})^{**}) = L(\mathcal{X})$ if and only if $\mathcal{X}$ is discrete.

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1. Introduction

This paper is concerned with the space of separately continuous quasi-multipliers on Banach algebras associated with hypergroups. Quasi-multipliers appeared as a generalization of the notion of multipliers in work of Akemann and Pedersen on C^*-algebras [1]. Indeed, Quasi-multipliers have been well interpreted as those elements of the enveloping von Neumann algebra $\mathcal{A}^{**}$ which are continuous on the set of states in $\mathcal{A}^*$ with respect to the weak*-topology. McKennon later in [2], introduced a topology, finer than both the topology of convergence of norms and weak*-topology, on the space of quasi-multipliers of a C^*-algebra $\mathcal{A}$ and deduced some basic characterizations. The notion of quasi-multiplier was a cornerstone in many different subjects such as C^*-algebras, operator spaces, Hilbert C^*-modules and Banach algebras, see [3], [4], [5] and [6] for more details. Motivated by this, quasi-multipliers were first studied within the framework of Banach algebras with a bounded approximate identity by K. McKennon [7].

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Definition 1.1. For a Banach algebra $\mathcal{A}$ a bilinear map $\mathfrak{m}\colon \mathcal{A}\times\mathcal{A}\longrightarrow\mathcal{A}$ is called a *quasi-multiplier* if for all $a,b,c,d\in\mathcal{A}$,

$$\mathfrak{m}(ab,cd)=a\mathfrak{m}(b,c)d.$$

We note that the quasi-multiplier algebra of $\mathcal{A}$ is the set of all separately continuous quasi-multipliers on $\mathcal{A}$ which is denoted by $\mathfrak{QM}(\mathcal{A})$. In the case where $\mathcal{A}$ admits a bounded approximate identity, it is shown in [7, Theorem 2] that $\mathfrak{QM}(\mathcal{A})$ is a Banach space with the norm defined by

$$\|\mathfrak{m}\|=\sup\{\|\mathfrak{m}(a,b)\|:a,b\in\mathcal{A},\|a\|=\|b\|=1\}.$$

Recall that an approximate identity (e_α) in a Banach algebra $\mathcal{A}$ will be called an *ultra-approximate identity* for $\mathcal{A}$ if for all $\mathfrak{m}\in\mathfrak{QM}(\mathcal{A})$ and $a\in\mathcal{A}$, the nets $(\mathfrak{m}(e_\alpha,a))$ and $(\mathfrak{m}(a,e_\alpha))$ are $\|\cdot\|$-Cauchy.

For a locally compact Hausdorff group G, K. McKennon identified quasi-multipliers on the group algebra $L^1(G)$. More precisely, it is proved in [7] that $\mathfrak{QM}(L^1(G))$ is isomorphic with the measure algebra $M(G)$. Morover in [8], Vasudevan and Goel showed that for Banach algebra $\mathcal{A}$ with a bounded approximate identity $\mathfrak{QM}(\mathcal{A})$ can be embedded into $\mathcal{A}^{**}$. This fact extended the well-known embedding of the left multipliers (right multipliers) of $\mathcal{A}$ into $\mathcal{A}^{**}$ [9]. More developments have arised by Grosser [10], Vasudevan and Goel and Takahashi [11], Kassem and Rowlands [12], Lin [5], Argün and Rowlands [13]. Recently in [14], Alinejad and Rostami obtained some results on quasi-multiplier of Banach algebras related to locally compact semigroups. In this paper, we concentrate on some classes of Banach algebras associated with hypergroups. The theory of hypergroups was initiataed independently by Dunkl [15], Jewett [16], and Spector [17] in 1970's. This theory has received a good deal of attention from harmonic analysts. Ghahramani and Medghalchi [18] investigated multipliers on Banach algebras associated with locally compact hypergroups and showed that for a foundation hypergroup $\mathcal{X}$, the multiplier algebras of $L(\mathcal{X})$ is isomorphic with $M(\mathcal{X})$ [18, Proposition 1]. Moreover, they characterized compact multipliers on $L(\mathcal{X})$, see [18] and [19] for more details. Motivated by these facts, the present paper is organized as follows:

In section 2, we characterize quasi-multipliers on $L(\mathcal{X})$, for a certain class of hypergroups $\mathcal{X}$. As the main result, we prove that $\mathfrak{QM}(L(\mathcal{X}))$ may be identified by $M(\mathcal{X})$. Indeed, this fact improves an interesting result of McKennon [7, Corollary of Theorem 22] in the setting of locally compact groups to a more general setting of locally compact hypergroups. In section 3, we concentrate on multiplier and quasi-multiplier algebra of the second dual of hypergroup algebras. Precisely, we show that the multiplier and quasi-multiplier algebra of $L_c(\mathcal{X})^{**}$ is isomorphic with $M(\mathcal{X})$ for certain hypergroups $\mathcal{X}$. As a consequence, we prove that $\mathfrak{QM}(L_c(\mathcal{X})^{**})=L(\mathcal{X})$ if and only if $\mathcal{X}$ is discrete.

2. Quasi-multiplier Algebra of Hypergroup Algebras

We first remark some standard notations and definitions that we shall need. We follow the hypergroup structure of Dunkl [15, Definition 1.1] without the commutativity assumption, that is more general than that of Jewett [16].

Throughout the paper, $\mathcal{X}$ denotes a locally compact Hausdorff space, $M(\mathcal{X})$ is the Banach space of all bounded complex-valued regular Borel measures on $\mathcal{X}$ with the total variation norm and $M_p(\mathcal{X})$ is the set of all probability Borel Measures on $\mathcal{X}$. Normally, we devote the symboles $C_b(\mathcal{X}), C_0(\mathcal{X})$, and $C_c(\mathcal{X})$ for the space of bounded continuous complex-valued functions on $\mathcal{X}$, those that vanish at infinity, and those that have compact support, respectively.

Definition 2.1. The space $\mathcal{X}$ is called a *hypergroup* if there is a map $\lambda\colon \mathcal{X}\times\mathcal{X} \longrightarrow M_p(\mathcal{X})$ with the following properties:

(i) the measures $\lambda_{(x,y)}$ have compact support for all $x, y \in \mathcal{X}$.

(ii) for each $f \in C_c(\mathcal{X})$, the mapping $(x, y) \mapsto \int_{\mathcal{X}} f(t)\, d\lambda_{(x,y)}(t)$ is in $C_b(\mathcal{X} \times \mathcal{X})$, and the mappings

$$x \mapsto \int_{\mathcal{X}} f(t)\, d\lambda_{(x,y)}(t), \quad x \mapsto \int_{\mathcal{X}} f(t)\, d\lambda_{(y,x)}(t)$$

are in $C_c(\mathcal{X})$ for all $y \in \mathcal{X}$.

(iii) the convolution $(\mu, \nu) \mapsto \mu * \nu$ on $M(\mathcal{X})$ defined by

$$\int_{\mathcal{X}} f(t)\, d(\mu * \nu)(t) = \int_{\mathcal{X}}\int_{\mathcal{X}}\int_{\mathcal{X}} f(t)\, d\lambda_{(x,y)}(t)\, d\mu(t)\, d\nu(t)$$

is associative, where $f \in C_0(\mathcal{X}), \mu, \nu \in M(\mathcal{X})$.

(iv) there is a unique point $e \in \mathcal{X}$ (say the identity) such that

$$\lambda_{(x,e)} = \delta_x = \lambda_{(e,x)} \quad (x \in \mathcal{X}),$$

where δ_x denotes the Dirac measure at x.

With the above definition, $M(\mathcal{X})$ can be regarded as a Banach algebra. Furthermore, Ghahramani and Medghalchi in [18] defined $L(\mathcal{X})$ as a subalgebra $M(\mathcal{X})$, consisting of all measures μ for which the mappings $x \mapsto |\mu| * \delta_x$ and $x \mapsto \delta_x * |\mu|$ from $\mathcal{X}$ to $M(\mathcal{X})$ are norm continuous. They have shown that $L(\mathcal{X})$ is a closed ideal in $M(\mathcal{X})$ and also that if $\mathcal{X}$ admits a left invariant measure m, then $L(\mathcal{X}) = L^1(\mathcal{X}, m)$ [18, Remark 1]. The hypergroup $\mathcal{X}$ is called *foundation* if

$$\mathcal{X} = \overline{\bigcup\{\operatorname{supp}(\mu) : \mu \in L(\mathcal{X})\}}.$$

In this case, the Banach algebra $L(\mathcal{X})$ has a bounded approximate identity, see [18] and [19]. We note that all hypergroups considered in this article are assumed to be foundation hypergroups without left invariant Haar measure. In [18, Lemma 1], it is proved that the Banach algebra $L(\mathcal{X})$ admits a bounded approximate identity with norm 1 carried by any compact neiborhood of e.

In the following, we show that $L(\mathcal{X})$ possesses an ultra-approximate identity.

Theorem 2.2. *Let $\mathcal{X}$ be a hypergroup. Then $L(\mathcal{X})$ admits an ultra-approximate identity.*

Proof. Let (e_α) be the bounded approximate identity for $L(\mathcal{X})$. Suppose that T is a left multiplier of $L(\mathcal{X})$. By [18, Proposition 1], it follows that the multiplier algebra of $L(\mathcal{X})$ is isometrically isomorphic with $M(\mathcal{X})$. We consider $\mu \in M(\mathcal{X})$ such that

$$T(\nu) = \mu * \nu \qquad (\nu \in L(\mathcal{X})).$$

Since $L(\mathcal{X})$ is an ideal in $M(\mathcal{X})$, then for all $\nu \in L(\mathcal{X})$,

$$\lim_\alpha \|\nu * T(\mu_\alpha) - \nu * \mu\| = \lim_\alpha \|(\nu * \mu) - (\nu * \mu) * \mu_\alpha\| = 0.$$

Therefore, $\{\nu * T(\mu_\alpha)\}$ is a Cauchy net in $L(\mathcal{X})$. For a right multiplier R of $L(\mathcal{X})$, it is shown similarly that, for each $\nu \in L(\mathcal{X})$, the net $\{R(\mu_\alpha) * \nu\}$ is Cauchy in $L(\mathcal{X})$. Hence, by [7, Theorem 8], we conclude that (e_α) is an ultra-approximate identity and the proof is complete. ∎

As a consequence of the above theorem and [7, Theorem 9], we give the following prompt result.

Corollary 2.3. *Let $\mathcal{X}$ be a hypergroup. Then $\mathfrak{QM}(L(\mathcal{X}))$ is identified with $M(\mathcal{X})$.*

For a Banach algebra $\mathcal{A}$ and continuous bilinear mapping $\mathfrak{m}\colon \mathcal{A}\times\mathcal{A} \longrightarrow \mathcal{A}$ we construct the maps $\mathfrak{m}^*\colon \mathcal{A}^*\times\mathcal{A} \longrightarrow \mathcal{A}^*$ and $\mathfrak{m}^{t^*}\colon \mathcal{A}\times\mathcal{A}^* \longrightarrow \mathcal{A}^*$ as follows:

$$\langle \mathfrak{m}^*(f,a),b\rangle = \langle f,\mathfrak{m}(a,b)\rangle, \quad \langle \mathfrak{m}^{t^*}(a,f),b\rangle = \langle f,\mathfrak{m}(b,a)\rangle,$$

for all $a,b\in\mathcal{A}$ and $f\in\mathcal{A}^*$.

In the sequel, we give a characterization of quasi-multipliers of hypergroup algebras as a generalization of Wendel's theorem.

Theorem 2.4. *Let $\mathcal{X}$ be a hypergroup. Then for a continuous bilinear mapping $\mathfrak{m}\colon L(\mathcal{X})\times L(\mathcal{X}) \longrightarrow L(\mathcal{X})$ the following statements are equivalent:*

(i) $\mathfrak{m}(\mu'*\mu,\nu*\nu') = \mu'*\mathfrak{m}(\mu,\nu)*\nu'$ *for all* $\mu,\mu',\nu,\nu'\in L(\mathcal{X})$.
(ii) $\mathfrak{m}(\delta_x*\mu,\nu*\delta_y) = \delta_x*\mathfrak{m}(\mu,\nu)*\delta_y$ *for all* $\mu,\nu\in L(\mathcal{X})$ *and* $x,y\in\mathcal{X}$.
(iii) $\mathfrak{m}(\mu'*\mu,\nu*\nu') = \mu'*\mathfrak{m}(\mu,\nu)*\nu'$ *for all* $\mu,\nu\in L(\mathcal{X})$ *and* $\mu',\nu'\in M(\mathcal{X})$.
(iv) there exists $\xi\in M(\mathcal{X})$ *such that* $\mathfrak{m}(\mu,\nu) = \mu*\xi*\nu$ *for all* $\mu,\nu\in L(\mathcal{X})$. *Moreover* $\|\mathfrak{m}\| = \|\xi\|$.

Proof. *(i)* ⇒ *(ii)*: Let $\mu,\nu\in L(\mathcal{X})$ and $x\in\mathcal{X}$ and let (e_α) be a bounded approximate identity in $L(\mathcal{X})$. Then for all $f\in C_0(\mathcal{S})$ we have

$$\begin{aligned}
\langle \mathfrak{m}(\delta_x*\mu,\nu),f\rangle &= \lim_\alpha\langle e_\alpha*\mathfrak{m}(\delta_x*\mu,\nu),f\rangle\\
&= \lim_\alpha\langle \mathfrak{m}(e_\alpha*(\delta_x*\mu),\nu),f\rangle\\
&= \lim_\alpha\langle \mathfrak{m}((e_\alpha*\delta_x)*\mu),\nu),f\rangle\\
&= \lim_\alpha\langle (e_\alpha*\delta_x)*\mathfrak{m}(\mu,\nu),f\rangle\\
&= \lim_\alpha\langle e_\alpha*(\delta_x*\mathfrak{m}(\mu,\nu)),f\rangle\\
&= \langle \delta_x*\mathfrak{m}(\mu,\nu),f\rangle.
\end{aligned}$$

Hence, $\mathfrak{m}(\delta_x*\mu,\nu) = \delta_x*\mathfrak{m}(\mu,\nu)$. Similarly, we can show that

$$\mathfrak{m}(\mu,\nu*\delta_y) = \mathfrak{m}(\mu,\nu)*\delta_y \quad (\mu,\nu\in L(\mathcal{X}), y\in\mathcal{X}),$$

and so the proof is complete.

(ii) ⇒ *(iii)*: Suppose that $\mu,\nu\in L(\mathcal{X})$ and $\mu',\nu'\in M(\mathcal{X})$. Therefore,

$$\begin{aligned}
\langle f,\mathfrak{m}(\mu'*\mu,\nu)\rangle &= \langle \mathfrak{m}^{t^*}(\nu,f),\mu'*\mu\rangle\\
&= \int_{\mathcal{X}}\langle \mathfrak{m}^{t^*}(\nu,f),\delta_x*\mu\rangle\, d\mu'(x)\\
&= \int_{\mathcal{X}}\langle f,\mathfrak{m}(\delta_x*\mu,\nu)\rangle\, d\mu'(x)\\
&= \int_{\mathcal{X}}\langle f,\delta_x*\mathfrak{m}(\mu,\nu)\rangle\, d\mu'(x)\\
&= \langle f,\mu'*\mathfrak{m}(\mu,\nu)\rangle,
\end{aligned}$$

holds for all $f \in C_0(\mathcal{X})$. Consequently, $\mathfrak{m}(\mu' * \mu, \nu) = \mu' * \mathfrak{m}(\mu, \nu)$. Also for each $f \in C_0(\mathcal{X})$, we get,

$$\begin{aligned}
\langle f, \mathfrak{m}(\mu, \nu * \nu') \rangle &= \langle \mathfrak{m}^*(f, \mu), \nu * \nu' \rangle \\
&= \int_{\mathcal{X}} \langle \mathfrak{m}^*(f, \mu), \nu * \delta_y \rangle \, d\nu'(y) \\
&= \int_{\mathcal{X}} \langle f, \mathfrak{m}(\mu, \nu * \delta_y \rangle \, d\nu'(y) \\
&= \int_{\mathcal{X}} \langle f, \mathfrak{m}(\mu, \nu) * \delta_y \rangle d\nu'(y) \\
&= \langle f, \mathfrak{m}(\mu, \nu) * \nu' \rangle.
\end{aligned}$$

Hence, $\mathfrak{m}(\mu, \nu * \nu') = \mathfrak{m}(\mu, \nu) * \nu'$, for all $\mu, \nu, \nu' \in L(\mathcal{X})$.
$(iii) \Rightarrow (i)$: Since $L(\mathcal{X}) \subset M(\mathcal{X})$, it is clear.
$(i) \Rightarrow (iv)$: Let (e_α) be a bounded approximate identity of norm 1 in $L(\mathcal{X})$. Since $\mathfrak{m}$ is jointly continuous [7, Theorem 1], without loss of generality, we may assume that $\mathfrak{m}(e_\alpha, e_\alpha) \longrightarrow \xi$ in the weak*-topology. Therefore, we have

$$\begin{aligned}
\langle f, \mathfrak{m}(\mu, \nu) \rangle &= \lim_\alpha \langle f, \mathfrak{m}(\mu * e_\alpha, e_\alpha * \nu) \rangle \\
&= \lim_\alpha \langle f, \mu * \mathfrak{m}(e_\alpha, e_\alpha) * \nu \rangle \\
&= \lim_\alpha \langle \nu \cdot f \cdot \mu, \mathfrak{m}(e_\alpha, e_\alpha) \rangle \\
&= \lim_\alpha \langle \nu \cdot f \cdot \mu, \xi \rangle \\
&= \langle f, \mu * \xi * \nu \rangle.
\end{aligned}$$

for all $f \in C_0(\mathcal{X})$ and $\mu, \nu \in M(\mathcal{X})$. This implies that $\mathfrak{m}(\mu, \nu) = \mu * \xi * \nu$. For the second part, let $\varepsilon > 0$ be arbitrary. Then there exists an $f \in C_0(\mathcal{X})$ such that $\|f\| \leq 1$ and $\|\xi\| - \varepsilon \leq |\langle \xi, f \rangle|$. On the other hand,

$$\begin{aligned}
|\langle \xi, f \rangle| &= |\lim_\alpha \langle \mathfrak{m}(e_\alpha, e_\alpha), f \rangle| \\
&\leq \lim_\alpha |\langle \mathfrak{m}(e_\alpha, e_\alpha), f \rangle| \\
&\leq \|\mathfrak{m}\|.
\end{aligned}$$

By this fact, we immediately conclude that $\|\mathfrak{m}\| = \|\xi\|$.
The implication $(iv) \Rightarrow (i)$ is straightforward. ■

Remark 2.5.
(i) Using the previous theorem we immediately conclude that for a locally compact group G, $\mathfrak{m} \in \mathfrak{QM}(L^1(G))$ if and only if there exists a measure $\mu \in M(G)$ such that $\|\mathfrak{m}\| = \|\mu\|$ and $\mathfrak{m}(f, g) = f * \mu * g$, for all $f, g \in L^1(G)$.
(ii) We note that without the continuity of bilinear map $\mathfrak{m}$, the clauses (i), (iii) and (iv) in Theorem 2.4 are again equivalent. Indeed, since $L(\mathcal{X})$ has a bounded approximate identity it follows that the bilinear map $\mathfrak{m}$ is automatically continuous, see [7, Theorem 1].

3. Some Results on Second Dual of Hypergroup Algebras

Let $L(\mathcal{X})^*$ and $L(\mathcal{X})^{**}$ be the first and second topological duals of $L(\mathcal{X})$, respectively, and set $B = L(\mathcal{X})^* \cdot L(\mathcal{X})$. It is shown in [20] that B^* is a Banach algebra equiped with the first Arens product $\square$ and also $L(\mathcal{X}) \subseteq B^*$. Let $\imath \colon B \longrightarrow L(\mathcal{X})^*$ be the natural embedding and also $\pi\colon L(\mathcal{X})^{**} \longrightarrow B^*$ is the adjoint of $\imath$. Then the quotient map π is identity on $L(\mathcal{X})$ and it is a weak*-weak* continuous homomorphism from $L(\mathcal{X})^{**}$ onto B^* [20]. Also, we have

$$F\Box G = F\Box\pi(G) \qquad (F, G \in L(\mathcal{X})^{**}).$$

We remark that $C_0(\mathcal{X}) \subseteq B$ and $B^* = M(\mathcal{X}) \oplus C_0(\mathcal{X})^{\perp}$, where the sum is the algebra direct sum of the subspace $M(\mathcal{X})$ and the two sided ideal $C_0(\mathcal{X})^{\perp}$ defined by

$$C_0(\mathcal{X})^{\perp} = \{F \in B^* : \ F|_{C_0(\mathcal{X})} = 0\}.$$

Definition 3.1. A compact set $K \subseteq \mathcal{X}$ is called a *compact carrier* for $F \in L(\mathcal{X})^{**}$ if we have

$$\langle F, f\rangle = \langle F, f\chi_K\rangle \quad (f \in L(\mathcal{X})^*),$$

where $f\chi_K \in L(\mathcal{X})^*$ defined by $\langle f\chi_K, \mu\rangle = \langle f, \mu\chi_K\rangle$.

Now, we set

$$L_c(\mathcal{X})^{**} = \mathrm{cl}_{L(\mathcal{X})^{**}}\{F \in L(\mathcal{X})^{**} : F \text{ has compact carrier}\}.$$

If $\mathcal{X} = G$ is a locally compact group, then $L_c(\mathcal{X})^{**} = L_0^{\infty}(G)^*$, where $L_0^{\infty}(G)$ is the introverted subspace of $L^{\infty}(G)$ consisting of all $f \in L^{\infty}(G)$ such that, for given $\varepsilon > 0$, there is a compact subset K of $\mathcal{X}$ for which $\|g\|_{G\setminus K} \leq \varepsilon$; see also [21] and [22]. It is well known that for a foundation hypergroup $\mathcal{X}$, $L(\mathcal{X})$ is a two-sided closed ideal in $L_c(\mathcal{X})^{**}$ [20, Theorem 14(d)]. Furthermore, an easy application of the Goldstine's theorem shows that $L(\mathcal{X})$ is weak* dense in $L_c(\mathcal{X})^{**}$. It is worthwhile to mention that Medghalchi in [20, Theorem 11] showed that

$$L(\mathcal{X})^{**} = L_c(\mathcal{X})^{**} \oplus \pi^{-1}(C_0(\mathcal{X})^{\perp}),$$

as Banach algebras. Moreover, it is proved that $\pi(L_c(\mathcal{X})^{**}) = M(\mathcal{X})$ [20, Proposition 13].

Definition 3.2. An element $E \in L_c(\mathcal{X})^{**}$ is called a *mixed identity* if

$$\mu\Box E = E\Diamond\mu = \mu \qquad (\mu \in L(\mathcal{X}),$$

where $\Diamond$ denotes the second Arens product on $L_c(\mathcal{X})^{**}$.

Let $\mathcal{X}$ be a hypergroup. We denote by $\mathcal{E}(\mathcal{X})$, the nonempty set of all mixed identities of $L_c(\mathcal{X})^{**}$, and $\mathcal{E}_1(\mathcal{X})$, consisting of those with norm 1. Note that $E \in \mathcal{E}_1(\mathcal{X})$ if and only if it is a weak*-cluster point of an approximate identity in $L(\mathcal{X})$ bounded by one. Moreover, when $\mathcal{X}$ is a hypergroup, and E is an element of $L_c(\mathcal{X})^{**}$ with norm 1, then E is a mixed identity if and only if E is a right identity for $L_c(\mathcal{X})^{**}$ and $\pi(E) = \delta_e$; that is,

$$F\Box E = F$$

for all $F \in L_c(\mathcal{X})^{**}$; see [20].

Let $\mathcal{A}$ be a Banach algebra and $\mathfrak{LM}(\mathcal{A})$ and $\mathfrak{RM}(\mathcal{A})$ denote the Banach algebra of left and right multipliers on $\mathcal{A}$, respectively. Define the linear isometry map$\lambda\colon \mathfrak{LM}(\mathcal{A}) \longrightarrow \mathfrak{QM}(\mathcal{A})$ for each $T \in \mathfrak{LM}(\mathcal{A})$ by

$$[\lambda(T)](x, y) = xT(y) \qquad (x, y \in \mathcal{A}).$$

Similarly, the function $\rho\colon \mathfrak{RM}(\mathcal{A}) \longrightarrow \mathfrak{QM}(\mathcal{A})$ defined by

$$[\rho(T)](x, y) = T(x)y \qquad (x, y \in \mathcal{A}),$$

is also a linear isometry [7, Theorem 4].

First, we obtain a characterization of multiplier algebra of $L_c(\mathcal{X})^{**}$ as follows.

Proposition 3.3. *Let $\mathcal{X}$ be a hypergroup. Then the multiplier algebra of $L_c(\mathcal{X})^{**}$ is isomorphic with $M(\mathcal{X})$.*

Proof. Let (e_α) be a bounded approximate identity of $L(\mathcal{X})$ with norm 1. So, if we assume that E is a weak* cluster points of (e_α), then E is a right identity in $L_c(\mathcal{X})^{**}$. Now, let $T\colon L_c(\mathcal{X})^{**} \longrightarrow L_c(\mathcal{X})^{**}$ be a right multiplier. We conclude that

$$T(m) = T(m\Box E) = m\Box T(E) \qquad (m \in L_c(\mathcal{X})^{**}).$$

Now, it follows that

$$T(m) = m\Box n = m\Box\pi(n).$$

Now, we know that the algebra consisting of elements $\pi(n)$ is exactly $M(\mathcal{X})$. On the other hand, since $M(\mathcal{X})$ has an identity we conclude that the multiplier algebra of $L_c(\mathcal{X})^{**}$ is $M(\mathcal{X})$. ∎

Now, we are ready to characterize quasi-multipliers of the second dual Banach algebra $L_c(\mathcal{X})^{**}$.

Theorem 3.4. *Let $\mathcal{X}$ be a hypergroup. Then the quasi-multiplier algebra of $L_c(\mathcal{X})^{**}$ is isomorphic with $M(\mathcal{X})$.*

Proof. Since $L(\mathcal{X})$ has a bounded approximate identity so $\mathcal{E}_1(\mathcal{X})$ is a nonempty set. Let $E \in \mathcal{E}_1(\mathcal{X})$ be a right identity of $L_c(\mathcal{X})^{**}$. For $\mathfrak{m} \in \mathfrak{QM}(L_c(\mathcal{X})^{**}$ define the linear operator $T\colon L_c(\mathcal{X})^{**} \longrightarrow L_c(\mathcal{X})^{**}$ by

$$T(F) = \mathfrak{m}(E, F) \qquad (F \in L_c(\mathcal{X})^{**}).$$

If $F, G \in L_c(\mathcal{X})^{**}$, then

$$\begin{aligned} T(F\Box G) &= \mathfrak{m}(E, F\Box G) \\ &= \mathfrak{m}(E, F)\Box G \\ &= T(F)\Box G. \end{aligned}$$

It follows that $T \in \mathfrak{LM}(L_c(\mathcal{X})^{**})$. Furthermore, for all $F, G \in L_c(\mathcal{X})^{**}$ we have

$$\begin{aligned} [\lambda(T)](F, G) &= F\Box T(G) \\ &= F\Box\mathfrak{m}(E, G) \\ &= \mathfrak{m}(F\Box E, G) \\ &= \mathfrak{m}(F, G). \end{aligned}$$

Thus $\lambda(T) = \mathfrak{m}$ and hence λ is surjective. This implies that the quasi-multiplier algebra of $L_c(\mathcal{X})^{**}$ is isomorphic to left multiplier algebra of $L_c(\mathcal{X})^{**}$. By using Proposition 3.3, we conclude that the quasi-multiplier algebra of $L_c(\mathcal{X})^{**}$ is isomorphic with $M(\mathcal{X})$. ∎

As a consequence, we obtain a necessary and sufficient condition for which the quasi-multiplier algebra $L_c(\mathcal{X})^{**}$ is equal to $L(\mathcal{X})$.

Corollary 3.5. *Let $\mathcal{X}$ be a hypergroup. Then the following assertions are equivalent.*

(i) $\mathfrak{QM}(L_c(\mathcal{X})^{**}) = L(\mathcal{X})$.
(ii) *$\mathcal{X}$ is discrete.*

Proof. Let $\mathcal{X}$ be discrete hypergroup. Thus by [20, Theorem 14(c)] we conclude that $M(\mathcal{X}) = L_c(\mathcal{X})^{**} = L(\mathcal{X})$. Now, it follows from Theorem 3.4 that

$$\mathfrak{QM}(L_c(\mathcal{X})^{**}) = M(\mathcal{X}) = L(\mathcal{X}).$$

Conversely, suppose that quasi-multiplier algebra of $L_c(\mathcal{X})^{**}$ is equal to $L(\mathcal{X})$. By Theorem 3.4, we deduce that $M(\mathcal{X}) = L(\mathcal{X})$ and so $\mathcal{X}$ is discrete. ∎

At the end, we obtain some results on the right annihilator of $L_c(\mathcal{X})^{**}$ which is denoted by $\mathrm{Ann}_r(L_c(\mathcal{X})^{**})$ and defined by

$$\mathrm{Ann}_r(L_c(\mathcal{X})^{**}) = \{R \in L_c(\mathcal{X})^{**} |\ \ L_c(\mathcal{X})^{**}\Box R = \{0\}\}.$$

Let $\mathcal{X}$ be a hypergroup. Then $\mathrm{Ann}_r(L_c(\mathcal{X})^{**})$ is exactly the weak*-closed ideal

$$\ker(\pi) = \{F - E\Box F : F \in L_c(\mathcal{X})^{**}\}$$

in $L_c(\mathcal{X})^{**}$ for all $E \in \mathcal{E}_1(\mathcal{X})$. Indeed, it follows immediately that if $\pi(R) = 0$ then R is a right annihilator. If R is a right annihilator, then in particular $\mu\Box R = 0$ for all $\mu \in L(\mathcal{X})$. So for $f \in C_0(\mathcal{X})$ we have $f = g \cdot \mu$ for some $g \in L(\mathcal{X})^*$ and $\mu \in L(\mathcal{X})$. Thus,

$$\begin{aligned} \langle \pi(R), f\rangle &= \langle \pi(R), g \cdot \mu\rangle \\ &= \langle \mu\Box\pi(R), g\rangle \\ &= 0. \end{aligned}$$

In the following, we show that the subspaces $L(\mathcal{X})$ and $\mathrm{Ann}_r(L_c(\mathcal{X})^{**})$ are invariant with respect to any quasi-multiplier on $L_c(\mathcal{X})^{**}$.

Theorem 3.6. *Let $\mathcal{X}$ be a hypergroup and $\mathfrak{m}\colon L_c(\mathcal{X})^{**} \times L_c(\mathcal{X})^{**} \longrightarrow L_c(\mathcal{X})^{**}$ be a quasi-multiplier. Then the following statements hold:*

(i) $\mathfrak{m}(L(\mathcal{X}) \times L(\mathcal{X})) \subseteq L(\mathcal{X})$.
(ii) $\mathfrak{m}(Ann_r(L_c(\mathcal{X})^{**}) \times Ann_r(L_c(\mathcal{X})^{**})) \subseteq Ann_r(L_c(\mathcal{X}))^{**})$.

Proof. (i) For each $\mu, \mu', \nu, \nu' \in L(\mathcal{X})$, we have

$$\mathfrak{m}(\mu' * \mu, \nu * \nu') = \mu'\Box\mathfrak{m}(\mu, \nu)\Box\nu'.$$

On the other hand, since $L(\mathcal{X})$ is an ideal in $L_c(\mathcal{X})^{**}$ and $L(\mathcal{X})^2 = L(\mathcal{X})$, it follows that $\mathfrak{m}(L(\mathcal{X}) \times L(\mathcal{X})) \subseteq L(\mathcal{X})$.

(ii) Let $R, S \in \mathrm{Ann}_r(L_c(\mathcal{X})^{**})$. Then we have

$$\mu\Box\mathfrak{m}(R, S)\Box\nu = \mathfrak{m}(\mu\Box R, S\Box\nu) = 0 \quad (\mu, \nu \in L(\mathcal{X})).$$

Hence

$$L(\mathcal{X})\Box\mathfrak{m}(R, S)\Box L(\mathcal{X}) = 0.$$

On the other hand, since the map $\pi : L(\mathcal{X})^{**} \longrightarrow B^*$ is a homomorphism and π acts as the identity map on $L(\mathcal{X})$ we conclude that

$$L(\mathcal{X}) * \pi(\mathfrak{m}(R, S)) * L(\mathcal{X}) = 0.$$

Now, since $L(\mathcal{X})$ has a bounded approximate identity it follows that $\mathfrak{m}(R, S) \in \ker(\pi) = \mathrm{Ann}_r(L_c(\mathcal{X})^{**})$, as required. ∎

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