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Bounds of the Derivative of Some Classes of Rational Functions

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Abstract Let $r(z)$ be a rational function with at most n poles, $a_1, a_2, \ldots, a_n$, where $|a_j| > 1$, $1 \le j \le n$. This paper investigates the estimate of the modulus of the derivative of a rational function $r(z)$ on the unit circle. We establish an upper bound when all zeros of $r(z)$ lie in $|z| \ge k \ge 1$ and a lower bound when all zeros of $r(z)$ lie in $|z| \le k \le 1$. In particular, when $k = 1$ and $r(z)$ has exactly n zeros, we obtain a generalization of results by Aziz and Shah.

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1. Introduction

Let $\mathcal{P}_n$ denote the class of all complex polynomials of degree at most n and let k be a positive real number. We denote $T_k = \{z : |z| = k\}, D_{k-} = \{z : |z| < k\}$, and $D_{k+} = \{z : |z| > k\}$. Consider a polynomial $p(z)$ of degree n. In 1926, Bernstein [1] presented the well-known inequality

$$\max_{|z|=1} |p'(z)| \le n \max_{|z|=1} |p(z)|. \tag{1.1}$$

Equality holds in (1.1) only for $p(z) = az^n$ where $a \neq 0$. If we restrict to the class of polynomials having no zeros in D_{1-}, inequality (1.1) can be sharpened. In fact, it was conjectured by Erdős and later proved by Lax [2] that if $p(z)$ has no zeros in D_{1-}, then

$$\max_{|z|=1} |p'(z)| \le \frac{n}{2} \max_{|z|=1} |p(z)|. \tag{1.2}$$

For the class of polynomials having no zeros in D_{1+}, Turán [3] proved that

$$\max_{|z|=1} |p'(z)| \ge \frac{n}{2} \max_{|z|=1} |p(z)|. \tag{1.3}$$

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For $a_j \in \mathbb{C}$ $(1 \leq j \leq n)$, we let $w(z) = \prod_{j=1}^{n}(z - a_j)$ and

$$B(z) = \prod_{j=1}^{n}\left(\frac{1-\overline{a_j}z}{z-a_j}\right), \quad \mathcal{R}_n = \mathcal{R}_n(a_1, a_2, \ldots, a_n) := \left\{\frac{p(z)}{w(z)} : p \in \mathcal{P}_n\right\}.$$

The product $B(z)$ is known as a *Blaschke product*. We can show that $|B(z)| = 1$ and $\frac{zB'(z)}{B(z)} = |B'(z)|$ for $z \in T_1$. Note that $\mathcal{R}_n$ is the set of rational functions with at most n poles, $a_1, a_2, \ldots, a_n$ and with finite limit at infinity. For f defined on T_1, we denote $||f|| = \sup_{z \in T_1} |f(z)|$, the *Chebyshev norm* of f on T_1. Throughout this paper, we always assume that all poles $a_1, a_2, \ldots, a_n$ are in D_{1+}.

In 1995, Li, Mohapatra, and Rodriguez [4] proved some inequalities similar to (1.2), and (1.3) for rational functions. Among other things they proved the following results.

Theorem 1.1 ([4]). *Let $r \in \mathcal{R}_n$ with all its zeros lying in $T_1 \cup D_{1+}$. Then for $z \in T_1$,*

$$|r'(z)| \leq \frac{1}{2}|B'(z)| \cdot ||r||. \tag{1.4}$$

Equality holds for $r(z) = aB(z) + b$ with $|a| = |b| = 1$.

Theorem 1.2 ([4]). *Let $r \in \mathcal{R}_n$, where r has exactly n poles at $a_1, a_2, \ldots, a_n$ and all its zeros lie in $T_1 \cup D_{1-}$. Then for $z \in T_1$,*

$$|r'(z)| \geq \left[\frac{1}{2}|B'(z)| - \frac{1}{2}(n-t)\right] \cdot |r(z)|, \tag{1.5}$$

where t is the number of zeros of r. Equality holds for $r(z) = aB(z) + b$ with $|a| = |b| = 1$.

Remark 1.3. In particular, if r has exactly n zeros in $T_1 \cup D_{1-}$, then inequality (1.5) yields Bernstein-type inequality, for $z \in T_1$,

$$|r'(z)| \geq \frac{1}{2}|B'(z)||r(z)|. \tag{1.6}$$

In 1997, inequalities (1.4) and (1.6) were improved by Aziz and Shah [5] under the same hypothesis. They obtained the following theorems.

Theorem 1.4 ([5]). *Let $r \in \mathcal{R}_n$ with all its zeros lying in $T_1 \cup D_{1+}$. Then for $z \in T_1$,*

$$|r'(z)| \leq \frac{1}{2}|B'(z)| \left(||r|| - m\right),$$

where $m = \min_{|z|=1} |r(z)|$. Equality holds for $r(z) = B(z) + he^{i\alpha}$ where $h \geq 1$ and α is real.

Theorem 1.5 ([5]). *Let $r \in \mathcal{R}_n$, where r has exactly n zeros and all its zeros lie in $T_1 \cup D_{1-}$. Then for $z \in T_1$,*

$$|r'(z)| \geq \frac{1}{2}|B'(z)| \left(|r(z)| + m\right)$$

where $m = \min_{|z|=1} |r(z)|$. Equality holds for $r(z) = B(z) + he^{i\alpha}$ where $h \leq 1$ and α is real.

In 1999, Aziz and Zarger [6] considered a class of rational functions $\mathcal{R}_n$ not vanishing in D_{k-}, where $k \geq 1$ and established the following generalization of Theorem 1.1.

Theorem 1.6 ([6]). *Let $r \in \mathcal{R}_n$ with all its zeros lying in $T_k \cup D_{k+}$, where $k \geq 1$. Then for $z \in T_1$,*

$$|r'(z)| \leq \frac{1}{2}\left[|B'(z)| - \frac{n(k-1)}{k+1} \cdot \frac{|r(z)|^2}{||r||^2}\right] \cdot ||r||.$$

Equality holds for $r(z) = \left(\frac{z+k}{z-a}\right)^n$ and $B(z) = \left(\frac{1-az}{z-a}\right)^n$ evaluated at $z = 1$, where $a > 1, k \geq 1$.

In 2004, Aziz and Shah [7] considered a class of rational functions $\mathcal{R}_n$ not vanishing in D_{k+}, where $k \leq 1$ and they proved the following generalization of Theorem 1.2.

Theorem 1.7 ([7]). *Let $r \in \mathcal{R}_n$ where r has exactly n poles at $a_1, a_2, \ldots, a_n$ and all its zeros lie in $T_k \cup D_{k-}$, $k \leq 1$. Then for $z \in T_1$,*

$$|r'(z)| \geq \frac{1}{2}\left[|B'(z)| + \frac{2t - n(1+k)}{1+k}\right] \cdot |r(z)|,$$

where t is the number of zeros of r. Equality holds for $r(z) = \frac{(z+k)^t}{(z-a)^n}$ and $B(z) = \left(\frac{1-az}{z-a}\right)^n$ evaluated at $z = 1$, where $a > 1, k \leq 1$.

As an immediate consequence of Theorem 1.7, they obtained the generalization of inequality (1.6), where r has exactly n zeros in $T_k \cup D_{k-}$, with $k \leq 1$.

Corollary 1.8 ([7]). *Let $r \in \mathcal{R}_n$ with all its zeros lying in $T_k \cup D_{k-}$, where $k \leq 1$. Then for $z \in T_1$,*

$$|r'(z)| \geq \frac{1}{2}\left[|B'(z)| + \frac{n(1-k)}{1+k}\right] \cdot |r(z)|.$$

Equality holds for $r(z) = \left(\frac{z+k}{z-a}\right)^n$ and $B(z) = \left(\frac{1-az}{z-a}\right)^n$ evaluated at $z = 1$, where $a > 1, k \leq 1$.

Next, we state our main results which generalize results by Aziz and Shah [5, 7] and Aziz and Zarger [6]. Their proofs will be presented in section 3. The first theorem gives an estimate of an upper bound of the modulus of the derivative of $r(z)$ on the unit circle when all zeros of $r(z)$ lie in $|z| \geq k \geq 1$.

Theorem 1.9. *Let $r(z) = p(z)/w(z) \in \mathcal{R}_n$ where $p(z)$ is a polynomial of degree n and all its zeros lie in $T_k \cup D_{k+}$, $k \geq 1$. Then for $z \in T_1$,*

$$|r'(z)| \leq \frac{1}{2}\left[|B'(z)| - \frac{n(k-1)(|r(z)| - m)^2}{(k+1)(||r|| - m)^2}\right](||r|| - m), \tag{1.7}$$

where $m = \min_{|z|=k} |r(z)|$. Equality holds for $r(z) = \left(\frac{z+k}{z-a}\right)^n$ and $B(z) = \left(\frac{1-az}{z-a}\right)^n$ evaluated at $z = 1$, where $a > 1, k \geq 1$.

In particular, for $k = 1$, Theorem 1.9 reduces to Theorem 1.4 and for $m = 0$, Theorem 1.9 reduces to Theorem 1.6.

The next theorem establishes an estimate of a lower bound of the modulus of the derivative of $r(z)$ on the unit circle when all zeros of r lie in $|z| \leq k \leq 1$.

Theorem 1.10. *Let $r \in \mathcal{R}_n$, where r has exactly n poles at $a_1, a_2, \ldots, a_n$ and all its zeros lie in $T_k \cup D_{k-}$, $k \leq 1$. Then for $z \in T_1$,*

$$|r'(z)| \geq \frac{1}{2}\left[|B'(z)| + \frac{2t - n(1+k)}{(1+k)}\right] \cdot (|r(z)| + m), \tag{1.8}$$

where t is the number of zeros of r with counting multiplicity and $m = \min_{|z|=k} |r(z)|$. Equality holds for $r(z) = \dfrac{(z+k)^t}{(z-a)^n}$ and $B(z) = \left(\dfrac{1-az}{z-a}\right)^n$ evaluated at $z = 1$, where $a > 1, k \leq 1$.

As an immediate consequence of Theorem 1.10, we have the following generalization of Theorem 1.5, where r has exactly n zeros in $T_1 \cup D_{1-}$.

Corollary 1.11. *Let $r(z) = p(z)/w(z) \in \mathcal{R}_n$ where $p(z)$ is a polynomial of degree n and all its zeros lie in $T_k \cup D_{k-}$, $k \leq 1$. Then for $z \in T_1$,*

$$|r'(z)| \geq \frac{1}{2}\left[|B'(z)| + \frac{n(1-k)}{(1+k)}\right] \cdot (|r(z)| + m),$$

where $m = \min_{|z|=k} |r(z)|$. Equality holds for $r(z) = \left(\dfrac{z+k}{z-a}\right)^n$ and $B(z) = \left(\dfrac{1-az}{z-a}\right)^n$ evaluated at $z = 1$, where $a > 1, k \leq 1$.

In particular, for $k = 1$, Corollary 1.11 reduces to Theorem 1.5, and for $m = 0$, Theorem 1.10 reduces to Theorem 1.7.

2. Lemmas

For the proof of our main theorems, we need the following lemmas. These three Lemmas are due to Li, Mohapatra, and Rodriguez [4].

Lemma 2.1 ([4])**.** *If $r \in \mathcal{R}_n$ and $r^*(z) = B(z)\overline{r(1/\overline{z})}$, then for $z \in T_1$,*

$$|(r^*(z))'| + |r'(z)| \leq |B'(z)| \cdot ||r||.$$

Equality holds for $r(z) = \lambda B(z)$ with $\lambda \in T_1$.

Lemma 2.2 ([4])**.** *Let $z \in \mathbb{C}$. Then*

$$\mathrm{Re}(z) \leq \frac{1}{2} \quad \textit{if and only if} \quad |z| \leq |z-1|.$$

Moreover, the statement holds when $\leq$ is replaced by $<$ at each occurrence.

Remark 2.3. Similar to the proof of Lemma 2.2, we can replace $\leq$ by $\geq$ and obtain that

$$\mathrm{Re}(z) \geq \frac{1}{2} \quad \text{if and only if} \quad |z| \geq |z-1|.$$

Lemma 2.4 ([4])**.** *Let $r \in \mathcal{R}_n$.*

(i) If all zeros of r lie in $T_1 \cup D_{1+}$, then for $z \in T_1$,

$$\mathrm{Re}\left(\frac{zr'(z)}{r(z)}\right) \leq \frac{|B'(z)|}{2},$$

where $r(z) \neq 0$.

(ii) If r has exactly n poles at $a_1, a_2, \dots, a_n$ and all its zeros lie in $T_1 \cup D_{1-}$, then for $z \in T_1$,

$$\operatorname{Re}\left(\frac{zr'(z)}{r(z)}\right) \geq \frac{|B'(z)|}{2} - \frac{1}{2}(n-t),$$

where t is the number of zeros of r with counting multiplicity and $r(z) \neq 0$.

The next lemma is due to Aziz and Zarger [6].

Lemma 2.5 ([6]). *If $z \in T_1$, then*

$$\operatorname{Re}\left(\frac{zw'(z)}{w(z)}\right) = \frac{n - |B'(z)|}{2}.$$

We need the following preliminary result for the proofs of Theorem 1.9 and Theorem 1.10.

Lemma 2.6. *Assume that $r \in \mathcal{R}_n$, where r has exactly n poles at $a_1, a_2, \dots, a_n$. Let t be the number of zeros of r with counting multiplicity.*

(i) If all zeros of r lie in $T_k \cup D_{k+}$, where $k \geq 1$, and $z \in T_1$ with $r(z) \neq 0$, then

$$\operatorname{Re}\left(\frac{zr'(z)}{r(z)}\right) \leq \frac{|B'(z)|}{2} + \frac{2t - n(1+k)}{2(1+k)}.$$

(ii) If all zeros of r lie in $T_k \cup D_{k-}$, where $k \leq 1$, and $z \in T_1$ with $r(z) \neq 0$, then

$$\operatorname{Re}\left(\frac{zr'(z)}{r(z)}\right) \geq \frac{|B'(z)|}{2} - \frac{n(1+k) - 2t}{2(1+k)}.$$

Proof. Let $r(z) = \dfrac{p(z)}{w(z)} \in \mathcal{R}_n$.
If $b_1, b_2, \dots, b_t$ are all zeros (may be not distinct) of $p(z)$, then $t \leq n$.

(i) Assume that $|b_j| \geq k \geq 1$, $j = 1, 2, \dots, t$. Then

$$\frac{zr'(z)}{r(z)} = \frac{zp'(z)}{p(z)} - \frac{zw'(z)}{w(z)} = \left[\sum_{j=1}^{t} \frac{z}{z - b_j}\right] - \frac{zw'(z)}{w(z)}.$$

For $z \in T_1$, this relation with the help of Lemma 2.5 gives

$$\operatorname{Re}\left(\frac{zr'(z)}{r(z)}\right) = \left(\sum_{j=1}^{t} \operatorname{Re}\left(\frac{z}{z - b_j}\right)\right) - \left(\frac{n - |B'(z)|}{2}\right). \tag{2.1}$$

For $z \in T_1$ with $z \neq b_j (1 \leq j \leq t)$, we consider two cases.

Case 1: $|b_j| = 1$. Then $k = 1$ and $\left|\dfrac{z}{z - b_j}\right| = \left|\dfrac{b_j}{z - b_j}\right| = \left|\dfrac{z}{z - b_j} - 1\right|$.

By Lemma 2.2, we obtain that $\operatorname{Re}\left(\dfrac{z}{z - b_j}\right) \leq \dfrac{1}{2} = \dfrac{1}{1+1} = \dfrac{1}{1+k}$.

Case 2: $|b_j| > 1$. A bilinear transformation $w_j(z) = \dfrac{z}{z - b_j}$ maps T_1 onto a circle $\left\{w : \left|w + \dfrac{1}{|b_j|^2 - 1}\right| = \dfrac{|b_j|}{|b_j|^2 - 1}\right\}$. Then

$$\operatorname{Re}\left(\frac{z}{z - b_j}\right) \leq \left(-\frac{1}{|b_j|^2 - 1}\right) + \frac{|b_j|}{|b_j|^2 - 1} = \frac{1}{|b_j| + 1} \leq \frac{1}{1+k}.$$

From both cases, $\mathrm{Re}\left(\dfrac{z}{z-b_j}\right) \le \dfrac{1}{1+k}$ for $|b_j| \ge k \ge 1$, $z \in T_1$ with $z \ne b_j$ $(1 \le j \le t)$. Substituting this relation into (2.1), we obtain that

$$\mathrm{Re}\left(\frac{zr'(z)}{r(z)}\right) \le \sum_{j=1}^{t}\left(\frac{1}{1+k}\right) - \left(\frac{n-|B'(z)|}{2}\right) = \frac{|B'(z)|}{2} + \frac{2t-n(1+k)}{2(1+k)}.$$

(ii) Assume that $|b_j| \le k \le 1$, $j = 1, 2, \ldots, t$. Then

$$\frac{zr'(z)}{r(z)} = \frac{zp'(z)}{p(z)} - \frac{zw'(z)}{w(z)} = \left[\sum_{j=1}^{t}\frac{z}{z-b_j}\right] - \frac{zw'(z)}{w(z)}.$$

For $z \in T_1$, this relation with the help of Lemma 2.5 gives

$$\mathrm{Re}\left(\frac{zr'(z)}{r(z)}\right) = \left(\sum_{j=1}^{t}\mathrm{Re}\left(\frac{z}{z-b_j}\right)\right) - \left(\frac{n-|B'(z)|}{2}\right). \tag{2.2}$$

For $z \in T_1$ with $z \ne b_j (1 \le j \le t)$, we consider two cases.

Case 1: $|b_j| = 1$. Then $k = 1$ and $\left|\dfrac{z}{z-b_j}\right| = \left|\dfrac{b_j}{z-b_j}\right| = \left|\dfrac{z}{z-b_j} - 1\right|$.

By Remark 2.3, we obtain that $\mathrm{Re}\left(\dfrac{z}{z-b_j}\right) \ge \dfrac{1}{2} = \dfrac{1}{1+1} = \dfrac{1}{1+k}$.

Case 2: $|b_j| < 1$. A bilinear transformation $w_j(z) = \dfrac{z}{z-b_j}$ maps T_1 onto a circle $\left\{\left|w - \left(\dfrac{1}{1-|b_j|^2}\right)\right| = \dfrac{|b_j|}{1-|b_j|^2}\right\}$. Then

$$\mathrm{Re}\left(\frac{z}{z-b_j}\right) \ge \left(\frac{1}{1-|b_j|^2}\right) - \frac{|b_j|}{1-|b_j|^2} = \frac{1}{1+|b_j|} \ge \frac{1}{1+k}.$$

From both cases, $\mathrm{Re}\left(\dfrac{z}{z-b_j}\right) \ge \dfrac{1}{1+k}$ for $|b_j| \le k \le 1$, $z \in T_1$ with $z \ne b_j$ $(1 \le j \le t)$. Substituting this relation into (2.2), we obtain that

$$\mathrm{Re}\left(\frac{zr'(z)}{r(z)}\right) \ge \sum_{j=1}^{t}\left(\frac{1}{1+k}\right) - \left(\frac{n-|B'(z)|}{2}\right) = \frac{|B'(z)|}{2} - \frac{n(1+k)-2t}{2(1+k)}.$$

■

3. Proofs of the Main Theorems

In this section, we present the proofs of our main results.

Proof of Theorem 1.9. Assume that $r \in \mathcal{R}_n$ has no zeros in $|z| < k$, where $k \ge 1$. Let $m = \min\limits_{|z|=k} |r(z)|$. If $r(z)$ has a zero on $|z| = k$, then $m = 0$ and hence for every α with $|\alpha| < 1$, we get $r(z) - \alpha m = r(z)$. In case $r(z)$ has no zeros on $|z| = k$, we have for every α with $|\alpha| < 1$ that $|-\alpha m| = |\alpha| \cdot m < |r(z)|$ for $|z| = k$. It follows from Rouche's theorem that $R(z) = r(z) - \alpha m$ and $r(z)$ have the same number of zeros in $\{|z| < k\}$. That is, for every α with $|\alpha| < 1$, $R(z)$ has no zeros in $|z| < k$. We assume that $R(z) \ne 0$. Observe that the number of zeros of $R(z)$ with counting multiplicity is n by Lemma 2.6 (i) yields that for $z \in T_1$,

$$\mathrm{Re}\left(\frac{zR'(z)}{R(z)}\right) \leq \frac{|B'(z)|}{2} + \frac{2t - n(1+k)}{2(1+k)} = \frac{|B'(z)|}{2} + \frac{n(1-k)}{2(1+k)}. \tag{3.1}$$

Note that $R^*(z) = B(z)\overline{R(1/\overline{z})} = B(z)\overline{R}(1/z)$. Then

$$\begin{aligned}(R^*(z))' &= \overline{R}(1/z)B'(z) + B(z)\left(\overline{R}(1/z)\right)' \\ &= B'(z)\overline{R}(1/z) + B(z)\left(\overline{R}'(1/z)\right)\left(-\frac{1}{z^2}\right) \\ &= B'(z)\overline{R}(1/z) - \frac{B(z)}{z^2}\cdot \overline{R}'(1/z).\end{aligned}$$

Then $z(R^*(z))' = zB'(z)\overline{R}(1/z) - \dfrac{B(z)}{z}\cdot\overline{R}'(1/z)$. Since $z \in T_1$, we have $\overline{z} = \dfrac{1}{z}$, $|B(z)| = 1$, $\dfrac{zB'(z)}{B(z)} = |B'(z)|$, and so

$$\begin{aligned}|z(R^*(z))'| &= \left|zB'(z)\overline{R(z)} - B(z)\overline{zR'(z)}\right| \\ &= \left|\frac{zB'(z)}{B(z)}\cdot\overline{R(z)} - \overline{zR'(z)}\right| \\ &= \left||B'(z)|\overline{R(z)} - \overline{zR'(z)}\right|.\end{aligned}$$

Since $|B'(z)|$ is real, we get $|z(R^*(z))'| = ||B'(z)|R(z) - zR'(z)|$. Then

$$\begin{aligned}\left|\frac{z(R^*(z))'}{R(z)}\right|^2 &= \left||B'(z)| - \frac{zR'(z)}{R(z)}\right|^2 \\ &= |B'(z)|^2 - 2|B'(z)|\cdot \mathrm{Re}\left(\frac{zR'(z)}{R(z)}\right) + \left|\frac{zR'(z)}{R(z)}\right|^2 \\ &\geq |B'(z)|^2 - 2|B'(z)|\left[\frac{|B'(z)|}{2} + \frac{n(1-k)}{2(1+k)}\right] + \left|\frac{zR'(z)}{R(z)}\right|^2 \\ &= \left|\frac{zR'(z)}{R(z)}\right|^2 + \frac{n(k-1)}{(1+k)}\cdot|B'(z)|,\end{aligned}$$

where the inequality comes from (3.1).
This implies that for $z \in T_1$,

$$\left[|R'(z)|^2 + \frac{n(k-1)}{(1+k)}\cdot|R(z)|^2|B'(z)|\right]^{\frac{1}{2}} \leq |(R^*(z))'|, \tag{3.2}$$

where $R^*(z) = B(z)\overline{R(1/\overline{z})} = r^*(z) - \overline{\alpha}mB(z)$.
Moreover, $(R^*(z))' = (r^*(z))' - \overline{\alpha}mB'(z)$ and $R'(z) = (r(z) - \alpha m)' = r'(z)$.
Apply these relations into (3.2), we obtain that

$$\left[|r'(z)|^2 + \frac{n(k-1)}{(1+k)}\cdot|r(z) - \alpha m|^2|B'(z)|\right]^{\frac{1}{2}} \leq |(r^*(z))' - \overline{\alpha}mB'(z)|, \tag{3.3}$$

for $z \in T_1$ and for α with $|\alpha| < 1$.
Choose the argument of α such that

$$|(r^*(z))' - \overline{\alpha} m B'(z)| = |(r^*(z))'| - m\,|\alpha|\,|B'(z)|, \tag{3.4}$$

for $z \in T_1$.
Triangle inequality yields that $|r(z) - m\alpha| \geq ||r(z)| - m|\alpha||$.
Note that $||r(z)| - m|\alpha||^2 = (|r(z)| - m|\alpha|)^2$ which implies that

$$|r(z) - m\alpha|^2 \geq (|r(z)| - m|\alpha|)^2. \tag{3.5}$$

Substituting relations (3.4) and (3.5) into (3.3), we obtain that

$$\left[|r'(z)|^2 + \frac{n(k-1)}{(1+k)} \cdot (|r(z)| - m|\alpha|)^2 |B'(z)|\right]^{\frac{1}{2}} \leq |(r^*(z))'| - m\,|\alpha|\,|B'(z)|.$$

Letting $|\alpha| \to 1$, we get

$$\left[|r'(z)|^2 + \frac{n(k-1)}{(1+k)} \cdot (|r(z)| - m)^2 |B'(z)|\right]^{\frac{1}{2}} \leq |(r^*(z))'| - m|B'(z)|.$$

Lemma 2.1 implies that

$$\left[|r'(z)|^2 + \frac{n(k-1)}{(1+k)} \cdot (|r(z)| - m)^2 |B'(z)|\right]^{\frac{1}{2}} \leq |B'(z)| \cdot ||r|| - |r'(z)| - m|B'(z)|.$$

Equivalently,

$$|r'(z)|^2 + \frac{n(k-1)}{(1+k)} \cdot (|r(z)| - m)^2 |B'(z)| \leq \left[(||r|| - m)|B'(z)| - |r'(z)|\right]^2.$$

Hence,

$$\begin{aligned}|r'(z)|^2 &+ \frac{n(k-1)}{(1+k)} \cdot (|r(z)| - m)^2 |B'(z)| \\ &\leq (||r|| - m)^2 |B'(z)|^2 - 2(||r|| - m)|B'(z)||r'(z)| + |r'(z)|^2.\end{aligned}$$

Then

$$2(||r|| - m)|r'(z)| \leq (||r|| - m)^2 |B'(z)| - \frac{n(k-1)}{(1+k)}(|r(z)| - m)^2.$$

Thus,

$$\begin{aligned}|r'(z)| &\leq \frac{(||r|| - m)^2 |B'(z)|}{2(||r|| - m)} - \frac{n(k-1)(|r(z)| - m)^2}{2(1+k)(||r|| - m)} \\ &= \frac{1}{2}\left[|B'(z)| - \frac{n(k-1)(|r(z)| - m)^2}{(1+k)(||r|| - m)^2}\right](||r|| - m),\end{aligned}$$

where $m = \min\limits_{|z|=k} |r(z)|$.
This proves inequality for $R(z) \neq 0$. In case $R(z) = 0$, we obtain that $r'(z) = 0$. This implies that the above inequality is trivially true.

Therefore, inequality holds for all $z \in T_1$.

To show that equality (1.7) holds for $r(z) = \dfrac{(z+k)^n}{(z-a)^n}$ and $B(z) = \left(\dfrac{1-az}{z-a}\right)^n$, where $a > 1$, $k \geq 1$ at $z = 1$, we observe that

$$||r|| = \sup_{z\in T_1} |r(z)| = \frac{(1+k)^n}{(1-a)^n} = |r(1)|,\ m = \min_{|z|=k} |r(z)| = 0, \text{ and } |B'(1)| = \frac{n(a+1)}{a-1}.$$

Since $r'(z) = \left[\dfrac{n}{z+k} + \dfrac{n}{a-z}\right]\left[\dfrac{(z+k)^n}{(z-a)^n}\right]$, we obtain that

$$|r'(1)| = \left[\frac{n}{1+k} + \frac{n}{a-1}\right]\left[\frac{(1+k)^t}{(1-a)^n}\right] = \left[\frac{n}{1+k} + \frac{n}{a-1}\right]||r||.$$

The right side of the relation (1.7) is

$$\begin{aligned}\frac{1}{2}\left[|B'(1)| - \frac{(n(k-1))(|r(1)|-m)^2}{(1+k)(||r||-m)^2}\right](||r||-m) &= \frac{1}{2}\left[\frac{n(a+1)}{a-1} - n + \frac{2t}{1+k}\right]||r|| \\ &= \frac{1}{2}\left[\frac{2n}{a-1} + \frac{2n}{1+k}\right]||r|| \\ &= |r'(1)|.\end{aligned}$$

This proves Theorem 1.9 completely. ∎

Remark 3.1. We show that our upper bound in Theorem 1.9 improves an upper bound in Theorem 1.4 as follows.
Since $k \geq 1$, we get that

$$\begin{aligned}\frac{(n(k-1))(|r(z)|-m)^2}{(1+k)(||r||-m)^2} &\geq \frac{(n(0))(|r(z)|-m)^2}{(1+k)(||r||-m)^2} \\ &= 0.\end{aligned}$$

Hence, $|B'(z)| - \dfrac{(n(k-1))(|r(z)|-m)^2}{(1+k)(||r||-m)^2} \leq |B'(z)|$.
In particular, if $k = 1$, then

$$|B'(z)| - \frac{(n(k-1))(|r(z)|-m)^2}{(1+k)(||r||-m)^2} = |B'(z)|.$$

Therefore, our upper bound in Theorem 1.9 is better than an upper bound in Theorem 1.4.

Next, we give the proof of the second main Theorem.

Proof of Theorem 1.10. Assume that $r \in \mathcal{R}_n$ has no zeros in $|z| > k$, where $k \leq 1$. Let $m = \min_{|z|=k} |r(z)|$ and t be the number of zeros of r with counting multiplicity. If $r(z)$ has a zero on $|z| = k$, then $m = 0$ and hence for every α with $|\alpha| < 1$, we get $r(z) + \alpha m = r(z)$. In case $r(z)$ has no zeros on $|z| = k$, we have for every α with $|\alpha| < 1$ that $|\alpha m| < |r(z)|$ for $|z| = k$. It follows from Rouche's theorem that $R(z) = r(z) + \alpha m$ and $r(z)$ have the same number of zeros in $|z| < k$. That is, for every α with $|\alpha| < 1$, $R(z)$ has no zeros in $|z| > k$. We assume that $R(z) \neq 0$ on $|z| = 1$. Lemma 2.6 (ii) implies that for $z \in T_1$,

$$\text{Re}\left(\frac{zR'(z)}{R(z)}\right) \geq \frac{|B'(z)|}{2} - \frac{n(1+k)-2t}{2(1+k)},$$

where $R(z) \neq 0$.
Then

$$\left|\frac{R'(z)}{R(z)}\right| = \left|\frac{zR'(z)}{R(z)}\right| \geq \text{Re}\left(\frac{zR'(z)}{R(z)}\right) \geq \frac{|B'(z)|}{2} - \frac{n(1+k)-2t}{2(1+k)}.$$

This implies that

$$|R'(z)| \geq \left[\frac{|B'(z)|}{2} - \frac{n(1+k)-2t}{2(1+k)}\right] \cdot |R(z)|, \qquad \text{for } z \in T_1.$$

Since $|R'(z)| = |r'(z)|$, we obtain that

$$|r'(z)| \geq \frac{1}{2}\left[|B'(z)| - \frac{n(1+k)-2t}{(1+k)}\right] \cdot |r(z) + \alpha m|, \qquad \text{for } z \in T_1.$$

Note that this inequality is trivially true for $R(z) = 0$ on $|z| = 1$. Therefore, this inequality holds for all $z \in T_1$. Choosing the argument of α suitably in the right side of the above inequality and noting that the left side is independent of α, we get that

$$|r'(z)| \geq \frac{1}{2}\left[|B'(z)| - \frac{n(1+k)-2t}{(1+k)}\right] \cdot (|r(z)| + |\alpha| m), \qquad \text{for } z \in T_1.$$

Letting $|\alpha| \to 1$, we get for $z \in T_1$ that

$$\begin{aligned} |r'(z)| &\geq \frac{1}{2}\left[|B'(z)| - \frac{n(1+k)-2t}{(1+k)}\right] \cdot (|r(z)| + m) \\ &= \frac{1}{2}\left[|B'(z)| + \frac{2t-n(1+k)}{(1+k)}\right] \cdot (|r(z)| + m). \end{aligned}$$

Similarly to the argument of the proof of Theorem 1.9, we can show that (1.8) becomes equality when $r(z) = \frac{(z+k)^t}{(z-a)^n}$ and $B(z) = \left(\frac{1-az}{z-a}\right)^n$, where $a > 1$, $k \leq 1$ at $z = 1$. ■

4. Conclusion

This paper investigates the estimate of the modulus of the derivative of $r(z)$ on the unit circle. We establish an upper bound when all zeros of $r(z)$ lie in $|z| \geq k \geq 1$ and a lower bound when all zeros of $r(z)$ lie in $|z| \leq k \leq 1$. In particular, if $r(z)$ has exactly n zeros and $k = 1$, our main theorems generalize results by Aziz and Shah [5] and Aziz and Zarger [6]. Furthermore, if $r(z)$ has a zero on T_k, the second main result generalizes a result by Aziz and Shah [7].

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