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Numerical Method Based on Wavelets, for the Solution of Multi Order Fractional Differential Equations

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Abstract In the present article, we are applying Chebyshev Wavelet Method (CWM) to find an approximate solution for multi order fractional differential equations. The fractional derivatives are defined in the Caputo sense. Numerical examples are presented to show the accuracy and reliability of the proposed method. Moreover, the results illustrate a strong agreement between the approximate and the exact solutions.

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1. Introduction

The history of fractional calculus is traced back to 1695, when Leibnitz mention fractional differential operator in a letter to L'Hospital. Later on the formal concepts and related theory regarding to fractional calculus have been presented by other mathematicians [1–3]. The subject of fractional calculus has taken greater importance due its bundle of applications in different branches of science and engineering. Some of these applications can be found in [4, 5]. Other applications are nonlinear oscillation of earthquakes [6], fluid dynamics traffic model [7], frequency dependent damping behavior of many viscoelastic materials [8, 9], Continuum and statistical mechanics [10], colored noise [11], Solid mechanics [12], Economics [13], Bio-Engineering [14–16], Anomalous transport [17] and Dynamics of interfaces between nano-particles and substrates [18].
The challenging work is to find the numerical solution of fractional order problems such as fractional differential equations, fractional partial differential equations, fractional integro-differential equations and dynamic system containing fractional derivatives. In this connection several methods have been used to solve these problems. The most important methods are Adomian decomposition method [19, 20], He's variation iteration method

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[21–23], homotopy perturbation method [24, 25], homotopy analysis method [26], collocation method [27], Galerkin method [28], Reproducing Kernel Hilbert Space method [29, 30], etc [31–34].
The numerical methods based wavelets have received considerable attention in dealing with various problems [35–38]. Among these methods, based on Chebyshev wavelets have gained much importance during the last decade. The simpleness, effectiveness, and straightforward implementation of this method is the real point of concentration towards the researcher. Therefore the researchers have paid greater attention to these methods to solve problems in different fields of science and engineering. For example Chebyshev Wavelet Operational Matrix (CWOM) [39], Chebyshev Finite Difference Method (CFDM) [40], Shifted Chebyshev polynomial Method (SCPM) [41] and Chebyshev Wavelet Method (CWM) [42].
In this article, an efficient Chebyshev Wavelet Method (CWM) is applied to obtain the numerical solutions of some fractional multi-order differential equations. The simulations performed by the proposed method are easily to compute. The absolute error is calculated, showing an efficient degree of accuracy.
This article is organized as follows: In section 2, we give some basic definitions about fractional calculus. In section 3, we give basic properties of Chebyshev Wavelets. A description of the Chebyshev Wavelet method (CWM) for solving fractional multi-order differential equations is given in section 4. Some numerical examples are carried out in section 5. This paper ends with a conclusion in section 6.

2. Preliminaries and Definitions

To continue with the current work we present some definitions and other mathematical preliminaries. These concepts play very massive role to complete the present work.

Definition 2.1. The Riemann fractional integral operator I^μ of order μ on the usual Lebesgue space $L_1[a,b]$ is given by

$$(I^\mu g)(t) = \frac{1}{\Gamma(\mu)} \int_0^t (t-\xi)^{\mu-1} g(\xi)\, d\xi, \quad \mu > 0,$$
$$(I^0 g)(t) = g(t),$$

This integral operator has the following properties

(a): $I^\mu I^\eta = I^{\mu+\eta}$,
(b): $I^\mu I^\eta = I^\eta I^\mu$,
(c): $I^\mu (t-a)^\nu = \dfrac{\Gamma(\nu+1)}{\Gamma(\mu+\nu+1)}(t-a)^{\mu+\nu}$

Where $\mu,\ \eta > 0,\ \nu > -1$.

Definition 2.2. The Riemann fractional derivative of order $\mu > 0$ is defined as

$$(D^\mu g)(t) = (\frac{d}{dt})^n (I^{n-\mu} g)(t),\ n-1 < \mu \leq n$$

where n is an integer.

However the Riemann fractional derivative has certain drawbacks due to which Caputo proposed a modified differential operator.

Definition 2.3. The Caputo definition of fractional differential operator is given by

$$(D^\mu g)(t) = \frac{1}{\Gamma(n-\mu)} \int_0^t (t-\xi)^{n-\mu-1} g^{(n)}(\xi)\ d\xi, \quad n-1 < \mu < n$$

Where $t > 0, n$ is an integer.
It has the following two basic properties

(a): $(D^\mu I^\mu g)(t) = g(t)$,

(b): $(I^\mu D^\mu g)(t) = \sum_{k=0}^{n} g^{(k)}(0^+)\frac{(t-a)^k}{k!},\ t > 0$

3. Properties of the Chebyshev Wavelets

Wavelets consist of family of functions generated from the dilation m and translation l of a single function $\psi(x)$ called the mother wavelet. When the dilation a and translation b change continuously then we get the following continuous family of Wavelet [43].

$$\psi_{m,l}(x) = |m|^{\frac{1}{2}}\psi(\frac{x-l}{m}),\ l, m \in R,\ m \neq 0,$$

If we restrict the parameters l and m to discrete values as

$$m = a_0^{-k},\ l = nb_0a_0^{-k},\ a_0 > 1,\ b_0 > 0$$

where $n,\ k$ are positive integers, then we have the following family of discrete wavelets

$$\psi_{k,n}(x) = |a|^{\frac{k}{2}}\psi(a_0^k x - nb_0),\ k, n \in Z$$

Where $\psi_{k,n}$ form a wavelet basis for $L^2(R)$.
Especially when $a_0 = 2$ and $b_0 = 1$, then $\psi_{k,n}(x)$ form an orthogonal basis.
The second kind of Chebyshev wavelets is constituted of four parameters, $\psi_{n,m}(x) = \psi(k, n, m, x)$, where $n = 1, 2, ..., 2^{k-1},\ k$ is any nonnegative integer, m is the degree of the second Chebyshev polynomial. The Chebyshev wavelets are defined on the interval $0 \leq x < 1$ as

$$\psi_{n,m} = \begin{cases} 2^{\frac{k}{2}}\tilde{T}_m(2^k x - 2n + 1) & , \ \frac{n-1}{2^{k-1}} \leq x \leq \frac{n}{2^{k-1}} \\ 0 & , \ otherwise \end{cases}$$

Where $\tilde{T}_m(x) = \sqrt{\frac{2}{\pi}}T_m(x),\ m = 0, 1, 2..., M-1$
Here $T_m(x)$ are second Chebyshev polynomials of degree m with respect to the weight function $w(x) = \sqrt{1-x^2}$ on the interval $[-1, 1]$, and satisfying the following recursive formula
$T_0(x) = 1,\ T_1(x) = 2x,$
$T_{m+1}(x) = 2xT_m(x) - T_{m-1}(x), m = 1, 2, 3, ...$

Lemma 3.1. *If the Chebyshev Wavelet expansion of a continuous function $f(x)$ converges uniformly, then the Chebyshev Wavelet expansion converges to the function $f(x)$.*

Proof. See [43] ■

Theorem 3.2. *A function $f(x) \in L_2[0, 1]$, with bounded second derivative, say $|f''(x)| \leq N$, can be expanded as an infinite sum of Chebyshev wavelets, and the series converges uniformly to $f(x)$, that is,*

$$f(x) = \sum_{n=1}^{\infty}\sum_{m=0}^{\infty} c_{nm}\psi_{nm}(x).$$

Proof. See [43] ■

4. Chebyshev Wavelet Method (CWM)

In this section, we consider the following fractional multi-order differential equation

$$y^{\alpha}(x) = F(x, y(x), D^{\beta_1}y(x), ..., D^{\beta_r}y(x)) = 0, \tag{4.1}$$

With the initial conditions given by

$$y^{(i)}(a) = d_i, \;\; i = 0, 1, ..., n$$

where $n < \alpha \leq n$, $0 < \beta_1 < \beta_2 < ... < \beta_r < \alpha$ and D^{α} denotes the Caputo fractional derivative of order α. The function F can be linear or nonlinear in general.
The solution to equation (4.1) can be extended by Chebyshev wavelets series as

$$y(x) = \sum_{n=1}^{\infty}\sum_{m=0}^{\infty} c_{n,m}\psi_{n,m}(x), \tag{4.2}$$

The series in equation (4.2) is truncated to finite number of terms that is

$$y_{k,M}(x) = \sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M-1} c_{n,m}\psi_{n,m}(x), \tag{4.3}$$

This shows that there are $2^{k-1}M$ conditions, $2^{k-1}M$ coefficients, $c_{i,j}$ to determine.
In the present paper, we first consider multi order linear fractional differential equatios of maximum order two. There are two initial conditions for this multi-order fractional differential equation. The initial conditions are approximated by Chebyshev wavelet method as given in the following equations.
The first initial condition is approximated as

$$\sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M-1} c_{n,m}\psi_{n,m}(0) = \alpha_0 \tag{4.4}$$

And the second is,

$$\sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M-1} c_{n,m}\psi'_{n,m}(0) = \beta_0 \tag{4.5}$$

The remaining $2^{k-1}M - 2$ conditions can be obtained by substituting equation (4.3) in equation (4.1), we get

$$\frac{d^{\alpha}}{dx^{\alpha}}\Big(\sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M-3} c_{n,m}\psi_{n,m}(x_i)\Big) = F\Bigg(x, \sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M-3} c_{n,m}\psi_{n,m}(x)\Big), \frac{d^{\beta_1}}{dx^{\beta_1}}\Big(\sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M-3} c_{n,m}\psi_{n,m}(x)\Big), \frac{d^{\beta_2}}{dx^{\beta_2}}\Big(\sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M-3} c_{n,m}\psi_{n,m}(x)\Big), ..., \frac{d^{\beta_r}}{dx^{\beta_r}}\Big(\sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M-3} c_{n,m}\psi_{n,m}(x)\Big)\Bigg) \tag{4.6}$$

Assume that equation (4.6) is exact at x_i points, then x_i points are calculated by the following formula $x_i = \dfrac{i - 0.5}{2^{k-1}M}$, $i = 1, 2, ..., \; 2^{k-1}M - 2$ The combination of equations (4.4), (4.5) and (4.6) form the linear system of $2^{k-1}M$ linear equations. The unknown

$c_{i,j}$ are calculated through the solution of this system of equations.
The same procedure can be applied for other multi order fractional integral equations.

5. Numerical Examples

Example 5.1. Consider the following fractional order linear initial value problem

$$\frac{d^2y}{dx^2}+\frac{d^{1.5}y}{dx^{1.5}}+y-x-1=0,\ 0\le x\le 1$$

with initial conditions

$$y(0)=1,\ y'(0)=1$$

The exact solution is $y(x)=x+1$.

Table 1. The numerical results of example 5.1

x_i	y (exact)	y (CWM)	Error (CWM)
0.0	1.0	1.00000	0.000000
0.1	1.1	1.0999999999560528885	4.39 E -11
0.2	1.2	1.1999999999124547729	8.75 E -11
0.3	1.3	1.2999999998695032436	1.3 E -10
0.4	1.4	1.3999999998274616388	1.72 E -10
0.5	1.5	1.4999999997865635722	2.13 E -10
0.6	1.6	1.5999999997470016908	2.52 E -10
0.7	1.7	1.6999999997088796221	2.91 E -10
0.8	1.8	1.7999999996720762002	3.27 E -10
0.9	1.9	1.8999999996359244178	3.64 E -10
1.0	2.0	1.9999999995985377773	4.01 E -10

In Table 1, the exact solution and approximate solution by Chebyshev wavelet method of Example 5.1 are represented by y(exact) and y(CWM) respectively. The Chebyshev wavelet method is applied for $M=19,\ K=1$. The approximate solutions are compared with the exact solution of problem. The errors associated with Chebyshev wavelet method is denoted by Error (CWM). The table shows that Chebyshev wavelet method has desire accuracy.
In Figure 1, the exact and (CWM) solutions of Example 5.1 are represented by y(exact) and y(CWM) respectively. It is cleared from the figure 1, that (CWM) has a close agreement with exact solution of the problem.
In Figure 2, the error associated with the proposed method of Example 5.1 is denoted by Error (CWM). The figure also indicates that the error is bounded by $0\le$ Error(CWM) $\le 4\times 10^{-10}$. Also there is a slight increase in the error as x move from 0 to 1.

Example 5.2. Consider the following nonlinear fractional initial value problem

$$\frac{d^3y}{dx^3}+\frac{d^{2.5}y}{dx^{2.5}}+y^2-x^4=0,\ 0\le x\le 1$$

with initial conditions

$$y(0)=1,\ y'(0)=0,\ y''(0)=2$$

Figure 1. The graph of exact solution verses Chebyshev wavelet approximation of Example 5.1

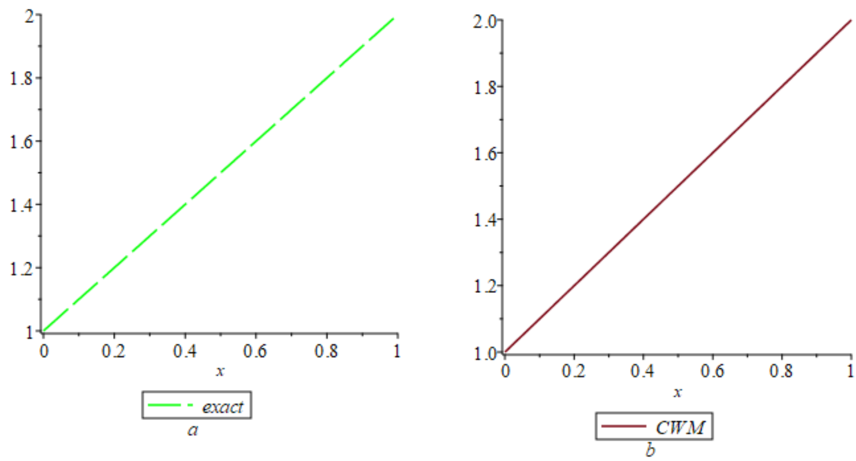
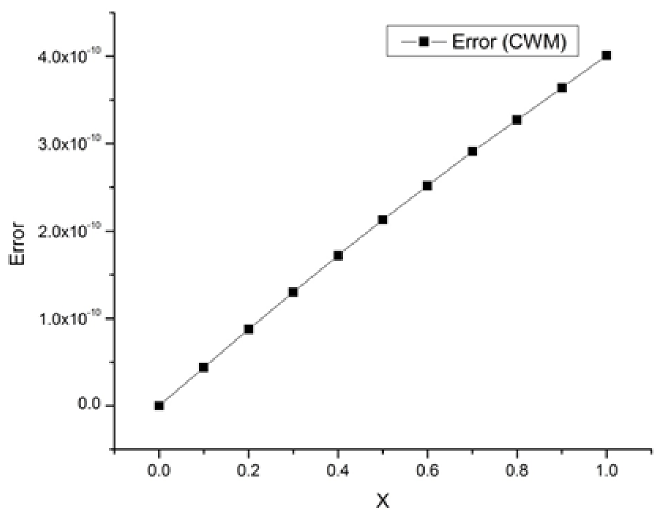


Figure 2. The graph of errors obtained by (CWM) of Example 5.1



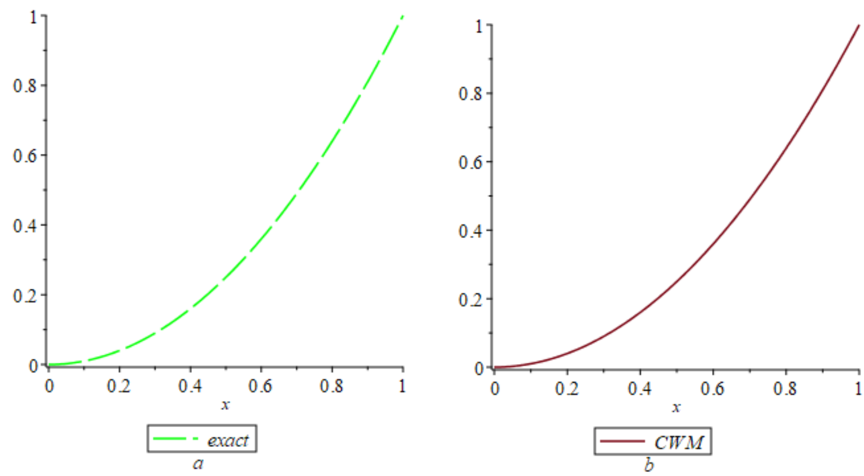
The exact solution is $y(x) = x^2$.

Table 2 analyzed the exact solution y(exact), the approximate solution by Chebyshev wavelet method y(CWM), Error by Chebyshev wavelet method Error (CWM). The table shows that the present method has an excellent accuracy. The numerical simulations are

TABLE 2. The numerical results of example 5.2

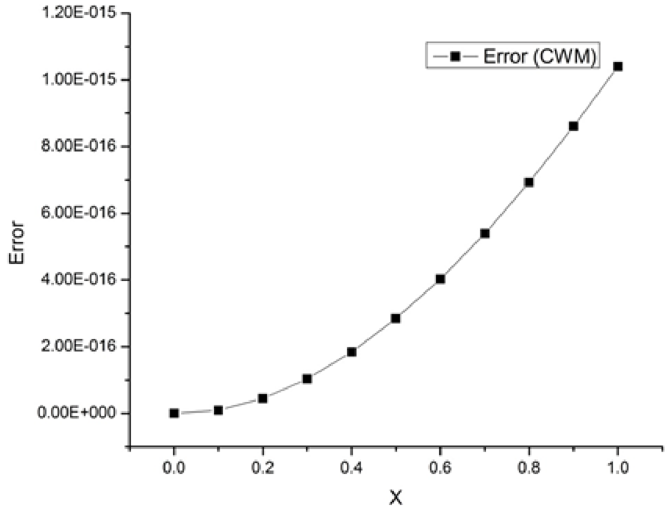
x_i	y (exact)	y (CWM)	Error (CWM)
0.0	0.0	-1.00 E -20	1.00 E -20
0.1	0.01	0.00999999999999999029	9.70 E -18
0.2	0.04	0.03999999999999995498	4.50 E -17
0.3	0.09	0.08999999999999989614	1.03 E -16
0.4	0.16	0.15999999999999981584	1.84 E -16
0.5	0.25	0.24999999999999971562	2.84 E -16
0.6	0.36	0.35999999999999959665	4.03 E -16
0.7	0.49	0.48999999999999946007	5.39 E -16
0.8	0.64	0.63999999999999930695	6.93 E -16
0.9	0.81	0.80999999999999913846	8.61 E-16
1.0	1.00	0.99999999999999895598	1.04 E-15

FIGURE 3. The graph of exact solution verses Chebyshev wavelet approximation of Example 5.2



done by using $k = 1$ and $M = 19$ in the current method.
In Figure 3, the exact and (CWM) solutions of Example 5.2 are shown by y(exact) and y(CWM) respectively. The figure reflects that (CWM) approximations are in close contact with the exact solution of the problem.
In Figure 4, the error associated with the (CWM) of Example 5.2 is represented by Error (CWM). The figure also indicates that the error is bounded by $0 \leq$ Error(CWM) $\leq 1.2 \times 10^{-15}$. Also the error gradually increases when x move from 0 to 1.

FIGURE 4. The graph of errors obtained by (CWM) of Example 5.2



Example 5.3. Consider the following nonlinear fractional initial value problem

$$\frac{d^4y}{dx^4} + \frac{d^{3.5}y}{dx^{3.5}} + y^3 - x^9 = 0,\ 0 \le x \le 1$$

with initial conditions

$$y(0) = 0,\ y'(0) = 0,\ y''(0) = 0,\ y'''(0) = 6$$

The exact solution is $y(x) = x^3$.

TABLE 3. The numerical results of example 5.3

x_i	y (exact)	y (CWM)	Error (CWM)
0.0	0.000	-5.66 E -21	5.66 E -21
0.1	0.001	0.00099999999999999999	9.04 E -21
0.2	0.008	0.00799999999999999988	1.14 E -19
0.3	0.027	0.02699999999999999954	4.51 E -19
0.4	0.064	0.06399999999999999885	1.14 E -18
0.5	0.125	0.12499999999999999765	2.35 E -18
0.6	0.216	0.21599999999999999571	4.29 E -18
0.7	0.343	0.34299999999999999218	7.82 E -18
0.8	0.512	0.51199999999999998471	1.52 E -17
0.9	0.729	0.72899999999999996696	3.30 E -17
1.0	1.000	0.99999999999999992328	7.67 E -17

Table 3 displayed the numerical results of Example 5.3 using Chebyshev wavelet method. The exact solution is represented by y(exact) and approximate solution obtained

FIGURE 5. The graph of exact solution verses Chebyshev wavelet approximation of Example 5.3

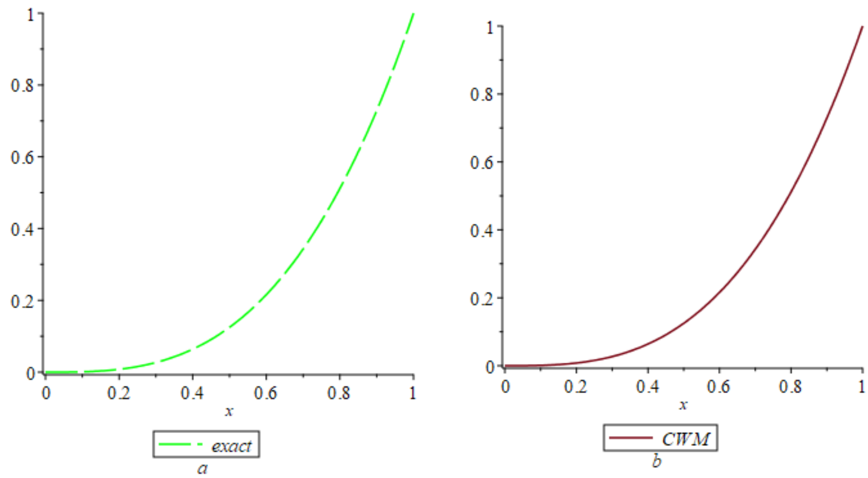
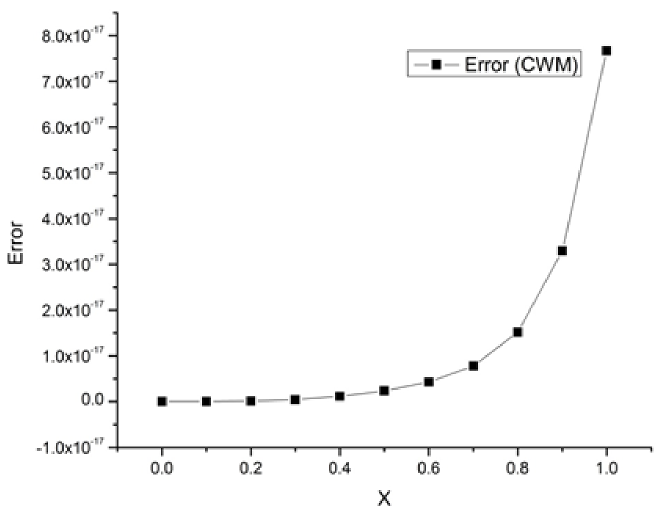


FIGURE 6. The graph of errors obtained by (CWM) of Example 5.3



by Chebyshev wavelet method is denoted by y(CWM). The error associated with CWM method is Error (CWM). The table shows the numerical results obtained by Chebyshev method having best accuracy.

Similarly, In Figure 5, both exact and (CWM) solutions of Example 5.3 are represented by y(exact) and y(CWM) respectively. The Figure shows that (CWM) is convergent to exact solution.
In Figure 6, the error associated with the (CWM) of Example 5.3 is denoted by Error (CWM). The figure shows that the error is bounded by $0 \leq$ Error(CWM) $\leq 8 \times 10^{-17}$. Also, the error has a very small increase as x move from 0 to 1.

Example 5.4. Consider the following nonlinear multi order fractional differential equation

$$\frac{d^{\alpha}y}{dx^{\alpha}} + \frac{d^{\beta}y}{dx^{\beta}} + y - \frac{6x^{3-\alpha}}{\Gamma(4-\alpha)} - \frac{6x^{4-\beta}}{\Gamma(4-\beta)} - x^3 - x = 0,\ 0 \leq x \leq 1$$

where $\alpha = 2.5,\ \beta = 1.9.$ with initial conditions

$$y(0) = 0,\ y'(0) = 1,\ y''(0) = 0$$

The exact solution is $y(x) = x^3 + x$.

Table 4. The numerical results of example 5.4

x_i	y (exact)	y (CWM)	Error (CWM)
0.0	0.000	-7.26 E -21	7.26 E -21
0.1	0.101	0.1010000000000000698	6.98 E -17
0.2	0.208	0.2080000000000002805	2.80 E -16
0.3	0.327	0.3270000000000005609	5.60 E -16
0.4	0.464	0.4640000000000006957	6.95 E -16
0.5	0.625	0.6249999999999992012	7.98 E -16
0.6	0.816	0.8159999999999886625	1.13 E -14
0.7	1.043	1.0429999999999411290	5.88 E -14
0.8	1.312	1.3119999999997706514	2.29 E -13
0.9	1.629	1.6289999999992500365	7.49 E -13
1.0	2.000	1.9999999999978446407	2.15 E -12

Table 4, shows the numerical results obtained by Chebyshev Wavelet Method (CWM). The numerical results obtained by (CWM) and exact solutions are respectively denoted by y(exact) and y(CWM). The algorithm is applied for $k = 1$ and $M = 19$.

FIGURE 7. The graph of exact solution verses Chebyshev wavelet approximation of Example 5.4

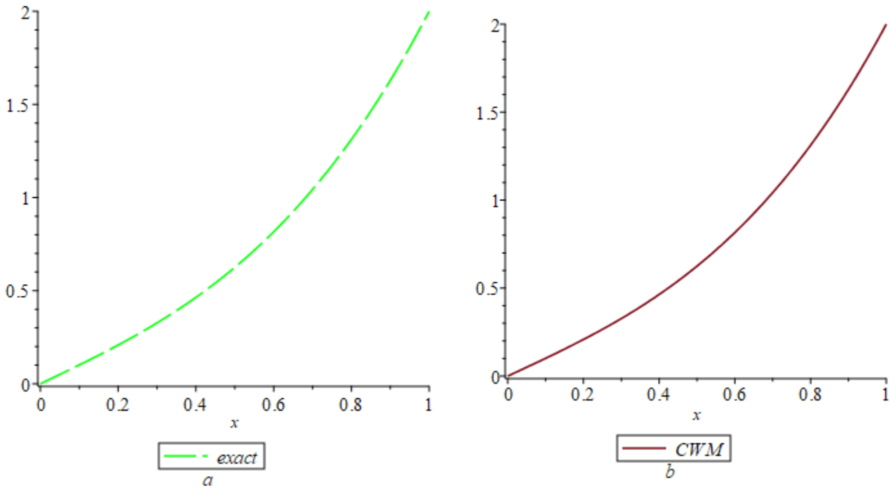
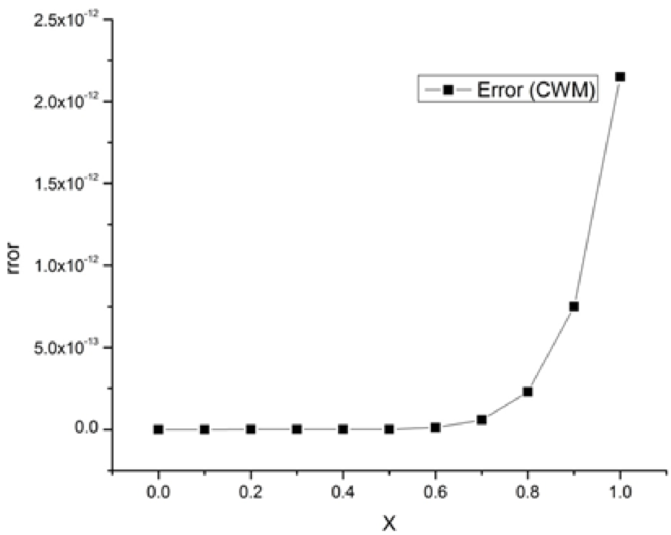


FIGURE 8. The graph of errors obtained by (CWM) of Example 5.4



In Figure 7, both exact and (CWM) solutions of Example 5.3 are represented by y(exact) and y(CWM) respectively. The Figure shows that (CWM) is convergent to exact solution.

In Figure 8, the error associated with the (CWM) of Example 5.4 is denoted by Error (CWM). The figure shows that the error is bounded by $0 \leq \text{Error(CWM)} \leq 8 \times 10^{-17}$. Also, the error has a very small increase as x move from 0 to 1.

6. Conclusion

The present work introduced an efficient method, Chebyshev Wavelet Method (CWM), to solve fractional multi order differential equations numerically. The proposed method has been implemented to solve several examples. It may be concluded that Chebyshev Wavelet Method (CWM) has higher degree of accuracy. For future work, we will continue with this technique for the numerical solution of other different high nonlinear multi order problems.

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