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Common Fixed Point Theorems for four mappings in d^* Metric spaces

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Abstract : In this paper, we introduce the concept of d^* metric space and prove some common fixed point theorems for four maps in d^* metric spaces. These theorems are version of some known results in ordinary metric spaces.

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1 Introduction and Preliminaries

In the present work, we introduce the concept of d^*-metric which is a probable modification of the definition of ordinary metric. In this section we give some properties about d^*-metric. In section 2, we prove two common fixed point theorem for four weakly compatible maps in d^*-metric spaces.

In what follows, $\mathbf{N}$ the set of all natural numbers and $\mathbf{R}^+$ the set of all positive real numbers.

Let binary operation $\diamond : \mathbf{R}^+ \times \mathbf{R}^+ \longrightarrow \mathbf{R}^+$ satisfies the following conditions:

(i) $\diamond$ is associative and commutative,
(ii) $\diamond$ is continuous.

Five typical examples are $a \diamond b = \max\{a, b\}$, $a \diamond b = a + b$, $a \diamond b = ab$, $a \diamond b = ab + a + b$ and $a \diamond b = \frac{ab}{\max\{a,b,1\}}$ for each $a, b \in \mathbf{R}^+$.

Definition 1.1. *The binary operation $\diamond$ is said to satisfy α-property if there exists a positive real number α such that*

$$a \diamond b \leq \alpha \max\{a, b\}$$

for every $a, b \in \mathbf{R}^+$.

Example 1.2. *(1) If define* $a \diamond b = a + b$*, for each* $a, b \in \mathbf{R}^+$*, then for* $\alpha \geq 2$*, we have* $a \diamond b \leq \alpha \max\{a, b\}$*.*

(2) If define $a \diamond b = \frac{ab}{\max\{a,b,1\}}$*, for each* $a, b \in \mathbf{R}^+$*, then for* $\alpha \geq 1$*, we have* $a \diamond b \leq \alpha \max\{a, b\}$*.*

Definition 1.3. *Let* X *be a nonempty set. A generalized metric (or* d^**-metric) on* X *is a function* $d^* : X^2 \longrightarrow \mathbf{R}^+$ *that satisfies the following conditions for each* $x, y, z \in X$*.*

(1) $d^*(x, y) \geq 0$*,*

(2) $d^*(x, y) = 0$ *if and only if* $x = y$*,*

(3) $d^*(x, y) = d^*(y, x)$*,*

(4) $d^*(x, y) \leq d^*(x, z) \diamond d^*(z, y)$*.*

The pair (X, d^*) *is called a generalized metric (or* d^**-metric) space.*

Some examples of such a function are

(a) Let X be a nonempty set. Define $d^*(x, y) = d(x, y)$, for each $x, y \in X$, where $a \diamond b = a + b$ for $a, b \in \mathbf{R}^+$ and d is an ordinary metric on X.

(b) If $X = \mathbf{R}^n$ then we define $d^*(x, y) = ||x - y||$ for every $x, y \in \mathbf{R}^n$, where $a \diamond b = ab + a + b$ for $a, b \in \mathbf{R}^+$ and $|| \cdot ||$ is a norm on $\mathbf{R}^n$.

(c) Let X be a nonempty set. Define

$$d^*(x, y) = \begin{cases} 0 & , \quad x = y \\ 1 & , \quad \text{otherwise} \end{cases},$$

for each $x, y \in X$, where $a \diamond b = \max\{a, b\}$ for $a, b \in \mathbf{R}^+$.

Let (X, d^*) be a d^*-metric space. For $r > 0$ define

$$B_{d^*}(x, r) = \{y \in X : d^*(x, y) < r\}.$$

Definition 1.4. *Let* (X, d^*) *be a* d^**-metric space and* $A \subset X$*.*

(1) If for every $x \in A$ *there exists* $r > 0$ *such that* $B_{d^*}(x, r) \subset A$*, then the subset* A *is called open subset of* X*. A subset* A *of* X *is said to be closed if the complement of* A *in* X *is open.*

(2) A subset A *of* X *is said to be* d^**-bounded if there exists* $r > 0$ *such that* $d^*(x, y) < r$ *for all* $x, y \in A$*.*

(3) A sequence $\{x_n\}$ *in* X *converges to* x *if and only if* $d^*(x_n, x) = d^*(x, x_n) \to 0$ *as* $n \to \infty$*. That is for each* $\epsilon > 0$ *there exists* $n_0 \in \mathbf{N}$ *such that*

$$\forall n \geq n_0 \Longrightarrow d^*(x, x_n) < \epsilon.$$

(4) A sequence $\{x_n\}$ *in* X *is called a Cauchy sequence if for each* $\epsilon > 0$ *, there exists* $n_0 \in \mathbf{N}$ *such that* $d^*(x_n, x_m) < \epsilon$ *for each* $n, m \geq n_0$*. The* d^**-metric space* (X, d^*) *is said to be complete if every Cauchy sequence is convergent.*

Let τ be the set of all $A \subset X$ with $x \in A$ if and only if there exists $r > 0$ such that $B_{d^*}(x, r) \subset A$. Then τ is a topology on X (induced by the d^*-metric d^*).

Lemma 1.5. *Let (X, d^*) be a d^*-metric space such that $\diamond$ satisfy α-property with $\alpha \leq 1$. If $r > 0$, then ball $B_{d^*}(x, r)$ is open in X.*

Proof. Let $z \in B_{d^*}(x, r)$, hence $d^*(x, z) < r$. Let $d^*(x, z) = \delta$ and $r' = r - \delta$. Let $y \in B_{d^*}(z, r')$, hence we have $d^*(y, z) < r' = r - \delta$. It follows, $d^*(y, z) + d^*(x, z) < r$. Thus

$$\begin{aligned} d^*(x, y) &\leq d^*(x, z) \diamond d^*(z, y) \\ &\leq \alpha \max\{d^*(x, z), d^*(z, y)\} \\ &\leq d^*(x, z) + d^*(z, y) < r. \end{aligned}$$

Therefore $B_{d^*}(z, r') \subseteq B_{d^*}(x, r)$. Hence the ball $B_{d^*}(x, r)$ is open. □

Lemma 1.6. *Let (X, d^*) be a d^*-metric space such that $\diamond$ satisfy α-property with $\alpha > 0$. If a sequence $\{x_n\}$ in X converges to x, then x is unique.*

Proof. Let $x_n \longrightarrow y$ and $y \neq x$. Since $\{x_n\}$ converges to x and y, then for each $\epsilon > 0$ there exists n_1, $n_2 \in \mathbf{N}$ such that for $n \geq n_1$, $d^*(x, x_n) < \frac{\epsilon}{\alpha}$ and for $n \geq n_2$, $d^*(y, x_n) < \frac{\epsilon}{\alpha}$. If we set $n_0 = \max\{n_1, n_2\}$, then for every $n \geq n_0$ we have by triangular inequality

$$\begin{aligned} d^*(x, y) &\leq d^*(x, x_n) \diamond d^*(x_n, y) \\ &\leq \alpha \max\{d^*(x, x_n), d^*(x_n, y)\} \\ &< \alpha \max\{\frac{\epsilon}{\alpha}, \frac{\epsilon}{\alpha}\} = \epsilon. \end{aligned}$$

Hence $d^*(x, y) = 0$ which is a contradiction. So, $x = y$. □

Lemma 1.7. *Let (X, d^*) be a d^*-metric space, such that $\diamond$ satisfy α-property with $\alpha > 0$. If a sequence $\{x_n\}$in X is converges to x, then the sequence $\{x_n\}$ is Cauchy.*

Proof. Since $x_n \longrightarrow x$, for each $\epsilon > 0$ there exists $n_0 \in \mathbf{N}$ such that for $n \geq n_0$ $d^*(x_n, x) < \frac{\epsilon}{\alpha}$. Then for every $n, m \geq n_0$, by triangular inequality, we have

$$\begin{aligned} d^*(x_n, x_m) &\leq d^*(x_n, x) \diamond d^*(x, x_m) \\ &\leq \alpha \max\{d^*(x_n, x), d^*(x, x_m)\} \\ &< \alpha \max\{\frac{\epsilon}{\alpha}, \frac{\epsilon}{\alpha}\} = \epsilon. \end{aligned}$$

Hence $\{x_n\}$ is a Cauchy sequence. □

In 1998, Jungck and Rhoades [4] introduced the concept of weak compatible mappings and proved some common fixed point theorems using this concept on ordinary metric spaces. After then, many fixed point results have been obtained using weakly compatible mappings on ordinary metric spaces (see [1], [2], [3], [5]). Similarly we can give the concept of weakly compatible mappings on d^*-metric spaces as follows.

Definition 1.8. *Let A and S be mappings from a d^*-metric space (X, d^*) into itself. Then the mappings are said to be weakly compatible if they are commute at their coincidence point, that is, $Ax = Sx$ implies that $ASx = SAx$.*

2 Main Results

Theorem 2.1. *Let (X, d^*) be a complete d^*-metric space, such that $\diamond$ satisfy α-property with $\alpha \leq 1$. If A, B, S and T be self mappings of X into itself satisfying:*
(i) $A(X) \subseteq T(X)$, $B(X) \subseteq S(X)$ and $T(X)$ or $S(X)$ is a closed subset of X,
(ii) The pairs (A, S) and (B, T) are wakly compatible,
(iii) for all $x, y \in X$,

$$\begin{aligned} d^*(Ax, By) &\leq k_1(d^*(Sx, Ty) \diamond d^*(Ax, Sx) \diamond d^*(By, Ty)) \\ &\quad + k_2(d^*(Sx, By) \diamond d^*(Ax, Ty)) \end{aligned}$$

where $k_1, k_2 > 0$ and $0 < k_1 + k_2 < 1$.
Then A, B, S and T have a unique common fixed point in X.

Proof. Let x_0 be an arbitrary point in X. By (i), we can choose a point x_1 in X such that $y_0 = Ax_0 = Tx_1$ and $y_1 = Bx_1 = Sx_2$. In general, there exists a sequence $\{y_n\}$ such that, $y_{2n} = Ax_{2n} = Tx_{2n+1}$ and $y_{2n+1} = Bx_{2n+1} = Sx_{2n+2}$, for $n = 1, 2, \cdots$. We claim that the sequence $\{y_n\}$ is a Cauchy sequence.

By (iii), we have

$$\begin{aligned} d^*(y_{2n}, y_{2n+1}) &= d^*(Ax_{2n}, Bx_{2n+1}) \\ &\leq k_1(d^*(Sx_{2n}, Tx_{2n+1}) \diamond d^*(Ax_{2n}, Sx_{2n}) \diamond d^*(Bx_{2n+1}, Tx_{2n+1})) \\ &\quad + k_2(d^*(Sx_{2n}, Bx_{2n+1}) \diamond d^*(Ax_{2n}, Tx_{2n+1})) \\ &\leq k_1(d^*(y_{2n-1}, y_{2n}) \diamond d^*(y_{2n}, y_{2n-1}) \diamond d^*(y_{2n+1}, y_{2n})) \\ &\quad + k_2(d^*(y_{2n-1}, y_{2n+1}) \diamond d^*(y_{2n}, y_{2n})). \end{aligned}$$

If we put $d_n = d^*(y_n, y_{n+1})$, then by above inequality we have

$$d_{2n} \leq k_1(d_{2n-1} \diamond d_{2n-1} \diamond d_{2n}) + k_2(d^*(y_{2n-1}, y_{2n+1}) \diamond 0).$$

Hence

$$\begin{aligned} d_{2n} &\leq k_1 \max\{d_{2n}, d_{2n-1}\} + k_2 \max\{d^*(y_{2n-1}, y_{2n+1}), 0\} \\ &\leq k_1 \max\{d_{2n}, d_{2n-1}\} + k_2(d^*(y_{2n-1}, y_{2n}) \diamond d^*(y_{2n}, y_{2n+1})) \\ &\leq k_1 \max\{d_{2n}, d_{2n-1}\} + k_2 \max\{d^*(y_{2n-1}, y_{2n}), d^*(y_{2n}, y_{2n+1})\} \\ &\leq k_1 \max\{d_{2n}, d_{2n-1}\} + k_2 \max\{d_{2n-1}, d_{2n}\} \\ &= (k_1 + k_2) \max\{d_{2n}, d_{2n-1}\}. \end{aligned}$$

If $d_{2n} \geq d_{2n-1}$, we have

$$d_{2n} \leq (k_1 + k_2)d_{2n} < d_{2n},$$

which is a contradiction. It follows that

$$d_{2n} < d_{2n-1}.$$

Similarly, it is easy to see that $d_{2n+1} < d_{2n}$. Therefore, $d_n < d_{n-1}$, for $n = 1, 2, \cdots$. Thus by the above inequalities we have

$$d_n \leq (k_1 + k_2)d_{n-1} = kd_{n-1},$$

where $k = k_1 + k_2 < 1$. Thus

$$d_n \leq kd_{n-1} \leq k^2 d_{n-2} \leq \cdots \leq k^n d_0.$$

That is

$$d^*(y_n, y_{n+1}) \leq k^n d^*(y_0, y_1) \longrightarrow 0$$

as $n \to \infty$. If $m \geq n$, then

$$\begin{aligned} d^*(y_n, y_m) &\leq d^*(y_n, y_{n+1}) \diamond d^*(y_{n+1}, y_{n+2}) \diamond \cdots \diamond d^*(y_{m-1}, y_m) \\ &\leq \max\{d^*(y_n, y_{n+1}), \cdots, d^*(y_{m-1}, y_m)\} \longrightarrow 0 \end{aligned}$$

as $n \to \infty$. It follows that, the sequence $\{y_n\}$ is a Cauchy sequence and by the completeness of X, $\{y_n\}$ converges to $y \in X$. Then

$$\lim_{n\to\infty} y_n = \lim_{n\to\infty} Ax_{2n} = \lim_{n\to\infty} Bx_{2n+1} = \lim_{n\to\infty} Sx_{2n+2} = \lim_{n\to\infty} Tx_{2n+1} = y.$$

Now let $T(X)$ is closed subset of X, then there exists $v \in X$ such that $Tv = y$. We now prove that $Bv = y$. By (iii), we get

$$\begin{aligned} \lim_{n\to\infty} d^*(Ax_{2n}, Bv) \leq &\lim_{n\to\infty} [k_1(d(Sx_{2n}, Tv) \diamond d^*(Ax_{2n}, Sx_{2n}) \diamond d^*(Bv, Tv)) \\ &+ k_2(d^*(Sx_{2n}, Bv) \diamond d^*(Ax_{2n}, Tv))] \end{aligned}$$

and so

$$\begin{aligned} d^*(y, Bv) &\leq k_1(d^*(y, Tv) \diamond d^*(y, y) \diamond d^*(Bv, y)) \\ &\quad + k_2(d^*(y, Bv) \diamond d^*(y, y)) \\ &\leq k_1 \max\{d^*(y, y), d^*(y, y), d^*(Bv, y)\} \\ &\quad + k_2 \max\{d^*(y, Bv), d^*(y, y)\} \\ &= (k_1 + k_2)d^*(y, Bv) \\ &< d^*(y, Bv), \end{aligned}$$

which is a contradiction if $d^*(y, Bv) > 0$. Hence $Bv = y = Tv$. Since B and T are weakly compatible mappings, then we have $BTv = TBv$ and so $By = Ty$.

Now, we prove that $By = y$. By (iii), we get

$$\begin{aligned}\lim_{n\to\infty} d^*(Ax_{2n}, By) &\leq \lim_{n\to\infty}[k_1(d(Sx_{2n}, Ty) \diamond d^*(Ax_{2n}, Sx_{2n}) \diamond d^*(By, Ty)) \\ &+ k_2(d^*(Sx_{2n}, By) \diamond d^*(Ax_{2n}, Ty))].\end{aligned}$$

Hence,

$$\begin{aligned}d^*(y, By) &\leq k_1(d^*(y, Ty) \diamond d^*(y, y) \diamond d^*(By, y)) \\ &\quad +k_2(d^*(y, By) \diamond d^*(y, y)) \\ &\leq k_1 \max\{d^*(y, By), d^*(y, y)\} \\ &\quad +k_2 \max\{d^*(y, By), d^*(y, y)\} \\ &= (k_1 + k_2)d^*(y, By) \\ &< d^*(y, By),\end{aligned}$$

which is a contradiction if $d^*(y, By) > 0$. Thus $By = y = Ty$.

Since $B(X) \subseteq S(X)$, there exists $w \in X$ such that $Sw = y$. We show that $Aw = y$. From (iii) we have

$$\begin{aligned}d^*(Aw, By) &\leq k_1(d^*(Sw, Ty) \diamond d^*(Aw, Sw) \diamond d^*(By, Ty)) \\ &+ k_2(d^*(Sw, By) \diamond d^*(Aw, Ty))\end{aligned}$$

and it follows that

$$\begin{aligned}d^*(Aw, y) &\leq k_1(d^*(y, y) \diamond d^*(Aw, y) \diamond d^*(y, y)) \\ &\quad +k_2(d^*(y, y) \diamond d^*(Aw, y)) \\ &\leq k_1 \max\{d^*(y, y), d^*(Aw, y)\} \\ &\quad +k_2 \max\{d^*(y, y), d^*(Aw, y)\} \\ &= (k_1 + k_2)d^*(Aw, y) \\ &< d^*(Aw, y),\end{aligned}$$

which is a contradiction if $d^*(Aw, y) > 0$. Thus $Aw = y$ and hence $Aw = y = Sw$. Since A and S are weakly compatible, then $ASw = SAw$ and so $Ay = Sy$.

Now, we show that $Ay = y$. From (iii), we get

$$\begin{aligned}d^*(Ay, By) &\leq k_1(d^*(Sy, Ty) \diamond d^*(Ay, Sy) \diamond d^*(By, Ty)) \\ &+ k_2(d^*(Sy, By) \diamond d^*(Ay, Ty))\end{aligned}$$

and it follows that

$$\begin{aligned}d^*(Ay, y) &\leq k_1(d^*(Sy, y) \diamond d^*(Ay, Sy) \diamond d^*(y, y)) \\ &\quad +k_2(d^*(Sy, y) \diamond d^*(Ay, y)) \\ &\leq k_1 \max\{d^*(Ay, y), d^*(Ay, Ay), d^*(y, y)\} \\ &\quad +k_2 \max\{d^*(Ay, y), d^*(Ay, y)\} \\ &= (k_1 + k_2)d^*(Ay, y) \\ &< d^*(Ay, y),\end{aligned}$$

which is a contradiction if $d^*(Ay, y) > 0$. Thus, $Ay = y$ and therefore $Ay = Sy = By = Ty = y$. That is y is a cammon fixed point for A, B, T, S.

The proof is similar when $S(X)$ is assumed to be a closed subset of X.

Now to prove uniqueness assume that x be another common fixed point of A, B, S and T. Then from (iii) we have

$$\begin{aligned} d^*(x, y) &= d^*(Ax, By) \\ &\leq k_1(d^*(Sx, Ty) \diamond d^*(Ax, Sx) \diamond d^*(By, Ty)) \\ &+ k_2(d^*(Sx, By) \diamond d^*(Ax, Ty)), \end{aligned}$$

and so

$$\begin{aligned} d^*(x, y) &\leq k_1 \max\{d^*(x, y), d^*(x, x), d^*(y, y)\} \\ &\quad + k_2 \max\{d^*(x, y), d^*(x, y)\} \\ &= (k_1 + k_2) d^*(x, y) \\ &< d^*(x, y), \end{aligned}$$

which is a contradiction if $d^*(x, y) > 0$. Thus $x = y$. This completes the proof. □

In the following theorem, let d is an ordinary metric on X.

Theorem 2.2. *Let (X, d) be a complete metric space, such that $\diamond$ satisfy α-property with $\alpha > 0$. If A, B, S and T be self mappings of X into itself, satisfying:*

(i) $A(X) \subseteq T(X)$, $B(X) \subseteq S(X)$ and $T(X)$ or $S(X)$ is a closed subset of X,

(ii) The pairs (A, S) and (B, T) are wakly compatible,

(iii) for all $x, y \in X$,

$$\begin{aligned} d(Ax, By) &\leq k_1(d(Sx, Ty) \diamond d(Ax, Sx)) + k_2(d(Sx, Ty) \diamond d(By, Ty)) \\ &\quad + k_3(d(Sx, Ty) \diamond \frac{d(Sx, By) + d(Ax, Ty)}{2}) \end{aligned}$$

where $k_1, k_2, k_3 > 0$ and $0 < \alpha(k_1 + k_2 + k_3) < 1$.

Then A, B, S and T have a unique common fixed point in X.

Proof. Let x_0 be an arbitrary point in X. By (i), we can choose a point x_1 in X such that $y_0 = Ax_0 = Tx_1$ and $y_1 = Bx_1 = Sx_2$. In general, there exists a sequence $\{y_n\}$ such that, $y_{2n} = Ax_{2n} = Tx_{2n+1}$ and $y_{2n+1} = Bx_{2n+1} = Sx_{2n+2}$, for $n = 1, 2, \cdots$. We claim that the sequence $\{y_n\}$ is a Cauchy sequence.

By (iii), we have

$$\begin{aligned}
d(y_{2n}, y_{2n+1}) &= d(Ax_{2n}, Bx_{2n+1}) \\
&\leq k_1(d(Sx_{2n}, Tx_{2n+1}) \diamond d(Ax_{2n}, Sx_{2n})) \\
&\quad +k_2(d(Sx_{2n}, Tx_{2n+1}) \diamond d(Bx_{2n+1}, Tx_{2n+1})) \\
&\quad +k_3(d(Sx_{2n}, Tx_{2n+1}) \diamond \frac{d(Sx_{2n}, Bx_{2n+1}) + d(Ax_{2n}, Tx_{2n+1})}{2}) \\
&= k_1(d(y_{2n-1}, y_{2n}) \diamond d(y_{2n}, y_{2n-1})) \\
&\quad +k_2(d(y_{2n-1}, y_{2n}) \diamond d(y_{2n+1}, y_{2n})) \\
&\quad +k_3(d(y_{2n-1}, y_{2n}) \diamond \frac{d(y_{2n-1}, y_{2n+1}) + d(y_{2n}, y_{2n})}{2}).
\end{aligned}$$

If we put $d_n = d(y_n, y_{n+1})$, then by above inequality we have

$$d_{2n} \leq k_1(d_{2n-1} \diamond d_{2n-1}) + k_2(d_{2n-1} \diamond d_{2n}) + k_3(d_{2n-1} \diamond \frac{d(y_{2n-1}, y_{2n+1})}{2}).$$

Hence

$$d_{2n} \leq k_1 \alpha d_{2n-1} + k_2 \alpha \max\{d_{2n-1}, d_{2n}\} + k_3 \alpha \max\{d_{2n-1}, \frac{d_{2n-1} + d_{2n}}{2}\}.$$

If $d_{2n} \geq d_{2n-1}$, then we have

$$d_{2n} \leq k_1 \alpha d_{2n} + k_2 \alpha d_{2n} + k_3 \alpha d_{2n} < d_{2n},$$

which is a contradiction. It follows that

$$d_{2n} < d_{2n-1}.$$

Similarly, it is easy to see that $d_{2n+1} < d_{2n}$. Therefore, $d_n < d_{n-1}$, for $n = 1, 2, \cdots$.

Thus by above inequality we get

$$d_n \leq \alpha(k_1 + k_2 + k_3) d_{n-1} = k d_{n-1},$$

where $\alpha(k_1 + k_2 + k_3) = k < 1$. Hence

$$d_n \leq k d_{n-1} \leq k^2 d_{n-2} \leq \cdots \leq k^n d_0.$$

That is

$$d(y_n, y_{n+1}) \leq k^n d(y_0, y_1) \longrightarrow 0$$

as $n \to \infty$. If $m \geq n$, then

$$\begin{aligned}
d(y_n, y_m) &\leq d(y_n, y_{n+1}) + d(y_{n+1}, y_{n+2}) + \cdots + d(y_{m-1}, y_m) \\
&\leq k^n d(y_0, y_1) + k^{n+1} d(y_0, y_1) \cdots + k^{m-1} d(y_0, y_1) \\
&= \frac{k^n}{1-k} d(y_0, y_1) \longrightarrow 0
\end{aligned}$$

as $n \to \infty$. It follows that, the sequence $\{y_n\}$ is Cauchy sequence and by the completeness of X, $\{y_n\}$ converges to $y \in X$. Then

$$\lim_{n\to\infty} y_n = \lim_{n\to\infty} Ax_{2n} = \lim_{n\to\infty} Bx_{2n+1} = \lim_{n\to\infty} Sx_{2n+2} = \lim_{n\to\infty} Tx_{2n+1} = y.$$

Let $T(X)$ is closed subset of X, then there exists $v \in X$ such that $Tv = y$.

We now prove that $Bv = y$. By (iii), we get

$$\begin{aligned}\lim_{n\to\infty} d(Ax_{2n}, Bv) \quad &\leq \quad \lim_{n\to\infty}[k_1(d(Sx_{2n}, Tv) \diamond d(Ax_{2n}, Sx_{2n})) \\ &\quad +k_2(d(Sx_{2n}, Tv) \diamond d(Bv, Tv)) \\ &\quad +k_3(d(Sx_{2n}, Tv) \diamond \frac{d(Sx_{2n}, Bv) + d(Ax_{2n}, Tv)}{2})]\end{aligned}$$

and so

$$\begin{aligned} d(y, Bv) \quad &\leq \quad k_1(d(y, Tv) \diamond d(y, y)) + k_2(d(y, Tv) \diamond d(Bv, Tv)) \\ &\quad +k_3(d(y, Tv) \diamond \frac{d(y, Bv) + d(y, Tv)}{2}) \\ &\leq \quad k_1\alpha \max\{d(y, Tv), 0\} + k_2\alpha \max\{0, d(Bv, y)\} \\ &\quad +k_3\alpha \max\{0, \frac{d(y, Bv) + 0}{2}\} \\ &< \quad d(y, Bv).\end{aligned}$$

It follows that $Bv = y = Tv$. Since B and T are two weakly compatible mappings, we have $BTv = TBv$ and so $By = Ty$.

Next, we prove that $By = y$. By (iii), we get

$$\begin{aligned}\lim_{n\to\infty} d(Ax_{2n}, By) \quad &\leq \quad \lim_{n\to\infty}[k_1(d(Sx_{2n}, Ty) \diamond d(Ax_{2n}, Sx_{2n})) \\ &\quad +k_2(d(Sx_{2n}, Ty) \diamond d(By, Ty)) \\ &\quad +k_3(d(Sx_{2n}, Ty) \diamond \frac{d(Sx_{2n}, By) + d(Ax_{2n}, Ty)}{2})].\end{aligned}$$

Hence,

$$\begin{aligned} d(y, By) \quad &\leq \quad k_1(d(y, Ty) \diamond d(y, y)) + k_2(d(y, Ty) \diamond d(By, Ty)) \\ &\quad +k_3(d(y, Ty) \diamond \frac{d(y, By) + d(y, Ty)}{2}) \\ &\leq \quad k_1\alpha \max\{d(y, Ty), d(y, y)\} + k_2\alpha \max\{d(y, Ty), d(By, Ty) \\ &\quad +k_3\alpha \max\{d(y, Ty), \frac{d(y, By) + d(y, Ty)}{2}\} \\ &< \quad d(y, By)\end{aligned}$$

and so $By = y$.

Since $B(X) \subseteq S(X)$, there exists $w \in X$ such that $Sw = y$. We prove that $Aw = y$. By (iii) we have

$$\begin{aligned} d(Aw, By) &\leq k_1(d(Sw, Ty) \diamond d(Aw, Sw)) + k_2(d(Sw, Ty) \diamond d(By, Ty)) \\ &\quad + k_3(d(Sw, Ty) \diamond \frac{d(Sw, By) + d(Aw, Ty)}{2}) \end{aligned}$$

and it follows that

$$\begin{aligned} d(Aw, y) &\leq k_1(d(Sw, y) \diamond d(Aw, Sw)) + k_2(d(Sw, y) \diamond d(y, y)) \\ &\quad + k_3(d(Sw, y) \diamond \frac{d(Sw, y) + d(Aw, y)}{2}) \\ &\leq k_1\alpha \max\{d(Sw, y), d(Aw, Sw)\} + k_2\alpha \max\{d(Sw, y), d(y, y)\} \\ &\quad + k_3\alpha \max\{d(Sw, y), \frac{d(Sw, y) + d(Aw, y)}{2}\} \\ &< d(Aw, y). \end{aligned}$$

This implies that $Aw = y$ and hence $Aw = Sw = y$. Since A and S are weakly compatible, then $ASw = SAw$ and so $Ay = Sy$.

Now, we prove that $Ay = y$. From (iii), we have

$$\begin{aligned} d(Ay, By) &\leq k_1(d(Sy, Ty) \diamond d(Ay, Sy)) + k_2 d(Sy, Ty) \diamond d(By, Ty)) \\ &\quad + k_3(d(Sy, Ty) \diamond \frac{d(Sy, By) + d(Ay, Ty)}{2}) \end{aligned}$$

and it follows that

$$\begin{aligned} d(Ay, y) &\leq k_1(d(Sy, y) \diamond d(Ay, Sy)) + k_2(d(Sy, y) \diamond d(y, y)) \\ &\quad + k_3(d(Sy, y) \diamond \frac{d(Sy, y) + d(Ay, y)}{2}) \\ &\leq k_1\alpha \max\{d(Sy, y), d(Ay, Sy)\} + k_2\alpha \max\{d(Sy, y), d(y, y)\} \\ &\quad + k_3\alpha \max\{d(Sy, y), \frac{d(Sy, y) + d(Ay, y)}{2}\} \\ &< d(Ay, y) \end{aligned}$$

and it follows that $Ay = y$ and therefore $Ay = Sy = By = Ty = y$. That is y is a common fixed point for A, B, T, S.

The proof is similar when $S(X)$ is assumed to be a closed subset of X.

Now to prove uniqueness assume that x be another common fixed point of A, B, S and T. Then

$$\begin{aligned} d(x, y) &= d(Ax, By) \\ &\leq k_1(d(Sx, Ty) \diamond d(Ax, Sx)) + k_2(d(Sx, Ty) \diamond d(By, Ty)) \\ &\quad + k_3(d(Sx, Ty) \diamond \frac{d(Sx, By) + d(Ax, Ty)}{2}) \end{aligned}$$

and so

$$\begin{aligned} d(x,y) &= d(Ax,By) \\ &\leq k_1(d(x,y)\diamond d(x,x)) + k_2(d(x,y)\diamond d(y,y)) \\ &\quad +k_3(d(x,y)\diamond \frac{d(x,y)+d(x,y)}{2}) \\ &< d(x,y). \end{aligned}$$

Thus it follows that $x = y$. □

Corollary 2.3. *Let (X, d) be a complete metric space. If A, B, S and T be self mappings of X into itself satisfying:*

(i) $A(X) \subseteq T(X)$, $B(X) \subseteq S(X)$ and $T(X)$ or $S(X)$ is a closed subset of X,

(ii) The pairs (A, S) and (B, T) are wakly compatible,

(iii) for all $x, y \in X$,

$$\begin{aligned} d(Ax,By) &\leq k_1(d(Sx,Ty)+d(Ax,Sx)) + k_2(d(Sx,Ty)+d(By,Ty)) \\ &\quad +k_3(d(Sx,Ty)+\frac{d(Sx,By)+d(Ax,Ty)}{2}), \end{aligned}$$

where $k_1, k_2, k_3 > 0$ and $0 < k_1 + k_2 + k_3 < \frac{1}{2}$.

Then A, B, S and T have a unique common fixed point in X.

Proof. If define $a \diamond b = a + b$ for each $a, b \in \mathbf{R}^+$, then for $\alpha \geq 2$, we have $a \diamond b \leq \alpha \max\{a, b\}$. Also if put $\alpha = 2$ then we get $0 < \alpha(k_1 + k_2 + k_3) < 1$, hence all conditions of Theorem 2.2 are holds. Thus A, B, S and T have a unique common fixed point in X. □

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