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# Intuitionistic Fuzzy PMS-Ideals in a PMS-Algebra

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**Abstract** In this paper, we use the notion of an intuitionistic fuzzy set to PMS-ideals in PMS-algebras. The concept of an intuitionistic fuzzy PMS-ideal of a PMS-algebra is given, along with some fundamental characteristics. We also define the level subsets of an intuitionistic fuzzy PMS-ideal of a PMS-algebra and characterize an intuitionistic fuzzy PMS-ideal in terms of its level subsets in a PMS-algebra.

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## 1. Introduction

In 1978, K. Iseki and S. Tanaka [1] introduced a type of abstract algebra called BCK-algebra, and later in 1980, K. Iseki [2] developed BCI-algebra as a proper subclass of BCK-algebra. In 2016, Sithar Selvam and Nagalakshmi [3] introduced a new class of algebra called PMS-algebra. In 1965, Zadeh [4] introduced the concept of the fuzzy set as an extension of the classical set to deal with uncertainties in the physical world. Following the introduction of the notion of fuzzy sets, several researchers looked into expanding it. As an extension of a fuzzy set, Atanassov [5, 6] introduced the concept of an intuitionistic fuzzy set to better deal with uncertainties. Sithar Selvam and Nagalakshmi [7] introduced fuzzy PMS-ideal in PMS-algebra and established various properties in detail. Kim and Jeong [9] studied the concept of an intuitionistic fuzzy B-subalgebra of a B-algebra in 2006. A. Zarandi and A. B. Saeid [11] investigated the intuitionistic fuzzy BG-subalgebras and intuitionistic fuzzy BG-ideals of the BG-algebra. In 2011, Mostafa and et al. [10] introduced the intuitionistic fuzzy KU-ideals in KU-algebra and investigated some related properties.

In this paper, we use the idea of an intuitionistic fuzzy set to PMS-ideals in a PMS-algebra. The concept of an intuitionistic fuzzy PMS-ideal of a PMS-algebra is given, along with

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some fundamental characteristics. We also define the level subsets of an intuitionistic fuzzy PMS-ideal of a PMS-algebra and characterize an intuitionistic fuzzy PMS-ideal in terms of its level subsets in a PMS-algebra.

## 2. Preliminaries

In this section, we consider some basic definitions, results and important concepts of PMS-algebras that are needed for our work.

**Definition 2.1.** [3]A PMS-algebra is a nonempty set $X$ with a constant 0 and a binary operation $*$ of type (2, 0) satisfying the following axioms:

$(i)$. $0 * x = x$,
$(ii)$. $(y * x) * (z * x) = z * y$ for all $x, y, z \in X$.

In $X$, we define a binary relation $\leq$ by $x \leq y$ if and only if $x * y = 0$.

**Definition 2.2.** [3] A nonempty subset $S$ of a PMS-algebra $X$ is called a PMS-subalgebra of $X$ if $x * y \in S$ for all $x, y \in S$.

**Definition 2.3.** [3] A nonempty subset $I$ of a PMS-algebra $(X, *, 0)$ is said to be a PMS-ideal of $X$ if it satisfies the following conditions:

$(i)$. $0 \in I$,
$(ii)$. $z * y, z * x \in I \Rightarrow y * x \in I$ for all $x, y, z \in X$.

**Proposition 2.4.** [3] *In any PMS-algebra* $(X, *, 0)$ *the following properties hold for all* $x, y, z \in X$.

$(i)$. $x * x = 0$,
$(ii)$. $(y * x) * x = y$,
$(iii)$. $x * (y * x) = y * 0$,
$(iv)$. $(y * x) * z = (z * x) * y$,
$(v)$. $(x * y) * 0 = y * x = (0 * y) * (0 * x)$.

**Definition 2.5.** [4] Let $X$ be a nonempty set. A fuzzy subset $A$ of the set $X$ is defined as $A = \{x, \mu_A(x) | x \in X\}$, where the mapping $\mu_A : X \to [0, 1]$ defines the degree of membership.

**Definition 2.6.** [7] A fuzzy set $A$ in a PMS-algebra $X$ is called a fuzzy PMS-subalgebra of $X$ if $\mu_A(x * y) \geq min\{\mu_A(x), \mu_A(y)\}$ for all $x, y, z \in X$

**Definition 2.7.** [7] A fuzzy set $A$ in a PMS-algebra $X$ is called a fuzzy PMS-ideal of $X$ if

$(i)$. $\mu_A(0) \geq \mu_A(x)$,
$(ii)$. $\mu_A(y * x) \geq min\{\mu_A(z * y), \mu_A(z * x)\}$ for all $x, y, z \in X$.

**Definition 2.8.** [5, 6] An intuitionistic fuzzy set (IFS) $A$ in a nonempty set $X$ is an object having the form $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle | x \in X\}$, where the functions $\mu_A : X \to [0, 1]$ and $\nu_A : X \to [0, 1]$ define the degree of membership and the degree of nonmembership of each element $x \in X$ to the set $A$ respectively, satisfying the condition $0 \leq \mu_A(x) + \nu_A(x) \leq 1$ for all $x \in X$.

**Remark 2.9.** Ordinary fuzzy sets over $X$ may be viewed as special intuitionistic fuzzy sets with the non membership function $\nu_A(x) = 1 - \mu_A(x)$. So each ordinary fuzzy set may be written as $\{\langle x, \mu_A(x), 1-\mu_A(x)\rangle | x \in X\}$ to define an intuitionistic fuzzy set. For the sake of simplicity, we write $A = (\mu_A, \nu_A)$ for an intuitionistic fuzzy set $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle | x \in X\}$.

**Definition 2.10.** [5, 6] Let $A = (\mu_A, \nu_A)$ and $B = (\mu_B, \nu_B)$ be any two intuitionistic fuzzy subsets of the set $X$, then

$(i)$. $A \subseteq B$ if and only if $\mu_A(x) \leq \mu_B(x)$ and $\nu_A(x) \geq \nu_B(x)$ for all $x \in X$,
$(ii)$. $A = B$ if and only if $A \subseteq B$ and $B \subseteq A$,
$(iii)$. $A \cap B = \{\langle x, min(\mu_A(x), \mu_B(x)), max(\nu_A(x), \nu_B(x))\rangle | x \in X\}$,
$(iv)$. $A \cup B = \{\langle x, max(\mu_A(x), \mu_B(x)), min(\nu_A(x), \nu_B(x))\rangle | x \in X\}$,
$(v)$. $\bar{A} = \{\langle x, \nu_A(x), \mu_A(x)\rangle | x \in X\}$,
$(vi)$. $\Box A = \{\langle x, \mu_A(x), 1-\mu_A(x)\rangle | x \in X\}$,
$(vii)$. $\Diamond A = \{\langle x, 1-\nu_A(x), \nu_A(x)\rangle | x \in X\}$.

**Lemma 2.11.** [8] *Let $A = (\mu_A.\nu_A)$ be an intutionistic fuzzy set in $X$. Then the following statements hold for any $x, y \in X$.*

$(i)$. $1 - max\{\mu_A(x), \mu_A(y)\} = min\{1-\mu_A(x), 1-\mu_A(y)\}$,
$(ii)$. $1 - min\{\mu_A(x), \mu_A(y)\} = max\{1-\mu_A(x), 1-\mu_A(y)\}$,
$(iii)$. $1 - max\{\nu_A(x), \nu_A(y)\} = min\{1-\nu_A(x), 1-\nu_A(y)\}$,
$(iv)$. $1 - min\{\nu_A(x), \nu_A(y)\} = max\{1-\nu_A(x), 1-\nu_A(y)\}$.

**Definition 2.12.** [8] An intuitionistic fuzzy subset $A = (\mu_A, \nu_A)$ of a PMS-algebra $X$ is called an intuitionistic fuzzy PMS-sub algebra of $X$ if

$(i)$. $\mu_A(x * y) \geq min\{\mu_A(x), \mu_A(y)\}$ and
$(ii)$. $\nu_A(x * y) \leq max\{\nu_A(x), \nu_A(y)\}$ for all $x, y \in X$.

## 3. Intuitionistic Fuzzy PMS-ideals

In this section, we introduce the notion of intuitionistic fuzzy PMS-ideals of a PMS-algebra. Some important properties related to intuitionistic fuzzy PMS-ideals are investigated. Throughout this paper, $X$ denotes a PMS-algebra unless otherwise specified.

**Definition 3.1.** An intuitionistic fuzzy set $A = (\mu_A, \nu_A)$ in $X$ is called an intuitionistic fuzzy PMS-ideal of $X$ if it satisfies the following conditions for all $x, y, z \in X$.

$(i)$. $\mu_A(0) \geq \mu_A(x)$ and $\nu_A(0) \leq \nu_A(x)$,
$(ii)$. $\mu_A(y * x) \geq min\{\mu_A(z * y), \mu_A(z * x)\}$,
$(iii)$. $\nu_A(y * x) \leq max\{\nu_A(z * y), \nu_A(z * x)\}$.

**Example 3.2.** Let $X = \{0, a, b, c\}$ such that $(X, *, 0)$ is a PMS algebra with the following table.

| $*$ | $0$ | $a$ | $b$ | $c$ |
|---|---|---|---|---|
| $0$ | $0$ | $a$ | $b$ | $c$ |
| $a$ | $b$ | $0$ | $a$ | $b$ |
| $b$ | $a$ | $b$ | $0$ | $a$ |
| $c$ | $c$ | $a$ | $b$ | $0$ |

Then $I = \{0, a, b\}$ is a PMS-ideal of $X$. Define an intuitionistic set $A = (\mu_A, \nu_A)$ in $X$ by

$$\mu_A(x) = \begin{cases} 1 & \text{if} \quad x = 0 \\ 0.5 & \text{if} \quad x = a, b \\ 0 & \text{if} \quad x = c \end{cases} \quad \text{and} \quad \nu_A(x) = \begin{cases} 0 & \text{if} \quad x = 0 \\ 0.3 & \text{if} \quad x = a, b \\ 1 & \text{if} \quad x = c. \end{cases}$$

Then by routine calculations we can see that $A = (\mu_A, \nu_A)$ is an intuitionistic fuzzy PMS-ideal of the PMS-algebra $X$.

**Theorem 3.3.** *Let $A = (\mu_A, \nu_A)$ be an intuitionistic fuzzy PMS-ideal of a PMS-algebra $X$ such that $x \leq y$, then $\mu_A(x) \geq \mu_A(y)$ and $\nu_A(x) \leq \nu_A(y)$, that is $\mu_A$ is order reversing and $\nu_A$ is order preserving, $\forall x, y \in X$.*

*Proof.* Let $A = (\mu_A, \nu_A)$ be an intuitionistic fuzzy PMS-ideal of a PMS-algebra $X$ such that $x \leq y, \forall x, y \in X$. Then we have $x * y = 0$.

$$\begin{aligned}
\text{Now, } \mu_A(x) = \mu_A(0 * x) &\geq min\{\mu_A(z * 0), \mu_A(z * x)\} \quad (\text{By Definitions } 2.1(i)) \text{ and } 3.1(ii)) \\
&= min\{\mu_A(z * 0), \mu_A((x * y) * (z * y))\} \quad (\text{By Definition } 2.1(ii)) \\
&= min\{\mu_A(z * 0), \mu_A(0 * (z * y))\} \quad (\therefore\ x \leq y \Rightarrow x * y = 0.) \\
&= min\{\mu_A(z * 0), \mu_A(z * y)\} \quad (\text{By Definition } 2.1(i)) \\
&= min\{\mu_A(0 * 0), \mu_A(0 * y)\} \quad (\text{Taking z} = 0) \\
&= min\{\mu_A(0), \mu_A(y)\} = \mu_A(y) \quad (\text{By Definition } 3.1(i))
\end{aligned}$$

and

$$\begin{aligned}
\nu_A(x) = \nu_A(0 * x) &\leq max\{\nu_A(z * 0), \nu_A(z * x)\} \quad (\text{By Definitions } 2.1(i)) \text{ and } 3.1(ii)) \\
&= max\{\nu_A(z * 0), \nu_A((x * y) * (z * y))\} \quad (\text{By Definition } 2.1(ii)) \\
&= max\{\nu_A(z * 0), \nu_A(0 * (z * y))\} \quad (\therefore\ x \leq y \Rightarrow x * y = 0.) \\
&= max\{\nu_A(z * 0), \nu_A(z * y)\} \quad (\text{By Definition } 2.1(i)) \\
&= max\{\nu_A(0 * 0), \nu_A(0 * y)\} \quad (\text{Taking z} = 0) \\
&= max\{\nu_A(0), \nu_A(y)\} = \nu_A(y). \quad (\text{By Definition } 3.1(i))
\end{aligned}$$

Hence $\mu_A(x) \geq \mu_A(y)$ and $\nu_A(x) \leq \nu_A(y)$. ■

**Theorem 3.4.** *Every intuitionistic fuzzy PMS-ideal $A = (\mu_A, \nu_A)$ of $X$ is an intuitionistic fuzzy PMS-sub algebra of $X$.*

*Proof.* Let $A = (\mu_A, \nu_A)$ be an intuitionistic fuzzy PMS-ideal of $X$ and $x, y \in X$. Then by Definition 3.1, we have

$$\begin{aligned}
\mu_A(x * y) &\geq min\{\mu_A(0 * x), \mu_A(0 * y)\} \\
&= min\{\mu_A(x), \mu_A(y)\}
\end{aligned}$$

and

$$\begin{aligned}
\nu_A(x * y) &\leq max\{\nu_A(0 * x), \nu_A(0 * y)\} \\
&= max\{\nu_A(x), \nu_A(y)\}.
\end{aligned}$$

Thus, $A = (\mu_A, \nu_A)$ is an intuitionistic fuzzy PMS-subalgebra of $X$. ■

**Theorem 3.5.** *Let $A = (\mu_A, \nu_A)$ be an intuitionistic fuzzy PMS-ideal of a PMS- algebra $X$. If $x * y \leq z$, then $\mu_A(x) \geq min\{\mu_A(y), \mu_A(z)\}$ and $\nu_A(x) \leq max\{\nu_A(y), \nu_A(z)\}$.*

*Proof.* Let $x, y, z \in X$ such that $x * y \leq z$. Then by the binary relation $\leq$ defined in $X$, we have $(x * y) * z = 0$. Then using Definition 2.1 $(i)$, Proposition 2.4 $(i, iv)$ and Theorem 3.4 , we have

$$\begin{aligned}\mu_A(x) = \mu_A(0*x) &= \mu_A(((x*y)*z)*x) \\ &= \mu_A(((z*y)*x)*x) \\ &= \mu_A((x*x)*(z*y)) \\ &= \mu_A(0*(z*y)) \\ &= \mu_A(z*y) \\ &\geq min\{\mu_A(z), \mu_A(y)\}. \\ \Rightarrow \mu_A(x) &\geq min\{\mu_A(z), \mu_A(y)\}.\end{aligned}$$

Similarly $\nu_A(x) = \nu_A(z*y) \leq max\{\nu_A(z), \nu_A(y)\}$.
Hence $\mu_A(x) \geq min\{\mu_A(y), \mu_A(z)\}$ and $\nu_A(x) \leq max\{\nu_A(y), \nu_A(z)\}$ for all $x, y, z \in X$. ∎

**Theorem 3.6.** *The intersection of any two intuitionistic fuzzy PMS-ideals of a PMS-algebra $X$ is also an intuitionistic fuzzy PMS-ideal of $X$.*

*Proof.* Let $A = (\mu_A, \nu_A)$ and $B = (\mu_B, \nu_B)$ be any two intuitionistic fuzzy PMS-ideal of $X$. Then we claim that $A \cap B$ is an intuitionistic fuzzy PMS-ideal of $X$. Let $x, y, z \in X$, then

$$\mu_{A\cap B}(0) = min\{\mu_A(0), \mu_B(0)\} \geq min\{\mu_A(x), \mu_B(x)\} = \mu_{A\cap B}(x) \text{ and}$$
$$\nu_{A\cap B}(0) = max\{\nu_A(0), \nu_B(0)\} \leq max\{\nu_A(x), \nu_B(x)\} = \nu_{A\cap B}(x).$$

Also,

$$\begin{aligned}\mu_{A\cap B}(y*x) &= min\{\mu_A(y*x), \mu_B(y*x)\} \\ &\geq min\{min\{\mu_A(z*y), \mu_A(z*x)\}, min\{\mu_B(z*y), \mu_B(z*x)\}\} \\ &= min\{min\{\mu_A(z*y), \mu_B(z*y)\}, min\{\mu_A(z*x), \mu_B(z*x)\}\} \\ &= min\{\mu_{A\cap B}(z*y), \mu_{A\cap B}(z*x)\}\end{aligned}$$

and

$$\begin{aligned}\nu_{A\cap B}(y*x) &= max\{\nu_A(y*x), \nu_B(y*x)\} \\ &\leq max\{max\{\nu_A(z*y), \nu_A(z*x)\}, max\{\nu_B(z*y), \nu_B(z*x)\}\} \\ &= max\{max\{\nu_A(z*y), \nu_B(z*y)\}, max\{\nu_A(z*x), \nu_B(z*x)\}\} \\ &= max\{\nu_{A\cap B}(z*y), \nu_{A\cap B}(z*x)\}.\end{aligned}$$

Hence $A \cap B$ is an intuitionistic fuzzy PMS-ideal of $X$. ∎

The above theorem can also be generalized to any set of intuitionistic fuzzy PMS-ideals as follows.

**Corollary 3.7.** *The intersection of any set of intuitionistic fuzzy PMS-ideals of a PMS-algebra $X$ is also an intuitionistic fuzzy PMS-ideal of $X$.*

**Corollary 3.8.** *The intersection of an intuitionistic fuzzy PMS-subalgebra and an intuitionistic fuzzy PMS-ideal of a PMS-algebra $X$ is an intuitionistic fuzzy PMS-subalgebra of $X$.*

**Theorem 3.9.** *An intuitionistic fuzzy subset $A = (\mu_A, \nu_A)$ is an intuitionistic fuzzy PMS-ideal of $X$ if and only if the fuzzy subsets $\mu_A$ and $\overline{\nu}_A$ are fuzzy PMS-ideals of $X$.*

*Proof.* Let $A = (\mu_A, \nu_A)$ be an intuitionistic fuzzy PMS-ideal of $X$. We need to show that the fuzzy subsets $\mu_A$ and $\overline{\nu}_A$ are fuzzy PMS-ideals of $X$. Clearly, $\mu_A$ is a fuzzy PMS-ideal of $X$ follows form the fact that $A = (\mu_A, \nu_A)$ is an intuitionistic fuzzy PMS-ideal of $X$. Now, it remains to show that $\overline{\nu}_A$ is a fuzzy PMS-ideal of $X$. Let $x, y, z \in X$, then we have

(i) $\overline{\nu}_A(0) = 1 - \nu_A(0) \geq 1 - \nu_A(x) = \overline{\nu}_A(x)$ and
$(ii)$ $\overline{\nu}_A(y*x) = 1 - \nu_A(y*x) \geq 1 - max\{\nu_A(z*y), \nu_A(z*x)\}$
$= min\{1 - \nu_A(z*y), 1 - \nu_A(z*x)\}$ (By Lemma 2.11(3))
$= min\{\overline{\nu}_A(z*y), \overline{\nu}_A(z*x)\}$.
Hence $\overline{\nu}_A$ is a fuzzy PMS-ideal of X.
Conversely, assume that $\mu_A$ and $\overline{\nu}_A$ are fuzzy PMS-ideals of $X$. Then for every $x, y, z \in X$, we get $\mu_A(0) \geq \mu_A(x)$ and $\overline{\nu}_A(0) \geq \overline{\nu}_A(x)$ (By Definition 2.7)
Now, $\overline{\nu}_A(0) \geq \overline{\nu}_A(x) \Rightarrow 1 - \nu_A(0) \geq 1 - \nu_A(x)$
$\Rightarrow \nu_A(0) \leq \nu_A(x)$
Also, $\mu_A(y*x) \geq min\{\mu_A(z*y), \mu_A(z*x)\}$ and $\overline{\nu}_A(y*x) \geq min\{\overline{\nu}_A(z*y), \overline{\nu}_A(z*x)\}$ (By Definition 2.7)

$$\begin{aligned} 1 - \nu_A(y*x) = \overline{\nu}_A(y*x) &\geq min\{\overline{\nu}_A(z*y), \overline{\nu}_A(z*x)\} \\ &= min\{1 - \nu_A(z*y), 1 - \nu_A(z*x)\} \\ &= 1 - max\{\nu_A(z*y), \nu_A(z*x)\}. \text{ (By Lemma 2.11(3)} \\ \Rightarrow \nu_A(y*x) &\leq max\{\nu_A(z*y), \nu_A(z*x)\}. \end{aligned}$$

Hence $A = (\mu_A, \nu_A)$ is an intuitionistic fuzzy PMS-ideal of $X$. ■

**Corollary 3.10.** *If $\mu_A$ is a fuzzy PMS-subalgebra of $X$, then $A = (\mu_A, \bar{\mu_A})$ is an intuitionistic fuzzy PMS-ideal of $X$.*

*Proof.* Suppose $\mu_A$ is a fuzzy PMS-ideal of $X$. Then we want to show that $A = (\mu_A, \bar{\mu_A})$ is an intuitionistic fuzzy PMS-ideal of $X$. Since $\mu_A$ is a fuzzy PMS-ideal of $X$, it follows that $\mu_A(0) \geq \mu_A(x)$ and $\mu_A(y*x) \geq min\{\mu_A(z*y), \mu_A(z*x)\}$ for all $x, y, z \in X$. Then it suffices to show that $\bar{\mu_A}(0) \leq \bar{\mu_A}(x)$ and $\bar{\mu_A}(y*x) \leq max\{\bar{\mu_A}(z*y), \bar{\mu_A}(z*x)\}$.
Now, $\bar{\mu_A}(0) = 1 - \mu_A(0) \leq 1 - \mu_A(x) = \bar{\mu_A}(x) \Rightarrow \bar{\mu_A}(0) \leq \bar{\mu_A}(x)$ and

$$\begin{aligned} \bar{\mu_A}(y*x) = 1 - \mu_A(y*x) &\leq 1 - min\{\mu_A(z*y), \mu_A(z*x)\} \\ &= max\{1 - \mu_A(z*y), 1 - \mu_A(z*x)\} \\ &= max\{\bar{\mu_A}(z*y), \bar{\mu_A}(z*x)\}. \\ \Rightarrow \bar{\mu_A}(y*x) &\leq max\{\bar{\mu_A}(z*y), \bar{\mu_A}(z*x)\}. \end{aligned}$$

Hence $A = (\mu_A, \bar{\mu_A})$ is an intuitionistic fuzzy PMS-ideal of $X$. ■

**Corollary 3.11.** *If $\bar{\nu_A}$ is a fuzzy PMS-ideal of $X$, then $A = (\bar{\nu_A}, \nu_A)$ is an intuitionistic fuzzy PMS-ideal of $X$.*

*Proof.* Similar to Corollary 3.10 ■

**Theorem 3.12.** *An intuitionistic fuzzy set $A = (\mu_A, \nu_A)$ is an intuitionistic fuzzy PMS-ideal of $X$ if and only if the intuitionistic fuzzy subset $\Box A = (\mu_A, \overline{\mu}_A)$ and $\Diamond A = (\overline{\nu}_A, \nu_A)$ are intuitionistic fuzzy PMS-ideals of $X$.*

*Proof.* Assume that $A = (\mu_A, \nu_A)$ is an intuitionistic fuzzy PMS-ideal of $X$. Then for any $x, y, z \in X$, we have $\mu_A(0) \geq \mu_A(x)$ and $\mu_A(y*x) \geq min\{\mu_A(z*y), \mu_A(z*x)\}$. Next we have to show that $\overline{\mu}_A$ satisfies the conditions

$$\overline{\mu}_A(0) \leq \overline{\mu}_A(x) \text{ and } \overline{\mu}_A(y*x) \leq max\{\overline{\mu}_A(z*y), \overline{\mu}_A(z*x)\}.$$

Thus for any $x, y, z \in X$, we have $\overline{\mu}_A(0) = 1 - \mu_A(0) \leq 1 - \mu_A(x) = \overline{\mu}_A(x)$ and

$$\begin{aligned}\overline{\mu}_A(y*x) &= 1 - \mu_A(y*x) \\ &\leq 1 - min\{\mu_A(z*y), \mu_A(z*x)\} \\ &= max\{1 - \mu_A(z*y), 1 - \mu_A(z*x)\} \quad \text{(By Lemma 2.11(2))} \\ &= max\{\overline{\mu}_A(z*y), \overline{\mu}_A(z*x)\}.\end{aligned}$$

Hence $\Box A$ is an intuitionistic fuzzy PMS-ideal of $X$.

Also, for any $x, y, z \in X$, we have $\nu_A(y*x) \leq max\{\mu_A(z*y), \nu_A(z*x)\}$. Now we have to show that $\overline{\nu}_A$ satisfies the conditions

$$\overline{\nu}_A(0) \geq \overline{\nu}_A(x) \text{ and } \overline{\nu}_A(y*x) \geq min\{\overline{\nu}_A(z*y), \overline{\nu}_A(z*x)\}.$$

So for any $x, y, z \in X$, we have $\overline{\nu}_A(0) = 1 - \nu_A(0) \geq 1 - \nu_A(x) = \overline{\nu}_A(x)$ and

$$\begin{aligned}\overline{\nu}_A(y*x) &= 1 - \nu_A(y*x) \\ &\geq 1 - max\{\nu_A(z*y), \nu_A(z*x)\} \\ &= min\{1 - \nu_A(z*y), 1 - \nu_A(z*x)\} \quad \text{(By Lemma 2.11(3))} \\ &= min\{\overline{\nu}_A(z*y), \overline{\nu}_A(z*x)\}.\end{aligned}$$

Hence $\Diamond A$ is an intuitionistic fuzzy PMS-ideal of $X$.

The proof of the converse follows from Definition 3.1 ∎

**Theorem 3.13.** *If $A = (\mu_A, \nu_A)$ is an intuitionistic fuzzy PMS-ideal of $X$, then $\bar{A}^c$ is also an intuitionistic fuzzy PMS-ideal of $X$.*

*Proof.* Let $A = (\mu_A, \nu_A)$ be an intuitionistic fuzzy PMS-ideal of $X$. Then $A^c = \{\langle x, \mu_{A^c}(x), \nu_{A^c}(x)\rangle | x \in X\}$ where $\mu_{A^c}(x) = 1 - \mu_A(x)$ and $\nu_{A^c}(x) = 1 - \nu_A(x)$ Therefore, $\bar{A}^c = \{\langle x, 1 - \nu_A(x), 1 - \mu_A(x)\rangle | x \in X\}$. Now for any $x, y, z \in X$, we have

$$\nu_{A^c}(0) = 1 - \nu_A(0) \geq 1 - \nu_A(x) = \nu_{A^c}(x) \Rightarrow \nu_{A^c}(0) \geq \nu_{A^c}(x)$$

and

$$\mu_{A^c}(0) = 1 - \mu_A(0) \leq 1 - \mu_A(x) = \mu_{A^c}(x) \Rightarrow \mu_{A^c}(0) \leq \mu_{A^c}(x).$$

$$\begin{aligned}\nu_{A^c}(y*x) = 1 - \nu_A(y*x) &\geq 1 - max\{\nu_A(z*y), \nu_A(z*x)\} \\ &= min\{1 - \nu_A(z*y), 1 - \nu_A(z*x)\} \\ &= min\{\nu_{A^c}(z*y), \nu_{A^c}(z*x)\} \\ \Rightarrow \nu_{A^c}(y*x) &\geq min\{\nu_{A^c}(z*y), \nu_{A^c}(z*x)\}\end{aligned}$$

and

$$\begin{aligned}\mu_{A^c}(y*x) = 1 - \mu_A(y*x) &\leq 1 - min\{\mu_A(z*y), \mu_A(z*x)\} \\ &= max\{1 - \mu_A(z*y), 1 - \mu_A(z*x)\} \\ &= max\{\mu_{A^c}(z*y), \mu_{A^c}(z*x)\} \\ \Rightarrow \mu_{A^c}(y*x) &\leq max\{\mu_{A^c}(z*y), \mu_{A^c}(z*x)\}.\end{aligned}$$

Hence then $\bar{A}^c$ is an intuitionistic fuzzy PMS-ideals of $X$. ∎

## 4. Level Subsets of Intuitionistic Fuzzy PMS-ideals

In this section, we study the notion of level subsets of intuitionistic fuzzy PMS-ideals of a PMS-algebra. Characterizations of intuitionistic fuzzy PMS-ideals interms of their level subsets are given.

**Theorem 4.1.** *If $A = (\mu_A, \nu_A)$ be an intuitionistic fuzzy PMS-ideal of $X$, then the sets $X_{\mu_A} = \{x \in X | \mu_A(x) = \mu_A(0)\}$ and $X_{\nu_A} = \{x \in X | \nu_A(x) = \nu_A(0)\}$ are PMS-ideals of $X$.*

*Proof.* Assume that $A = (\mu_A, \nu_A)$ is an intuitionistic fuzzy PMS-ideal of $X$. Clearly, both $X_{\mu_A}$ and $X_{\nu_A}$ contain the element zero.
Let $z * y, z * x \in X_{\mu_A}$ for $x, y, z \in X$. Then $\mu_A(z * y) = \mu_A(0) = \mu_A(z * x)$.
So,
$$\begin{aligned}\mu_A(y * x) &\geq min\{\mu_A(z * y), \mu_A(z * x)\}\\ &= min\{\mu_A(0), \mu_A(0)\} = \mu_A(0).\\ &\Rightarrow \mu_A(y * x) \geq \mu_A(0).\end{aligned}$$
By Definition 3.1 $(i)$, we get that $\mu_A(y * x) = \mu_A(0)$ which implies that $y * x \in X_{\mu_A}$.
Also, let $z * y, z * x \in X_{\nu_A}$ for $x, y, z \in X$. Then $\nu_A(z * y) = \nu_A(0) = \nu_A(z * x)$, and
So,
$$\begin{aligned}\nu_A(y * x) &\leq max\{\nu_A(z * y), \nu_A(z * x)\}\\ &= max\{\nu_A(0), \nu_A(0)\} = \nu_A(0).\\ &\Rightarrow \nu_A(y * x) \leq \nu_A(0).\end{aligned}$$
By Definition 3.1 $(i)$, we get that $\nu_A(y * x) = \nu_A(0)$ which implies that $y * x \in X_{\nu_A}$.
Hence, the sets $X_{\mu_A}$ and $X_{\nu_A}$ are PMS-ideals of $X$. ∎

**Definition 4.2.** For any nonempty subset $S$ of a PMS-algebra $X$, the intuitionistic fuzzy characteristic function of $S$, denoted by $\chi_S = \{\langle x, \mu_{\chi_S}(x), \nu_{\chi_S}(x)\rangle | x \in X\}$ is the intuitionistic fuzzy subset in $X$ where $\mu_{\chi_S} : X \to [0, 1]$ and $\nu_{\chi_S} : X \to [0, 1]$ are fuzzy subsets defined by

$$\mu_{\chi_S}(x) = \begin{cases} 1 & \text{if } x \in S \\ 0 & \text{if } x \notin S \end{cases} \quad \text{and} \quad \nu_{\chi_S}(x) = \begin{cases} 0 & \text{if } x \in S \\ 1 & \text{if } x \notin S. \end{cases}$$

**Theorem 4.3.** *Let $S$ be any nonempty subset of a PMS-algebra $X$. Then the intuitionistic fuzzy characteristic function $\chi_S = \langle \mu_{\chi_S}, \nu_{\chi_S} \rangle$ of $S$ is an intuitionistic fuzzy PMS-ideal of $X$ if and only if $S$ is a PMS-ideal of $X$.*

**Theorem 4.4.** *Let $S$ be a nonempty subset of $X$ and $A = (\mu_A, \nu_A)$ be an intuitionistic fuzzy set in $X$ defined by*

$$\mu_A(x) = \begin{cases} p & \text{if } x \in S \\ q & \text{if } x \notin S \end{cases} \quad \text{and} \quad \nu_A(x) = \begin{cases} r & \text{if } x \in S \\ s & \text{if } x \notin S. \end{cases}$$

*for all $p, q, r, s \in [0, 1]$ with $p \geq q, r \leq s$ and $0 \leq p + r \leq 1, 0 \leq q + s \leq 1$. Then $A$ is an intuitionistic fuzzy PMS-ideal of $X$ if and only if $S$ is a PMS-ideal of $X$.*

*Proof.* Let $A$ be an intuitionistic fuzzy PMS-ideal of $X$. Let $x \in X$ such that $x \in S$. Then $\mu_A(0) \geq \mu_A(x) = p$ and $\nu_A(0) \leq \nu_A(x) = r$. Hence $0 \in S$.
Let $z * y, z * x \in S$, then $\mu_A(y * x) \geq min\{\mu_A(z * y), \mu_A(z * x)\} = min\{p, p\} = p$ and $\nu_A(y * x) \leq max\{\nu_A(z * y), \nu_A(z * x)\} = max\{r, r\} = r$. Hence $y * x \in S$. So, $S$ is a PMS-ideal of $X$.
Conversely, suppose that $S$ is a PMS-ideal of $X$ and let $x \in X$. Since $0 \in S$, $\mu_A(0) = p$ and $\nu_A(0) = r$. Clearly, $p \geq \mu_A(x)$ and $r \leq \nu_A(x)$ for all $x \in X$. Hence $\mu_A(0) \geq \mu_A(x)$

and $\nu_A(0) \leq \nu_A(x)$ for all $x \in X$.
Now consider the following cases.

Case($i$). $z*y, z*x \in S$. Then $y*x \in S$. Thus $\mu_A(y*x) = p = min\{\mu_A(z*y), \mu_A(z*x)\}$ and $\nu_A(y*x) = r = max\{\nu_A(z*y), \nu_A(z*x)\}$.

Case($ii$). $z*y \notin S$ or $z*x \notin S$. Then $\mu_A(y*x) \geq q = min\{\mu_A(z*y), \mu_A(z*x)\}$ and $\nu_A(y*x) \leq s = max\{\nu_A(z*y), \nu_A(z*x)\}$.

Hence $A = (\mu_A, \nu_A)$ is an intuitionistic fuzzy PMS-ideal of $X$. ■

**Definition 4.5.** For any $t, s \in [0, 1]$ and an intuitionistic fuzzy set $A = (\mu_A, \nu_A)$ in a PMS-algebra $X$, the set $U(\mu_A, t) = \{x \in X | \mu_A(x) \geq t\}$ is called an upper $t$-level set of $A$ and the set $L(\nu_A, s) = \{x \in X | \nu_A(x) \leq s\}$ is called a lower $s$-level set of $A$.

The following theorem characterizes an intuitionistic fuzzy PMS-ideal in terms of its level sets.

**Theorem 4.6.** *An intuitionistic fuzzy subset $A = (\mu_A, \nu_A)$ of a PMS-algebra $X$ is an intuitionistic fuzzy PMS-ideal of $X$ if and only if the nonempty level subsets $U(\mu_A, t)$ and $L(\nu_A, s)$ of $A$ are PMS-ideals of $X$ for all $t, s \in [0, 1]$ with $0 \leq t + s \leq 1$.*

*Proof.* Let $A = (\mu_A, \nu_A)$ be an intuitionistic fuzzy PMS-ideal of $X$ such that $U(\mu_A, t) \neq \emptyset$ and $L(\nu_A, s) \neq \emptyset$ for all $t, s \in [0, 1]$. Then there exist $a \in U(\mu_A, t)$ and $b \in L(\nu_A, s)$. Thus $\mu_A(a) \geq t$ and $\nu_A(b) \leq s$. Since $A = (\mu_A, \nu_A)$ is an intuitionistic fuzzy PMS-ideal of $X$, we have $\mu_A(0) \geq \mu_A(x)$ and $\nu_A(0) \leq \nu_A(x)$ for all $x \in X$. Thus $\mu_A(0) \geq \mu_A(a) \geq t$ and $\nu_A(0) \leq \nu_A(b) \leq s$. So, $0 \in U(\mu_A, t)$ and $0 \in L(\nu_A, s)$.
Suppose $x, y, z \in X$ such that $z * y, z * x \in U(\mu_A, t)$. Therefore, $\mu_A(z * y) \geq t$ and $\mu_A(z * x) \geq t$. Since $A = (\mu_A, \nu_A)$ is an intuitionistic fuzzy PMS-ideal of $X$, we have
$\mu_A(y * x) \geq min\{\mu_A(z * y), \mu_A(z * x)\} \geq min\{t, t\} = t$.

$$\Rightarrow \mu_A(y * x) \geq t$$
$$\Rightarrow y * x \in U(\mu_A, t).$$

Hence $U(\mu_A, t)$ is a PMS-ideal of X.
Also, let $x, y, z \in X$ such that $z * y, z * x \in L(\nu_A, s)$.Therefore, $\nu_A(z * y) \leq s$ and $\nu_A(z * x) \leq s$. Since $A = (\mu_A, \nu_A)$ is an intuitionistic fuzzy PMS-ideal of $X$, we have
$\nu_A(y * x) \leq max\{\nu_A(z * y), \nu_A(z * x)\} \leq max\{s, s\} = s$.

$$\Rightarrow \nu_A(y * x) \leq s$$
$$\Rightarrow y * x \in L(\nu_A, s).$$

Hence $L(\nu_A, s)$ is a PMS-ideal of $X$.
Conversely, assume that $U(\mu_A, t)$ and $L(\nu_A, s)$ are PMS-ideals of $X$ for each $s, t \in [0, 1]$. Let $x \in X$ such that $\mu_A(x) = t$ and $\nu_A(x) = s$. Since $U(\mu_A, t)$ and $L(\nu_A, s)$ are PMS-ideals of $X$, we have $0 \in U(\mu_A, t)$ and $0 \in L(\nu_A, s)$. Then, $\mu_A(0) \geq t = \mu_A(x)$ and $\nu_A(0) \leq s = \nu_A(x)$. Hence $\mu_A(0) \geq \mu_A(x)$ and $\nu_A(0) \leq \nu_A(x)$ for all $x \in X$. Next we need to show that $\mu_A(y * x) \geq min\{\mu_A(z * y), \mu_A(z * x)\}$ for all $x, y, z \in X$. Assume on contrary that $\mu_A(y * x) \geq min\{\mu_A(z * y), \mu_A(z * x)\}$ is not true. Then there exist $x_0, y_0, z_0 \in X$ such that $\mu_A(y_0 * x_0) < min\{\mu_A(z_0 * y_0), \mu_A(z_0 * x_0)\}$.

Take $t = \frac{1}{2}[\mu_A(y_0 * x_0) + min\{\mu_A(z_0 * y_0), \mu_A(z_0 * x_0)\}]$

Thus $t \in [0, 1]$ and $\mu_A(y_0 * x_0) < t < min\{\mu_A(z_0 * y_0), \mu_A(z_0 * x_0)\}$.

$\Rightarrow \mu_A(y_0 * x_0) < t, \mu_A(z_0 * y_0) > t$ and $\mu_A(z_0 * x_0) > t$

$\Rightarrow z_0 * y_0, z_0 * x_0 \in U(\mu_A, t)$ but $y_0 * x_0 \notin U(\mu_A, t)$, which is a contradiction, since $U(\mu_A, t)$ is a PMS-ideal of $X$.

Therefore, $\mu_A(y*x) \geq min\{\mu_A(z*y), \mu_A(z*x)\}$ for all $x, y, z \in X$.
Similarly, assume that $\nu_A(y*x) \leq max\{\nu_A(z*y), \nu_A(z*x)\}$ is not true. Then there exist $a, b, c \in X$ such that $\nu_A(b*a) > max\{\nu_A(c*b), \nu_A(c*a)\}$. Then by taking s$=\frac{1}{2}[\nu_A(b*a)+max\{\nu_A(c*b), \nu_A(c*a)\}]$, we get, $max\{\nu_A(c*b), \nu_A(c*a)\} < s < \nu_A(b*a)$. Therefore, $c*b, c*a \in L(\nu_A, s)$ but $(b*a) \notin L(\nu_A, s)$, which makes a contradiction since $L(\nu_A, s)$ is a PMS-ideal of $X$. Thus $\nu_A(y*x) \leq max\{\nu_A(z*y), \nu_A(z*x)\}$ for all $x, y, z \in X$. Therefore $A = (\mu_A, \nu_A)$ is an intuitionistic fuzzy PMS-ideal of $X$. ∎

**Definition 4.7.** Let $X$ be a PMS-algebra and $A = (\mu_A, \nu_A)$ be an intuitionistic fuzzy PMS-ideal of $X$, for $t, s \in [0,1]$ with $0 \leq t+s \leq 1$, the ideals $U(\mu_A, t) = \{x \in X : \mu_A(x) \geq t\}$ and $L(\nu_A, t) = \{x \in X : \nu_A(x) \leq s\}$ are called the level PMS-ideals of $X$.

**Corollary 4.8.** *An intuitionistic fuzzy subset $A = (\mu_A, \nu_A)$ of a PMS-algebra $X$ is an intuitionistic fuzzy PMS-ideal if and only if the level subsets $U(\mu_A, t)$ and $L(\nu_A, s)$ are PMS-ideals of $X$ for all $t \in Im(\mu_A), s \in Im(\nu_A)$ with $0 \leq t+s \leq 1$.*

*Proof.* Assume that $A = (\mu_A, \nu_A)$ is an intuitionistic fuzzy PMS-ideal of $X$. Clearly, $U(\mu_A, t) \neq \emptyset$ and $L(\nu_A, s) \neq \emptyset$ , Then there exist $a \in U(\mu_A, t)$ and $b \in L(\nu_A, s)$ such that $\mu_A(a) \geq t$ and $\nu_A(b) \leq s$. Since $A = (\mu_A, \nu_A)$ is an intuitionistic fuzzy PMS-ideal we have $\mu_A(0) \geq \mu_A(x)$ and $\nu_A(0) \leq \nu_A(x), \forall x \in X$. This implies $\mu_A(0) \geq \mu_A(a) \geq t$ and $\nu_A(0) \leq \nu_A(b) \leq s$. Hence $0 \in U(\mu_A, t)$ and $0 \in (\nu_A, s)$.
Let $x, y, z \in X$ such that $z*y, z*x \in U(\mu_A, t)$ . Then, $\mu_A(z*y) \geq t$ and $\mu_A(z*x) \geq t$. Since $A = (\mu_A, \nu_A)$ is an intuitionistic fuzzy PMS-ideal of $X$, we have

$$\begin{aligned}\mu_A(y*x) &\geq min\{\mu_A(z*y), \mu_A(z*x)\} \geq min\{t,t\} = t.\\ &\Rightarrow \mu_A(y*x) \geq t\\ &\Rightarrow y*x \in U(\mu_A, t).\end{aligned}$$

Hence $U(\mu_A, t)$ is a PMS-ideal of $X$.
Also, let $x, y, z \in X$ such that $z*y, z*x \in L(\nu_A, s)$ . Then, $\nu_A(z*y) \leq s$ and $\nu_A(z*x) \leq s$. Since $A = (\mu_A, \nu_A)$ is an intuitionistic fuzzy PMS-ideal of $X$, we have

$$\begin{aligned}\nu_A(y*x) &\leq max\{\nu_A(z*y), \nu_A(z*x)\} \leq max\{s,s\} = s.\\ &\Rightarrow \nu_A(y*x) \leq s\\ &\Rightarrow y*x \in L(\nu_A, s).\end{aligned}$$

Hence $L(\nu_A, s)$ is a PMS-ideal of $X$.
Conversely assume that, the level subset $U(\mu_A, t)$ and $L(\nu_A, s)$ is a PMS-ideal of $X$ for any $t \in Im(\mu_A)$ and $s \in Im(\nu_A)$ with $0 \leq t+s \leq 1$. Then $0 \in U(\mu_A, t)$ and $0 \in (\nu_A, s)$. Let $x \in X$ such that $\mu_A(x) = t$ and $\nu_A(x) = s$. So we have $\mu_A(0) \geq t = \mu_A(x)$ and $\mu_A(0) \leq s = \nu_A(x)$. Therefore, $\mu_A(0) \geq \mu_A(x)$ and $\mu_A(0) \leq \nu_A(x)$ for all $x \in X$.
Let $x, y, z \in X$ and let $t \in Im(\mu_A)$ such that $t = min\{\mu_A(z*y), \mu_A(z*x)\}$.
Therefore, $\mu_A(z*y) \geq t$ and $\mu_A(z*x) \geq t \Rightarrow z*y, z*x \in U(\mu_A, t)$.
Since $U(\mu_A, t)$ is a PMS-ideal of $X$, we have $y*x \in U(\mu_A, t)$.

$$\Rightarrow \mu_A(y*x) \geq t = min\{\mu_A(z*y), \mu_A(z*x)\}.$$

Also, let $x, y, z \in X$ and let $s \in Im(\nu_A)$ such that $s = max\{\nu_A(z*y), \nu_A(z*x)\}$.
Therefore, $\nu_A(z*y) \leq s$ and $\nu_A(z*x) \leq s \Rightarrow z*y, z*x \in U(\nu_A, s)$.
Since $L(\nu_A, s)$ is a PMS-ideal of $X$, we have $y*x \in L(\nu_A, s)$.

$$\Rightarrow \nu_A(y*x) \leq s = max\{\nu_A(z*y), \nu_A(z*x)\}.$$

Hence $A = (\mu_A, \nu_A)$ is an intuitionistic fuzzy PMS-ideal of $X$. ∎

The following Theorem shows that every PMS-ideal can be characterized as level PMS-ideal of an intuitionistic fuzzy PMS-ideal.

**Theorem 4.9.** *Every PMS-ideal of $X$ is a level PMS-ideal of an intuitionistic fuzzy PMS-ideal $A = (\mu_A, \nu_A)$ of $X$.*

*Proof.* Let $S$ be a PMS-ideal of a PMS -algebra $X$ and $A = (\mu_A, \nu_A)$ be an intuitionistic fuzzy set in $X$ defined by

$$\mu_A(x) = \begin{cases} t & \text{if } x \in S \\ 0 & \text{if } x \notin S \end{cases} \quad \text{and} \quad \nu_A(x) = \begin{cases} s & \text{if } x \in S \\ 1 & \text{if } x \notin S \end{cases} \quad \forall\, t, s \in [0,1], t + s \le 1.$$

Clearly $U(\mu_A, t) = S = L(\nu_A, s)$. Since $0 \in S$, we have $\mu_A(0) = t$ and $\nu_A(0) = s$.
Thus $\mu_A(0) \ge \mu_A(x)$ and $\nu_A(0) \le \nu_A(x)$ for all $x \in X$.
Now, consider the following cases, to prove that $\mu_A(y * x) \ge min\{\mu_A(z * y), \mu_A(z * x)\}$ and $\nu_A(x * y) \le max\{\nu_A(z * y), \nu_A(z * x)\}$ for all $x, y, z \in X$.

Case $(i)$. If $z * y, z * x \in S$, then $y * x \in S$, since $S$ is a PMS-ideal of a PMS-algebra $X$. Then $\mu_A(z*y) = \mu_A(z*x) = \mu_A(y*x) = t$ and $\nu_A(z*y) = \nu_A(z*x) = \nu_A(y*x) = s$. Hence $\mu_A(y*x) = min\{\mu_A(z*y), \mu_A(z*x)\}$ and $\nu_A(y*x) = max\{\nu_A(z*y), \nu_A(z*x)\}$.

Case $(ii)$. If $z*y \in S, z*x \notin S$, then we have $\mu_A(z*y) = t, \mu_A(z*x) = 0$ and $\nu_A(z*y) = s$, $\nu_A(z * x) = 1$. Then $\mu_A(y * x) \ge 0 = min\{t, 0\} = min\{\mu_A(z * y), \mu_A(z * x)\}$ and $\nu_A(y * x) \le 1 = max\{t, 1\} = max\{\nu_A(z * y), \nu_A(z * x)\}$. Thus, $\mu_A(y * x) \ge min\{\mu_A(z * y), \mu_A(z * x)\}$ and $\nu_A(y * x) \le max\{\nu_A(z * y), \nu_A(z * x)\}$.

Case $(iii)$. If $z * y \notin S, z * x \in S$, then we get similar result as in Case (ii).

Case $(iv)$. If $z * y, z * x \notin S$, then $\mu_A(z * y) = 0 = \mu_A(z * x)$ and $\nu_A(z * y) = 1 = \nu_A(z * x)$.Then $\mu_A(x*y) \ge 0 = min\{\mu_A(z*y), \mu_A(z*x)\}$ and $\nu_A(x*y) \le 1 = max\{\nu_A(z*y), \nu_A(z * x)\}$. So, in all cases we get $\mu_A(y * x) \ge min\{\mu_A(z * y), \mu_A(z * x)\}$ and $\nu_A(x * y) \le max\{\nu_A(z * y), \nu_A(z * x)\}$ for all $x, y, z \in X$.

Thus, $A = (\mu_A, \nu_A)$ is an intuitionistic fuzzy PMS-ideal of a PMS-algebra $X$. Hence S is a level PMS-ideal of $X$ corresponding to an intuitionistic fuzzy PMS-ideal $A = (\mu_A, \nu_A)$ of $X$. ■

We can also prove the following Theorem as a generalization of the above Theorem.

**Theorem 4.10.** *Let $\{S_i : i = 0, 1, 2, ..., n\}$ be any family of a PMS-ideal of a PMS-algebra $X$ such that $S_0 \subset S_1 \subset S_3 \subset ... \subset S_n = X$, then there exists an intuitionistic fuzzy PMS-ideal $A = (\mu_A, \nu_A)$ of $X$ whose level PMS-ideals are exactly the PMS-ideals $\{S_i\}$ of $X$.*

*Proof.* Consider a set of numbers $t_0 > t_1 > ... > t_n$ and $s_0 < s_1 < ... < s_n$ where each $t_i, s_i \in [0,1]$ with $0 \le t_i + s_i \le 1$. Let $A = (\mu_A, \nu_A)$ be an intuitionistic fuzzy set defined by

$$\mu_A(x) = \begin{cases} t_0 & \text{if } x \in S_0 \\ t_i & \text{if } x \in S_i - S_{i-1} \end{cases} \quad \text{and} \quad \nu_A(x) = \begin{cases} s_0 & \text{if } x \in S_0 \\ s_i & \text{if } x \in S_i - S_{i-1} \end{cases} \quad 0 < i \le n.$$

Clearly, $0 \in S_i$ for $i = 0, 1, 2, , n$, since each $S_i$ is a PMS-ideal of a PMS-algebra $X$. So, $\mu_A(0) = t_0 = \mu_A(x)$ and $\nu_A(0) = s_0 = \nu_A(x)$ if $x \in S_0$, and $\mu_A(0) = t_i = \mu_A(x)$ and $\nu_A(0) = s_i = \nu_A(x)$ if $x \in S_i - S_{i-1}$. In any case $\mu_A(0) \ge \mu_A(x)$ and $\nu_A(0) \le \nu_A(x)$, for all $x \in X$. To show that $\mu_A(y * x) \ge min\{\mu_A(z * y), \mu_A(z * x)\}$ and $\nu_A(y * x) \le max\{\nu_A(z * y), \nu_A(z * x)\}$, we consider the following two cases.

Case $(i)$. Let $z * y, z * x \in S_i - S_{i-1}$. Therefore, $\mu_A(z * y) = \mu_A(z * x) = t_i$ and $\nu_A(z*y) = \nu_A(z*x) = s_i$. Since $S_i$ is a PMS-ideal of $X$, we have $y*x \in S_i$ and so either $y*x \in S_i - S_{i-1}$ or $y*x \in S_{i-1}$. In any case we can conclude that $\mu_A(y*x) \geq t_i = min\{\mu_A(z * y), \mu_A(z * x)\}$ and $\nu_A(y * x) \leq s_i = max\{\nu_A(z * y), \nu_A(z * x)\}$.

Case $(ii)$. Let $z * y \in S_i - S_{i-1}$ and $z * x \in S_j - S_{j-1}$ for $i > j$. Therefore, $\mu_A(z * y) = t_i$ and $\mu_A(z * x) = t_j$ and $\nu_A(z * y) = s_i$ and $\nu_A(z * x) = s_j$. Thus $z * y \in S_i$ and $z * x \in S_j$. Since $S_j \subseteq S_i$ it follows that $z * x \in S_i$ and thus $y * x \in S_i$ since $S_i$ is a PMS-ideal of $X$. Hence $\mu_A(y * x) \geq t_i = min\{\mu_A(z * y), \mu_A(z * x)\}$ and $\nu_A(y*x) \leq s_i = max\{\nu_A(z*y), \nu_A(z*x)\}$. Thus $A = (\mu_A, \nu_A)$ is an intuitionistic fuzzy PMS-ideal of $X$.

Also, from the definition of $A = (\mu_A, \nu_A)$, we observe that

$Im(\mu_A) = \{t_0, t_1, ..., t_n\}$ and $Im(\nu_A) = \{s_0, s_1, ..., s_n\}$.

So, the level ideals of $A = (\mu_A, \nu_A)$ are given by the chain of PMS-ideals.
$U(\mu_A, t_0) \subset U(\mu_A, t_1) \subset ... \subset U(\mu_A, t_n) = X$ and $L(\nu_A, s_0) \subset ... \subset (\nu_A, s_n) = X$.
Now $U(\mu_A, t_0) = \{x \in X | \mu_A(x) \geq t_0\} = S_0 = \{x \in X | \nu_A(x) \leq s_0\} = L(\nu_A, s_0)$.
Finally, we have to prove that $U(\mu_A, t_i) = S_i = L(\nu_A, s_i)$ for $0 < i \leq n$.
Now let $x \in S_i$, then $\mu_A(x) = t_i$ and $\nu_A(x) = s_i$. This implies $x \in U(\mu_A, t_i)$ and $x \in L(\nu_A, s_i)$. Hence $S_i \subseteq U(\mu_A, t_i)$ and $S_i \subseteq L(\nu_A, s_i)$. If $x \in U(\mu_A, t_i)$ and $x \in L(\nu_A, s_i)$, then $\mu_A(x) \geq t_i$ and $\nu_A(x) \leq s_i$ which implies that $x \notin S_j$ for $j > i$. Otherwise, if $x \in S_j$ for $j > i$, we get $\mu_A(x) \geq t_j$ and $\nu_A(x) \leq s_j$. This gives $t_i > \mu_A(x) \geq t_j$ and $s_i < \nu_A(x) \leq s_j$, which is a contradiction to the assumption that $x \in U(\mu_A, t_i)$ and $x \in L(\nu_A, s_i)$. Hence $\mu_A(x) \in \{t_0, t_1, ..., t_n\}$ and $\nu_A(x) \in \{s_0, s_1, ..., s_n\}$. So, $x \in S_k$ for some $k \leq i$. As $S_k \subseteq S_i$, it follows that $x \in S_i$, which implies $U(\mu_A, t_i) \subseteq S_i$ and $L(\nu_A, s_i) \subseteq S_i$.

Hence $U(\mu_A, t_i) = S_i = L(\nu_A, s_i)$ for $0 < i \leq n$. ∎

Note that, if $X$ is a finite PMS-algebra, then the number of PMS-ideals of $X$ is finite, whereas the number of level PMS-ideal of a fuzzy PMS-ideal $A$ appears to be infinite. But, since every level PMS-ideal is indeed a PMS-ideal of $X$, not all these level PMS-ideals are distinct. This condition is characterized by the next theorem.

**Theorem 4.11.** *Let $A = (\mu_A, \nu_A)$ be an intuitionistic fuzzy PMS-ideal of $X$. Then*

*(i). the upper level PMS-ideals $U(\mu_A, t_1)$ and $U(\mu_A, t_2)$,(with $t_1 < t_2$) of an intuitionistic fuzzy PMS-ideal $A$ of $X$ are equal if and only if there is no $x \in X$ such that $t_1 \leq \mu_A(x) < t_2$, and*

*(ii). the lower level PMS-ideals $L(\nu_A, s_1)$ and $L(\nu_A, s_2)$,(with $s_1 > s_2$) of an intuitionistic fuzzy PMS-ideal $A$ of $X$ are equal if and only if there is no $x \in X$ such that $s_1 \geq \nu_A(x) > s_2$.*

*Proof.* $(i)$ Suppose that $A = (\mu_A, \nu_A)$ be an intuitionistic fuzzy PMS-ideal of $X$ and $U(\mu_A, t_1) = U(\mu_A, t_2)$ for some $t_1 < t_2$. Assume that there exists $x \in X$ such that $t_1 \leq \mu_A(x) < t_2$. This implies $x \in (\mu_A, t_1)$ but $x \notin U(\mu_A, t_2)$, which contradicts the assumption $U(\mu_A, t_1) = U(\mu_A, t_2)$. Hence there is no $x \in X$ such that $t_1 \leq \mu_A(x) < t_2$.
Conversely, suppose there is no $x \in X$ such that $t_1 \leq \mu_A(x) < t_2$. Since $t_1 < t_2$, we get

$$U(\mu_A, t_2) \subseteq (\mu_A, t_1). \tag{1}$$

Now, $x \in (\mu_A, t_1) \Rightarrow \mu_A(x) \geq t_1$. So $\mu_A(x) \geq t_2$, because $\mu_A(x)$ does not lie between $t_1$ and $t_2$. Hence, $x \in U(\mu_A, t_2)$. Therefore,

$$U(\mu_A, t_1) \subseteq U(\mu_A, t_2). \tag{2}$$

Thus, from (1) and (2), we get $U(\mu_A, t_1) = U(\mu_A, t_2)$.
$(ii)$. The proof of $(ii)$ is similar to $(i)$. ■

**Corollary 4.12.** *Let $A = (\mu_A, \nu_A)$ be an intuitionistic fuzzy PMS-ideals of $X$ with finite images. If $U(\mu_A, t_1) = U(\mu_A, t_2)$ and $L(\nu_A, s_1) = L(\nu_A, s_2)$, for any $t_1, t_2 \in Im(\mu_A)$ and $s_1, s_2 \in Im(\nu_A)$, then $t_1 = t_2$ and $s_1 = s_2$.*

*Proof.* Let $A = (\mu_A, \nu_A)$ be an intuitionistic fuzzy PMS-ideal of $X$ with finite images such that $U(\mu_A, t_1) = U(\mu_A, t_2)$ and $L(\nu_A, s_1) = L(\nu_A, s_2)$, for some $t_1, t_2 \in Im(\mu_A)$ and $s_1, s_2 \in Im(\nu_A)$. So we need to show that $t_1 = t_2$ and $s_1 = s_2$.
Assume on contrary that $t_1 \neq t_2$ and $s_1 \neq s_2$. Without loss of generality assume that $t_1 < t_2$ and $s_1 > s_2$.
Let $x \in U(\mu_A, t_2)$, then $\mu_A(x) \geq t_2 > t_1$.
$\Rightarrow \mu_A(x) > t_1$
Hence $x \in U(\mu_A, t_1)$.
Let $x \in X$ such that $t_1 < \mu_A(x) < t_2$. Then $x \in U(\mu_A, t_1)$ but $x \notin U(\mu_A, t_2)$
$\Rightarrow U(\mu_A, t_2) \subset U(\mu_A, t_1)$
$\Rightarrow U(\mu_A, t_1) \neq U(\mu_A, t_2)$ which contradicts the hypothesis that $U(\mu_A, t_1) = U(\mu_A, t_2)$.
Therefore, $t_1 = t_2$. Similarly, we prove that $s_1 = s_2$. ■

**Theorem 4.13.** *Let $A = (\mu_A, \nu_A)$ be an intuitionistic fuzzy PMS-ideal of $X$ and let $x \in X$. Then*

*(i). $\mu_A(x) = t_1$ if and only if $x \in U(\mu_A, t_1)$ but $x \notin U(\mu_A, t_2), \forall\, t_2 > t_1$,*
*(ii). $\nu_A(x) = s_1$ if and only if $x \in L(\nu_A, s_1)$ but $x \notin L(\nu_A, t_2), \forall\, s_2 < s_1$.*

*Proof.* Let $A = (\mu_A, \nu_A)$ be an intuitionistic fuzzy PMSideal of $X$ and let $x \in X$.

$(i)$. Assume $\mu_A(x) = t_1$. So that $x \in U(\mu_A, t_1)$. If possible, let $x \in U(\mu_A, t_2)$ for $t_2 > t_1$. Then $\mu_A(x) \geq t_2 > t_1$. This contradicts the fact that $\mu_A(x) = t_1$.
Hence $x \in U(\mu_A, t_1)$ but $x \notin U(\mu_A, t_2), \forall\, t_2 > t_1$.
Conversely, let $x \in (\mu_A, t_1)$ but $x \notin U(\mu_A, t_2), \forall\, t_2 > t_1$,
then $x \in U(\mu_A, t_1) \Rightarrow \mu_A(x) \geq t_1$. Since $x \notin U(\mu_A, t_2), \forall\, t_2 > t_1$, we have $\mu_A(x) = t_1$

$(ii)$. Assume $\nu_A(x) = s_1$. So that $x \in L(\nu_A, s_1)$. If possible, let $x \in L(\nu_A, s_2)$ for $s_2 < s_1$. Then $\nu_A(x) \leq s_2 < s_1$. This contradicts the fact that $\nu_A(x) = s_1$.
Hence $x \in L(\nu_A, s_2)$ but $x \notin L(\nu_A, t_2), \forall\, s_2 < s_1$.

Conversely, let $x \in (\nu_A, t_1)$ but $x \notin L(\nu_A, s_2), \forall\, s_2 < s_1$, then $x \in L(\nu_A, s_1)$
$\Rightarrow \nu_A(x) \leq s_1$. Since $x \notin L(\nu_A, s_2), \forall s_2 < s_1$, we have $\nu_A(x) = s_1$. ■

## 5. Conclusion

In this paper, by using the concept of an intuitionistic fuzzy set to a PMS-ideal in PMS-algebra we introduced the notion of an intuitionistic fuzzy PMS-ideal of PMS-algebra along with some fundamental properties and, we established some related results as well. We also defined the level subsets of an intuitionistic fuzzy PMS-ideal of a PMS-algebra and described it in terms of its level subsets in a PMS-algebra.

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