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Fixed Points of Various Types of Operators in Modular Metric Spaces

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Abstract This paper is mainly devoted to the study of fixed point theorems for generalized contractive type mappings over a complete modular metric space. A new approach for obtaining fixed point result using a Cantor's Intersection like Theorem on modular metric spaces has been investigated. The notion of orbital completeness has been exploited in search of fixed point for Caristi-type mapping. We also explore a result on fixed points for Ćirić operator over a complete modular metric space. Further we give a result on common fixed point which extends a result due to Jungck.

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1. Introduction

A new direction in fixed point theory was initiated in 2010 when Chistyakov ([1],[2]) introduced the concept of modular metric spaces. He first developed some theory related to these spaces and then gave some fixed point results [3] on modular metric spaces. This study continued further with Abdou and Khamsi ([4], [5], [6]) proving many new theorems. Mongkolkeha et al. [7] explored contraction mappings in this setting to find fixed point theorems. The theory has evolved with works done by authors like Abdou [8], Abobaker and Ryan [9], Mitrovic et al. [10] and Hussain [11]. Hitherto many researchers have been pursuing the study of fixed points in the realm of modular metric spaces (see [12], [13], [14], [15], [16], [17], [18], [19]).

First we recall briefly the basic concepts and facts in modular metric spaces.

Let X be any arbitrary non-empty set and $\omega : (0, \infty) \times X \times X \to [0, \infty]$ be written as $\omega_\lambda (x, y) = \omega (\lambda, x, y)$, for each $\lambda > 0$ and $x, y \in X$.

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Definition 1.1. [1, Definition 2.1] Let X be a non-empty set and $\omega : (0,\infty) \times X \times X \to [0,\infty]$ satisfy the following:

(1) $\omega_\lambda(x,y) = 0$ for all $\lambda > 0$ if and only if $x = y$;
(2) $\omega_\lambda(x,y) = \omega_\lambda(y,x)$;
(3) $\omega_{\lambda+\mu}(x,z) \le \omega_\lambda(x,y) + \omega_\mu(y,z)$;

for all $\lambda, \mu > 0$ and for all $x, y, z \in X$. Then ω is said to be a metric modular on X.

For $x, y \in X$, the function $0 < \lambda \mapsto \omega_\lambda(x,y)$ is nonincreasing on $(0,\infty)$.

A metric modular ω on X is said to be convex if, instead of (3), it satisfies the inequality

$$\omega_{\lambda+\mu}(x,z) \le \frac{\lambda}{\lambda+\mu}\omega_\lambda(x,y) + \frac{\mu}{\mu+\lambda}\omega_\mu(y,z)$$

for all $\lambda, \mu > 0$ and for all $x, y, z \in X$.

Given a modular ω on X and a point x_0 in X, the two sets

$$X_\omega \equiv X_\omega(x_0) = \{x \in X : \omega_\lambda(x,x_0) \to 0 \text{ as } \lambda \to \infty\}$$

and

$$X^*_\omega \equiv X^*_\omega(x_0) = \{x \in X : \exists\lambda = \lambda(x) > 0 \text{ such that } \omega_\lambda(x,x_0) < \infty\}$$

are each known as modular spaces (around x_0).

In general, $X_\omega(x_0) \subset X^*_\omega(x_0)$. The modular space X_ω can be equipped with a metric d_ω, generated by ω and given by

$$d_\omega(x,y) = \inf\{\lambda > 0 : w_\lambda(x,y) \le \lambda\}, \quad \text{for all } \ x, y \in X_\omega.$$

It is known that d_ω is a well defined metric on X^*_ω also. If ω is convex then $X_\omega(x_0) = X^*_\omega(x_0)$ and this common set can be endowed with a metric d^*_ω given by

$$d^*_\omega(x,y) = \inf\{\lambda > 0 : \omega_\lambda(x,y) \le 1\}, \quad \text{for all } \ x, y \in X^*_\omega.$$

Definition 1.2. Let X_ω be a modular metric space.

(1) A sequence $\{x_n\}$ in X_ω is said to be modular convergent or ω-convergent to an element $x \in X$ if there exists a number $\lambda > 0$, possibly depending on $\{x_n\}$ and x, such that

$$\lim_{n\to\infty} \omega_\lambda(x_n, x) = 0.$$

Here x is called a modular limit of the sequence $\{x_n\}$.

(2) A sequence $\{x_n\}$ in X_ω is said to be modular Cauchy or ω-Cauchy if there exists a number $\lambda > 0$, possibly depending on the sequence, such that

$$\omega_\lambda(x_m, x_n) \to 0 \text{ as } m, n \to \infty.$$

(3) The modular space X_ω is said to be modular complete or ω-complete if every ω-Cauchy sequence from X_ω is ω-convergent.

Let (X,d) be a metric space with at least two points. There are several ways to define a metric modular on X.

Example 1.3. [9, Examples 2.1 - 2.3] We take (X,d) to be a metric space.

(1) Let for all $x, y \in X$, $\omega_\lambda(x,y) = d(x,y)$. In this case, property (3) in the definition of a modular is just the triangle inequality for the metric. This modular is not convex as we can see by taking $z = y$ and $\mu = \lambda$.

(2) Let $\omega_\lambda(x,y) = \frac{d(x,y)}{\lambda}$ for all $\lambda > 0$. In this case, we can think of $\omega_\lambda(x,y)$ as the average velocity required to travel from x to y in time λ. A simple calculation with the triangle inequality shows that this modular is convex.

(3) Let $\omega_\lambda(x,y) = \frac{d(x,y)}{\lambda+d(x,y)}$ for all $\lambda > 0$. It can be shown that this modular is not convex if we take $z = y$ and $\mu = \lambda$.

Example 1.4. [7, Example 3.7] Let $X = \{(a,0) \in \mathbb{R}^2 : 0 \leq a \leq 1\} \cup \{(0,b) \in \mathbb{R}^2 : 0 \leq b \leq 1\}$. Define the mapping $\omega : (0,\infty) \times X \times X \to [0,\infty]$ by

$$\begin{aligned}\omega_\lambda((a_1,0),(a_2,0)) &= \frac{4|a_1-a_2|}{3\lambda},\\ \omega_\lambda((0,b_1),(0,b_2)) &= \frac{|b_1-b_2|}{\lambda},\\ \omega_\lambda((a,0),(0,b)) &= \frac{4a}{3\lambda} + \frac{b}{\lambda} = \omega_\lambda((0,b),(a,0)).\end{aligned}$$

We note that $\omega_\lambda((0,0),(0,0)) = 0$ is satisfied by all three conditions above. Here $X = X_\omega$ and X_ω is a ω-complete modular metric space.

Let ω be a metric modular on X and X_ω be a modular metric space induced by ω. For any $x \in X_\omega$ and $r \geq 0$, the set $\omega\text{-}B_\delta(x) = \{y \in X_\omega : \omega_\delta(x,y) < r; \delta > 0\}$ is called a modular open ball. A modular closed ball is defined as $\omega\text{-}B_\delta[x] = \{y \in X_\omega : \omega_\delta(x,y) \leq r; \delta > 0\}$.

Definition 1.5.

(1) A subset M of X_ω is said to be ω-closed if the ω-limit of any ω-convergent sequence of M is in M.

(2) A subset M of X_ω is said to be ω-bounded if

$$\sup\{\omega_\lambda(x,y) : x,y \in X, \lambda > 0\} < \infty.$$

(3) A function $f : X_\omega \to \mathbb{R}$ is said to be ω-lower semicontinuous at $u \in X_\omega$ if given $\epsilon > 0$, there is a $\delta > 0$ such that

$$f(x) > f(u) - \epsilon \quad \text{for} \quad x \in \omega\text{-}B_\delta(u).$$

2. Main Results

Here we establish a Cantor's Intersection like theorem in a complete modular metric space. We begin with the following definition.

Definition 2.1. The diameter of an ω-bounded subset M of X_ω is denoted by ω-Diam(M) and is defined by

$$\omega\text{-Diam}(M) = \sup\{\omega_\lambda(x,y) : x,y \in X, \lambda > 0\}.$$

Lemma 2.2. *Let ω be a metric modular on X and X_ω be a modular metric space induced by ω. Let F be a ω-bounded subset of X_ω. Then its closure is bounded and ω-Diam$(\bar{F}) = \omega$-Diam(F).*

Proof. Since $F \subseteq \bar{F}$, we have

$$0 \leq \sup\{\omega_\lambda(x,y) : x,y \in F, \lambda > 0\} \leq \sup\{\omega_\lambda(x,y) : x,y \in \bar{F}, \lambda > 0\}$$
$$\Rightarrow \omega\text{-Diam}(F) \leq \omega\text{-Diam}(\bar{F}). \tag{2.1}$$

Let $u, v \in \bar{F}$ be such that $\omega_\lambda(u,v) \geq 0$, for $\lambda > 0$. Then given $\epsilon > 0$, there exist $z_1, z_2 \in F$ to satisfy $\omega_{\frac{\lambda}{3}}(u, z_1) < \frac{\epsilon}{2}$ and $\omega_{\frac{\lambda}{3}}(v, z_2) < \frac{\epsilon}{2}$. Therefore

$$\begin{aligned} \omega_\lambda(u,v) &\leq \omega_{\frac{\lambda}{3}}(u,z_1) + \omega_{\frac{\lambda}{3}}(z_1,z_2) + \omega_{\frac{\lambda}{3}}(z_2,v) \\ &< \epsilon + \omega_{\frac{\lambda}{3}}(z_1,z_2), \\ \Rightarrow \omega_\lambda(u,v) &< \epsilon + \omega\text{-Diam}(F), \\ \Rightarrow \omega\text{-Diam}(\bar{F}) &< \epsilon + \omega\text{-Diam}(F). \end{aligned}$$

As $\epsilon > 0$ is arbitrary, we get

$$\omega\text{-Diam}(\bar{F}) \leq \omega\text{-Diam}(F). \tag{2.2}$$

Combining (2.1) and (2.2), we get ω-Diam$(F) = \omega$-Diam$(\bar{F})$. ■

Theorem 2.3. *(Cantor's Intersection like Theorem) Let ω be a metric modular on X and X_ω be a modular metric space induced by ω. Let $\{F_n\}$ be a monotonically decreasing sequence of non-empty closed subsets of X_ω such that ω-Diam$(F_n) \to 0$ as $n \to \infty$. Then $F = \bigcap_{n=1}^{\infty} F_n$ contains exactly one point if and only if X_ω is a complete modular metric space.*

Proof. We construct a sequence $\{x_n\}$ in X_ω by selecting a point $x_n \in F_n$ for each n. Since the sets $\{F_n\}$ are nested, so $x_n \in F_m$, for all $n \geq m$. Let $\epsilon > 0$ be given. Since ω-Diam$(F_n) \to 0$, there exists a positive integer N such that ω-Diam$(F_N) < \epsilon$. It is clear that for all $n, m \geq N$, $x_n, x_m \in F_N$ and as such we have $\omega_\lambda(x_m, x_n) \leq \omega$-Diam$(F_N) < \epsilon$ for all $n, m \geq N$. Thus $\{x_n\}$ is a ω-Cauchy sequence in X_ω. Since X_ω is a complete modular metric space, so there exists $x \in X_\omega$ such that $x_n \to x$.

We now claim that $x \in \bigcap F_n$.

Let n be fixed. Then the subsequence $\{x_n, x_{n+1}, x_{n+2}, \ldots\}$ of $\{x_n\}$ is contained in F_n and still converges to x. But F_n being a closed subspace of the complete modular metric space X_ω, it is complete and so $x \in F_n$. This is true for each $n \in \mathbb{N}$. Hence $x \in \bigcap F_n$. This shows that $\bigcap F_n$ is nonempty.

Finally to prove that x is the only point in the intersection $\bigcap F_n$. Let $x \in \bigcap F_n$ and $y \in \bigcap F_n$. Then x and y both are in F_n, for each $n \in \mathbb{N}$. Therefore $0 \leq \omega_\lambda(x,y) \leq \omega$-Diam$(F_n) \to 0$ as $n \to \infty$. Then $\omega_\lambda(x,y) = 0 \Rightarrow x = y$. Hence such $x \in X$ is unique and consequently $F = \bigcap F_n$ is a singleton set.

Conversely, let for every decreasing sequence $\{F_n\}$ of non-empty closed sets with ω-Diam$(F_n) \to 0$ as $n \to \infty$ has exactly one point in its intersection. Let $\{x_n\}$ be any ω-Cauchy sequence in X_ω. Let G_n be the range of the sequence $\{x_n, x_{n+1}, x_{n+2}, \ldots\}$. Obviously $G_1 \supseteq G_2 \supseteq G_3 \ldots$ and so $\{x_n\}$ is ω-Cauchy. This yields ω-Diam$(G_n) \to 0$ as $n \to 0$ and hence ω-Diam$(\bar{G}_n) \to 0$ as $n \to 0$. Then by hypothesis, $\bigcap \bar{G}_n$ consists of a single point x (say). Thus

$$\omega_\lambda(x, x_n) \leq \omega\text{-Diam}(\bar{G}_n) \to 0.$$

This gives

$$\omega_\lambda(x, x_n) \to 0 \text{ as } n \to \infty.$$

Hence $\{x_n\}$ converges to x in X_ω. Therefore X_ω is complete. ■

Next we prove a fixed point theorem for a mixed type mapping employing Cantor's Intersection like theorem.

Lemma 2.4. *Let ω be a metric modular on X and X_ω be a modular metric space induced by ω and T be a self mapping on ω satisfying the following condition:*

$$\omega_\lambda(Tx, Ty) \leq \alpha\omega_\lambda(x, Tx) + \beta\omega_\lambda(y, Ty) + \gamma\omega_\lambda(x, y),$$

where $\alpha + \beta + \gamma < 1$ and $\alpha, \beta, \gamma \geq 0$ for all $x, y \in X_\omega$.

Let $\{\alpha_n\}$ be a sequence of reals with $0 < \alpha_n < 1$ for all n and $\lim_{n\to\infty} \alpha_n = 0$. For each $n \in \mathbb{N}$, if the set

$$G_n = \{x \in X_\omega : \omega_\lambda(x, Tx) \leq \alpha_n, \lambda > 0\}$$

is nonempty, then $\{G_n\}$ is a decreasing sequence of sets with ω-Diam$(G_n) \to 0$.

Proof. Clearly, $\{G_n\}$ is a monotone decreasing sequence. Let x, y be elements in G_n so that $\omega_\lambda(x, Tx) \leq \alpha_n$ and $\omega_\lambda(y, Ty) \leq \alpha_n$. Now

$$\begin{aligned}
\omega_{3\lambda}(x, y) &\leq \omega_{2\lambda}(x, Ty) + \omega_\lambda(Ty, y) \\
&\leq \omega_\lambda(x, Tx) + \omega_\lambda(Tx, Ty) + \omega_\lambda(Ty, y) \\
&\leq 2\alpha_n + \omega_\lambda(Tx, Ty) \\
&\leq 2\alpha_n + \alpha\omega_\lambda(x, Tx) + \beta\omega_\lambda(y, Ty) + \gamma\omega_\lambda(x, y) \\
&\leq (2 + \alpha + \beta)\alpha_n + \gamma\omega_\lambda(x, y) \\
&\leq (2 + \alpha + \beta)\alpha_n + \gamma\ \omega\text{-Diam}(G_n).
\end{aligned}$$

Then

$$\sup\{\omega_{3\lambda}(x, y) : x, y \in G_n, \lambda > 0\} \leq (2 + \alpha + \beta)\ \alpha_n + \gamma\ \omega\text{-Diam}(G_n),$$

which gives ω-Diam$(G_n)(1 - \gamma) \leq (2 + \alpha + \beta)\ \alpha_n$, and so

$$\omega\text{-Diam}(G_n) \leq \frac{(2 + \alpha + \beta)}{1 - \gamma}\ \alpha_n \to 0 \text{ as } n \to \infty.$$

■

Lemma 2.5. *Let ω be a metric modular on X and X_ω be a modular metric space induced by ω. If the function $f : X_\omega \to \mathbb{R}^+$ defined by $f(x) = \omega_\lambda(x, Tx)$ is a ω-lower semicontinuous function then the sets G_n as constructed in Lemma 2.4 are ω-closed.*

Proof. It is a consequence of ω-lower semicontinuity property of f. ■

Lemma 2.6. *Let ω be a metric modular on X and X_ω be a modular metric space induced by ω. A function $T : X_\omega \to X_\omega$ be a mapping satisfying the condition of Lemma 2.4. Then $T(G_n) \subset G_n$, where the sets G_n appear there.*

Proof. Let $x \in G_n$. Then $\omega_\lambda(x, Tx) \leq \alpha_n$. Now

$$\begin{aligned}
\omega_\lambda(Tx, T^2x) &= \omega_\lambda(Tx, T(Tx)) \\
&\leq \alpha\omega_\lambda(x, Tx) + \beta\omega_\lambda(Tx, T^2x) + \gamma\omega_\lambda(x, Tx), \\
\text{or, } (1 - \beta)\omega_\lambda(Tx, T^2x) &\leq (\alpha + \gamma)\omega_\lambda(x, Tx), \\
\text{or, } \omega_\lambda(Tx, T^2x) &\leq \frac{\alpha + \gamma}{1 - \beta}\ \omega_\lambda(x, Tx), \\
\text{or, } \omega_\lambda(Tx, T^2x) &\leq \frac{\alpha + \gamma}{1 - \beta}\ \alpha_n < \alpha_n \text{ (since } \alpha + \beta + \gamma < 1).
\end{aligned}$$

Then $Tx \in G_n$ and therefore $T(G_n) \subset G_n$. ■

Theorem 2.7. *Let ω be a metric modular on X and X_ω be a modular metric space induced by ω. Let X_ω be complete and $T : X_\omega \to X_\omega$ be a self mapping on X_ω which satisfies the following conditions :*

(1) $\omega_\lambda(Tx, Ty) \leq \alpha\omega_\lambda(x, Tx) + \beta\omega_\lambda(y, Ty) + \gamma\omega_\lambda(x, y)$, where $\alpha + \beta + \gamma < 1$ and $\alpha, \beta, \gamma \geq 0$, for all $x, y \in X_\omega$, and

(2) $\omega_\lambda(x, Tx)$ is a ω-lower semicontinuous function on X_ω and $\omega_\lambda(x, Tx) < \infty$ for all $x \in X_\omega$.

Then T has a fixed point in X_ω.

Proof. Let $x_0 \in X_\omega$. Also let $x_{n+1} = Tx_n$ for all $n \in \mathbb{N} \cup \{0\}$. So we get

$$\begin{aligned} \omega_\lambda(x_1, x_2) &= \omega_\lambda(Tx_0, Tx_1) \\ &\leq \alpha\omega_\lambda(x_0, Tx_0) + \beta\omega_\lambda(x_1, Tx_1) + \gamma\omega_\lambda(x_0, x_1) \\ &= \alpha\omega_\lambda(x_0, Tx_0) + \beta\omega_\lambda(x_1, Tx_1) + \gamma\omega_\lambda(x_0, x_1), \end{aligned}$$

which implies

$$\omega_\lambda(x_1, x_2) \leq \left(\frac{\alpha+\gamma}{1-\beta}\right) \omega_\lambda(x_0, x_1).$$

Similarly, $\omega_\lambda(x_2, x_3) \leq (\frac{\alpha+\gamma}{1-\beta})^2 \omega_\lambda(x_0, x_1)$ and so on.

Proceeding in this way, we obtain

$$\omega_\lambda(x_n, x_{n+1}) \leq \left(\frac{\alpha+\gamma}{1-\beta}\right)^n \omega_\lambda(x_0, x_1) \to 0 \text{ as } n \to \infty,$$

since $(\frac{\alpha+\gamma}{1-\beta}) < 1$ and $\omega_\lambda(x, Tx) < \infty$ for all $x \in X_\omega$.

Let $\{\alpha_n\}$ be a sequence of real numbers such that $\lim_{n\to\infty} \alpha_n = 0$ where $0 < \alpha_n < 1$ for all n. Let us construct the sets $G_n = \{x \in X_\omega : \omega_\lambda(x, Tx) \leq \alpha_n\}$. Then $G_n \neq \phi$ for all n and by Lemma 2.4, $\{G_n\}$ is monotone decreasing with ω-Diam$(G_n) \to 0$. By condition (2) and Lemma 2.5, it follows that the sets G_n are ω-closed. Now we apply Theorem 2.3 (Cantor's Intersection like Theorem) to obtain $G = \bigcap G_n$ to be a singleton set $\{u\}$ (say). Using Lemma 2.6 we obtain $Tu = u$. Therefore u is a fixed point of T. ∎

When operator $T : X_\omega \to X_\omega$ in this theorem is purely of contractive type, i.e., when $\alpha = \beta = 0$, the hypothesis of completeness of the modular metric space is not redundant as supported by the example below.

Example 2.8. Take $X_\omega = \mathbb{N}$, the set of natural numbers. We take $\omega : (0, \infty) \times X \times X \to [0, \infty)$ given by

$$\omega_\lambda(m, n) = \begin{cases} 0, & \text{if } m = n, \\ \frac{1}{n\lambda} + \frac{1}{m\lambda}, & \text{if } m \neq n. \end{cases}$$

Then X_ω is a modular metric space which is not complete. For, if we consider the sequence $\{n\}$, then $\omega_\lambda(m, n) = \frac{1}{n\lambda} + \frac{1}{m\lambda} \longrightarrow 0$ as $n, m \longrightarrow \infty$, $\lambda > 0$ showing that $\{n\}$ is a Cauchy sequence in X_ω. But for a fixed number n_0,

$$\lim_{n\to\infty} \omega_\lambda(n, n_0) = \lim_{n\to\infty} \left(\frac{1}{n\lambda} + \frac{1}{n_0\lambda}\right) = \frac{1}{n_0\lambda} > 0.$$

But $\lim_{m,n\to\infty} \omega_\lambda(n,m) = 0$. which shows that $\{n\}$ does not ω-converge to any point of $\mathbb{N}$. So X_ω is not ω-complete. Now let us consider a self-mapping T on $\mathbb{N}$ defined by

$$Tn = 2n, \text{ for all } n \in \mathbb{N}.$$

For any $m, n \in \mathbb{N}$, $m \neq n$, we have

$$\omega_\lambda(Tm,Tn) = \frac{1}{2m\lambda} + \frac{1}{2n\lambda} = \frac{1}{2}\omega_\lambda(m,n) < \frac{3}{4}\omega_\lambda(m,n)$$

so that T satisfies the condition (1) of Theorem 2.7 with $\alpha = 0 = \beta$, $\gamma = \frac{3}{4}$. Also condition (2) of Theorem 2.7 is satisfied. But T does not have a fixed point in X_ω.

Let X_ω be modular metric space and $T : X_\omega \to X_\omega$ be a self-map. Let $x \in X_\omega$. The set $O(x) = \{T^n(x), n = 0, 1, 2, 3, \ldots\}$ is called the orbit of x. The mapping T is called orbitally continuous if $\lim_{i\to\infty} T^{n_i}x = z$ implies $\lim_{i\to\infty} TT^{n_i}x = Tz$ for each $x \in X_\omega$. The modular space X_ω is called T-orbitally complete if every ω-Cauchy sequence of the form $\{T^n(x), n = 0, 1, 2, \ldots\}$, $x \in X_\omega$ converges in X_ω.

Theorem 2.9. *Let ω be a metric modular on X and X_ω be a modular metric space induced by ω. Let X_ω be a T-orbitally complete modular metric space and $T : X_\omega \to X_\omega$ be an operator satisfying*

$$\begin{aligned}\omega_\lambda(Tx,Ty) \quad &\leq \quad \alpha[\omega_\lambda(x,Tx) + \omega_\lambda(y,Ty)] + \beta\omega_\lambda(x,y) \\ &\quad\quad +\gamma\max\{\omega_{2\lambda}(x,Ty), \omega_{2\lambda}(y,Tx)\},\end{aligned}$$

for all $x, y \in X_\omega$, where $\alpha, \beta, \gamma \geq 0$ and $(2\alpha + \beta + 2\gamma) < 1$. If $\omega_\lambda(Tx,x)$ is a ω-lower semicontinuous function in X_ω and $\omega_\lambda(Tx,x) < \infty$ for all $x \in X_\omega$, then there exists a point $u \in X_\omega$ such that $Tu = u$.

Proof. Let $x_0 \in X_\omega$ and we construct the iterative sequence $x_n = Tx_{n-1}$, $n = 1, 2, 3, \cdots$. We have

$$\begin{aligned}\omega_\lambda(x_2,x_1) \quad &= \quad \omega_\lambda(Tx_1,Tx_0) \\ &\leq \quad \alpha[\omega_\lambda(x_1,Tx_1) + \omega_\lambda(x_0,Tx_0)] + \beta\omega_\lambda(x_1,x_0) \\ &\quad\quad +\gamma\max\{\omega_{2\lambda}(x_1,Tx_0), \omega_{2\lambda}(x_0,Tx_1)\} \\ &= \quad \alpha[\omega_\lambda(x_1,x_2) + \omega_\lambda(x_0,x_1)] + \beta\omega_\lambda(x_1,x_0) \\ &\quad\quad +\gamma\omega_{2\lambda}(x_0,Tx_1) \\ &\leq \quad \alpha[\omega_\lambda(x_1,x_2) + \omega_\lambda(x_0,x_1)] + \beta\omega_\lambda(x_1,x_0) \\ &\quad\quad +\gamma[\omega_\lambda(x_0,x_1) + \omega_\lambda(x_1,Tx_1)].\end{aligned}$$

It follows that

$$(1 - \alpha - \gamma)\omega_\lambda(x_2,x_1) \leq (\alpha + \beta + \gamma)\omega_\lambda(x_1,x_0)$$

and so

$$\omega_\lambda(x_2,x_1) \leq \left(\frac{\alpha + \beta + \gamma}{1 - \alpha - \gamma}\right)\omega_\lambda(x_1,x_0).$$

Proceeding in this way, we obtain $\omega_\lambda(x_{n+1},x_n) \leq r^n\omega_\lambda(x_1,x_0)$, where $r = \left(\frac{\alpha+\beta+\gamma}{1-\alpha-\gamma}\right) < 1$. So $\omega_\lambda(x_{n+1},x_n) \to 0$ as $n \to \infty$ and $\omega_\lambda(Tx,x) < \infty$.

Now, for every $m, n \in \mathbb{N}$ such that $m > n$, we have

$$\begin{aligned}\omega_\lambda(x_n, x_m) &\leq \omega_{\lambda_1}(x_n, x_{n+1}) + \omega_{\lambda_1}(x_{n+1}, x_{n+2}) + \cdots + \omega_{\lambda_1}(x_{m-1}, x_m)\\ &\leq (r^n + r^{n+1} + \cdots + r^{m-1})\omega_{\lambda_1}(x_1, x_0)\\ &< r^n(1 + r + r^2 + \cdots\cdots)\omega_{\lambda_1}(x_1, x_0)\\ &= \frac{r^n}{1-r}\omega_{\lambda_1}(x_1, x_0),\end{aligned}$$

where $\lambda_1 = \frac{\lambda}{m-n} > 0$. Since $0 \leq r < 1$ and $\omega_\lambda(Tx, x) < \infty$ for all $\lambda > 0$, letting $m, n \longrightarrow \infty$, we conclude that $\{x_n\}$ is a ω-cauchy sequence in X_ω. As X_ω is a ω-complete, there exists a point $u \in X_\omega$ such that $\lim_{n\to\infty} x_n = u$.

Therefore

$$\lim_{m,n\to\infty} \omega_\lambda(x_m, x_n) = \lim_{n\to\infty} \omega_\lambda(x_n, u) = \omega_\lambda(u, u)$$

$$\Rightarrow 0 = \lim_{n\to\infty} \omega_\lambda(x_n, u) = \omega_\lambda(u, u).$$

Since $\omega_\lambda(Tx, x)$ is a ω-lower semicontinuous function on X_ω, given $\epsilon > 0$, we find a $\delta > 0$ such that

$$\omega_\lambda(Tx, x) > \omega_\lambda(Tu, u) - \epsilon, \text{ where } x \in \omega\text{-}B_\delta(u).$$

Now $\lim_{n\to\infty} \omega_\lambda(x_n, u) = 0$ implies that $x_n \in \omega\text{-}B_\delta(u)$ eventually, i.e.,

$$\omega_\lambda(x_n, u) < \delta \text{ for all } n > m_0 \text{ for some } m_0 \in \mathbb{N}.$$

So $\omega_\lambda(Tx_n, x_n) > \omega_\lambda(Tu, u) - \epsilon$ which further implies that

$$\omega_\lambda(Tu, u) < \omega_\lambda(Tx_n, x_n) + \epsilon.$$

As $n \to \infty$, $\omega_\lambda(Tx_n, x_n) \to 0$, we have $\omega_\lambda(u, Tu) \leq \epsilon$. As $\epsilon > 0$ is arbitrary, we have $\omega_\lambda(u, Tu) = 0$ and we get $u = Tu$. Therefore u is a fixed point of T and the proof is complete. ∎

The statement of Caristi like theorem in modular metric spaces is as follows.

Theorem 2.10. *(Caristi like Theorem) Let ω be a metric modular on X and X_ω be a modular metric space induced by ω. Let X_ω be T-orbitally complete, where T is a self mapping on X_ω and let $\phi : X_\omega \to \mathbb{R}^+$, where $\mathbb{R}^+ = \{x \in \mathbb{R} : x > 0\}$ satisfy the following condition :*

$$\omega_\lambda(Tx, x) \leq \phi(x) - \phi(Tx), \forall x \in X_\omega.$$

If T is orbitally continuous at a point $x_0 \in X_\omega$, then $\lim_{n\to\infty} T^n x_0 = u$ for some $u \in X_\omega$ such that $Tu = u$.

Proof. Let $x_0 \in X_\omega$ be arbitrary. Let us consider the orbit

$$O(x_0) = \{T^n(x_0), n = 0, 1, 2, \ldots\}$$

and assume $x_{n+1} \neq x_n$, where $x_n = T^n(x_0)$. Then

$$\begin{aligned}\omega_\lambda(x_{n+1}, x_n) &= \omega_\lambda(Tx_n, x_n)\\ &\leq \phi(x_n) - \phi(Tx_n)\\ &= \phi(x_n) - \phi(x_{n+1}).\end{aligned}$$

Therefore $\sum_{i=1}^{n} \omega_\lambda(x_{i+1}, x_i) \leq \phi(x_1) - \phi(x_{n+1}) \leq \phi(x_1)$. That means the series $\sum_{n=1}^{\infty} \omega_\lambda(x_{n+1}, x_n)$ is ω-convergent. If m, n are two positive integers, and $m > n$ then

$$\omega_{\lambda_1}(x_m, x_n) \leq \omega_\lambda(x_m, x_{m-1}) + \omega_\lambda(x_{m-1}, x_{m-2}) + ... + \omega_\lambda(x_{n+1}, x_n),$$

where $\lambda_1 = (m-n)\lambda > 0$, i.e., $\omega_{\lambda_1}(x_m, x_n) \leq \sum_{i=n}^{m-1} \omega_\lambda(x_{i+1}, x_i)$. Since the series $\sum_{n=1}^{\infty} \omega_\lambda(x_{n+1}, x_n)$ is convergent, so for arbitrary $\epsilon > 0$, there exists a positive integer n_0 such that

$$\sum_{i=n}^{m-1} \omega_\lambda(x_{i+1}, x_i) < \epsilon,$$

when $m > n \geq n_0$. So when $m > n \geq n_0$, we get from the above that $\omega_{\lambda_1}(x_m, x_n) < \epsilon$. This implies that $\{x_n\}$ is an ω-Cauchy sequence in X_ω. Since X_ω is T-orbitally complete, there exists $u \in X_\omega$ such that

$$\lim_{n\to\infty} x_n = u \Rightarrow \lim_{n\to\infty} T^n x_0 = u.$$

Since T is orbitally continuous at x_0, so we have $\lim_{n\to\infty} T(T^n x_0) = Tu$, or, $\lim_{n\to\infty} x_{n+1} = Tu$, i.e., $u = Tu$. Therefore u is a fixed point of T. ∎

The next example justifies the necessity of incorporation of the function ϕ in the above theorem.

Example 2.11. Let $X_\omega = [0, 1]$ and the metric modular $\omega : (0, \infty) \times X \times X \to [0, \infty]$ is defined by

$$\omega_\lambda(x, y) = \max\left\{\frac{x}{\lambda}, \frac{y}{\lambda}\right\}.$$

Let $T : X_\omega \to X_\omega$ be an operator on X_ω where

$$Tx = \begin{cases} 1, & \text{if } x = 0, \\ 0, & \text{otherwise.} \end{cases}$$

Clearly T has no fixed point and T is orbitally continuous at $0 \in X_\omega$.
We suppose that there exists $\phi : X_\omega \longrightarrow \mathbb{R}^+$ which satisfies

$$\omega_\lambda(Tx, x) \leq \phi(x) - \phi(Tx), \text{ for all } x \in X_\omega.$$

Then if we take $x = 0$, we find that

$$\omega_\lambda(1, 0) = \omega_\lambda(T0, 0) \leq \phi(0) - \phi(1), \quad (1)$$

and taking $x = 1$, we get

$$\omega_\lambda(0, 1) \leq \phi(1) - \phi(0). \quad (2)$$

The two inequalities (1) and (2) cannot hold simultaneously. So there is no such function ϕ as wanted in Theorem 2.10.

Also the assumption of orbital continuity of T is not redundant in Caristi like theorem as seen in the example below.

Example 2.12. Let $X_\omega = [0,1]$ and the metric modular be as in Example 2.11. We define an operator $T : X_\omega \to X_\omega$ by

$$Tx = \begin{cases} 1, & \text{if } x = 0, \\ \frac{x}{2}, & \text{if } 0 < x \le 1. \end{cases}$$

Then T has no fixed point in X_ω and T is not orbitally continuous at any $x \in X_\omega$. However there is a function $\phi : X_\omega \longrightarrow \mathbb{R}^+$ which satisfies the condition

$$\omega_\lambda(Tx, x) \le \phi(x) - \phi(Tx), \text{ for all } x \in X,$$

as assumed in Theorem 2.10. Let us take $\phi : X_\omega \to \mathbb{R}^+$ given by

$$\phi(x) = \begin{cases} \frac{3}{\lambda}, & \text{if } x = 0, \\ \frac{2x}{\lambda}, & \text{if } 0 < x \le 1. \end{cases}$$

This auxiliary function ϕ serves the purpose.

Now we deal with a Ćirić operator in this setting and prove a fixed point result in modular metric spaces.

Definition 2.13. An operator $T : X_\omega \to X_\omega$, where X_ω is a modular metric space associated with the metric modular ω on X, is said to be a Ćirić Operator if

$$\omega_\lambda(T^n x, T^n y) \le q^n(x,y)\delta(x,y), n = 1, 2, 3, \dots,$$

for all $x, y \in X_\omega$, where q and δ are two non-negative real valued functions over $X_\omega \times X_\omega$ satisfying $q(x,y) < 1$ for all $(x,y) \in X_\omega \times X_\omega$ with $\sup_{x,y \in X_\omega} q(x,y) = 1$ and $\delta(x, Tx) < \infty$ for all $x \in X_\omega$.

Theorem 2.14. *Let ω be a metric modular on X and X_ω be a modular metric space induced by ω. Let X_ω be T-orbitally complete where T is a self mapping on X_ω. Let us suppose further $T : X_\omega \to X_\omega$ is a Ćirić operator satisfying the condition*

$$\begin{aligned} \omega_\lambda(Tx, Ty) \le\ & \alpha\omega_\lambda(x, Tx) \\ & + \beta \max\{\omega_\lambda(y, Ty) + \omega_{3\lambda}(x, y), \omega_{2\lambda}(x, Ty), \omega_{2\lambda}(y, Tx)\}, \end{aligned}$$

for all $x, y \in X_\omega$, where $\alpha \ge 0, 0 \le \beta < 1$. Then T has a fixed point in X_ω.

Proof. Let $x_0 \in X_\omega$ and $x_n = T^n(x_0)$ where $n = 1, 2, 3, \dots$. Then

$$\begin{aligned} & \omega_\lambda(T^m x_0, T^n x_0) \\ = \ & \omega_\lambda(T(T^{m-1}x_0), T(T^{n-1}x_0)) \\ \le \ & \alpha\omega_\lambda(T^{m-1}x_0, T^m x_0) + \beta \max\{\omega_\lambda(T^{n-1}x_0, T^n x_0) \\ & + \omega_{3\lambda}(T^{m-1}x_0, T^{n-1}x_0), \omega_{2\lambda}(T^{m-1}x_0, T^n x_0), \omega_{2\lambda}(T^{n-1}x_0, T^m x_0)\}. \end{aligned}$$

So

$$\begin{aligned}
&\omega_\lambda(T^m x_0, T^n x_0)\\
\leq\ & \alpha\omega_\lambda(T^{m-1}x_0, T^m x_0)\\
&+\beta\max\{\omega_\lambda(T^{n-1}x_0, T^n x_0)+\omega_\lambda(T^{m-1}x_0, T^m x_0)\\
&+\omega_\lambda(T^m x_0, T^n x_0)+\omega_\lambda(T^n x_0, T^{n-1}x_0)\}, \omega_\lambda(T^{m-1}x_0, T^m x_0)\\
&+\omega_\lambda(T^m x_0, T^n x_0), \omega_\lambda(T^{n-1}x_0, T^n x_0)+\omega_\lambda(T^n x_0, T^m x_0)\}\\
=\ & \alpha\omega_\lambda(T^{m-1}x_0, T^m x_0)\\
&+\beta[2\omega_\lambda(T^{n-1}x_0, T^n x_0)+\omega_\lambda(T^{m-1}x_0, T^m x_0)+\omega_\lambda(T^m x_0, T^n x_0)]\\
\leq\ & \left(\frac{\alpha+\beta}{1-\beta}\right)\omega_\lambda(T^{m-1}x_0, T^m x_0)+\left(\frac{2\beta}{1-\beta}\right)\omega_\lambda(T^{n-1}x_0, T^n x_0)\\
\leq\ & \left(\frac{\alpha+\beta}{1-\beta}\right)q^{m-1}(x_0, Tx_0)\delta(x_0, Tx_0)\\
&+\left(\frac{2\beta}{1-\beta}\right)q^{n-1}(x_0, Tx_0)\delta(x_0, Tx_0)\\
&\to 0 \text{ as } m, n\to\infty \text{ and } \delta(x, Tx)<\infty.
\end{aligned}$$

Then $\{T^n x_0\}$ is a ω-Cauchy sequence in X_ω which is T-orbitally complete. So there exists $u \in X_\omega$ such that

$$\lim_{m,n\to\infty}\omega_\lambda(T^m x_0, T^n x_0)=\lim_{n\to\infty}\omega_\lambda(T^n x_0, u)=\omega_\lambda(u,u)=0.$$

This implies that $\lim_{n\to\infty}[\omega_\lambda(T^n x_0, u)]=0$. Now

$$\begin{aligned}
&\omega_\lambda(T^n x_0, Tu)\\
=\ & \omega_\lambda(T(T^{n-1}x_0), Tu)\\
\leq\ & \alpha\omega_\lambda(T^{n-1}x_0, T^n x_0)+\beta\max\{\omega_\lambda(u, Tu)+\omega_{3\lambda}(T^{n-1}x_0, u),\\
&\omega_{2\lambda}(T^{n-1}x_0, Tu), \omega_{2\lambda}(u, T^n x_0)\}\\
\leq\ & \alpha\omega_\lambda(T^{n-1}x_0, T^n x_0)+\beta\max\{\omega_\lambda(u, Tu)+\omega_\lambda(T^{n-1}x_0, u),\\
&\omega_{2\lambda}(T^{n-1}x_0, Tu), \omega_\lambda(u, T^n x_0)\}.
\end{aligned}$$

Passing on limit as $n\to\infty$,

$$\lim_{n\to\infty}\omega_\lambda(T^n x_0, Tu)\leq\beta\max\{\omega_\lambda(u, Tu), \lim_{n\to\infty}\omega_{2\lambda}(T^{n-1}x_0, Tu)\}.$$

If $\max\{\omega_\lambda(u, Tu), \lim_{n\to\infty}\omega_{2\lambda}(T^{n-1}x_0, Tu)\}=\omega_\lambda(u, Tu)$, then

$$\lim_{n\to\infty}\omega_\lambda(T^n x_0, Tu)\leq\beta\omega_\lambda(u, Tu)<\omega_\lambda(u, Tu) \text{ as } \beta<1,$$

which yields

$$\lim_{n\to\infty}\omega_{2\lambda}(T^n x_0, Tu)\leq\lim_{n\to\infty}\omega_\lambda(T^n x_0, Tu)<\omega_\lambda(u, Tu),$$

and so

$$\lim_{n\to\infty}\{\omega_\lambda(T^n x_0, u)+\omega_\lambda(u, Tu)\}<\omega_\lambda(u, Tu),$$

giving rise to $\omega_\lambda(u, Tu)<\omega_\lambda(u, Tu)$, which is not true. Hence

$$\max\left\{\omega_\lambda(u, Tu), \lim_{n\to\infty}\omega_{2\lambda}(T^{n-1}x_0, Tu)\right\}=\lim_{n\to\infty}\omega_{2\lambda}(T^{n-1}x_0, Tu).$$

Then

$$\lim_{n\to\infty} \omega_{2\lambda}(T^n x_0, Tu) \leq \beta \lim_{n\to\infty} \omega_{2\lambda}(T^{n-1} x_0, Tu).$$

Since $\beta < 1$, we have $\lim_{n\to\infty} \omega_{2\lambda}(T^n x_0, Tu) = 0$ which gives us

$$\lim_{n\to\infty} \{\omega_\lambda(T^n x_0, u) + \omega_\lambda(u, Tu)\} = 0.$$

Then we get $\omega_\lambda(u, Tu) = 0$. Therefore we have

$$\omega_\lambda(u, u) = \omega_\lambda(Tu, Tu) = \omega_\lambda(u, Tu) = 0,$$

which implies that $u = Tu$. Hence u is a fixed point of T. ■

We finish this section with an analogue of Jungck theorem [20] for modular metric spaces.

Theorem 2.15. *(Jungck Theorem in a modular metric space) Let ω be a metric modular on X and X_ω be a modular metric space induced by ω. Let X_ω be a complete modular metric space, and T and I be commuting mappings of X_ω into itself satisfying the inequality*

$$\omega_\lambda(Tx, Ty) \leq \mu \omega_\lambda(Ix, Iy), \tag{2.3}$$

for all $x, y \in X_\omega$ and $0 < \mu < 1$. If $\omega_\lambda(Ix, Iy) < \infty$ for all $x, y \in X_\omega$ and the range of I contains the range of T and further I is continuous, then T and I have a unique common fixed point.

Proof. Let $x_0 \in X_\omega$ be arbitrary. Then Tx_0 and Ix_0 are well defined. Since $Tx_0 \in T(X_\omega)$ and the range of I contains the range of T, there exists $x_1 \in X_\omega$ such that $Ix_1 = Tx_0$. In general, if $x_n \in X_\omega$ is chosen, then there exists a point $x_{n+1} \in X_\omega$ such that $Ix_{n+1} = Tx_n$. As $\omega_\lambda(Tx, Ty) \leq \mu \omega_\lambda(Ix, Iy)$ for all $x, y \in X_\omega$ and $0 < \mu < 1$, we have

$$\omega_\lambda(Ix_{m+k}, Ix_{n+k}) = \omega_\lambda(Tx_{m+k-1}, Tx_{n+k-1}) \leq \mu \omega_\lambda(Ix_{m+k-1}, Ix_{n+k-1}).$$

So, for all $k \in \mathbb{N}$,

$$\omega_\lambda(Ix_{m+k}, Ix_{n+k}) \leq \mu^k \omega_\lambda(Ix_m, Ix_n). \tag{2.4}$$

.

Case-I: If $Ix_{n+1} = Ix_n$ for some n, then $Tx_n = Ix_n = p$. Here p is a common fixed point of T and I. Indeed, $Tp = T(Ix_n) = I(Tx_n) = Ip$. If we consider $\omega_\lambda(p, Tp) > 0$, then we have

$$\begin{aligned}
\omega_\lambda(p, Tp) =& \omega_\lambda(Tx_n, Tp) \\
\leq& \mu \omega_\lambda(Ix_n, Ip) \\
=& \mu \omega_\lambda(p, Ip) \\
=& \mu \omega_\lambda(p, Tp) \\
<& \omega_\lambda(p, Tp),
\end{aligned}$$

which is a contradiction.

Case-II: If $Ix_{n+1} \neq Ix_n$ for all $n \geq 0$, then $Ix_{n+k} \neq Ix_n$ for all $n \geq 0$ and $k \geq 1$, viz., if $Ix_n = Ix_{n+k}$ for some $n \geq 0$ and $k \geq 1$, then we have

$$\omega_\lambda(Ix_{n+1}, Ix_{n+k+1}) = \omega_\lambda(Tx_n, Tx_{n+k}) \leq \mu \omega_\lambda(Ix_n, Ix_{n+k}) = 0.$$

So we have $Ix_{n+k} = Ix_{n+k+1}$. Then (2.4) implies that

$$\begin{aligned}\omega_\lambda(Ix_{n+1}, Ix_n) &= \omega_\lambda(Ix_{n+k+1}, Ix_{n+k})\\ &\leq \mu^k \omega_\lambda(Ix_{n+1}, Ix_n)\\ &< \omega_\lambda(Ix_{n+1}, Ix_n),\end{aligned}$$

which is a contradiction.

Thus we assume that $Ix_n \neq Ix_m$ for all distinct $n, m \in \mathbb{N}$. Note that $Ix_{m+k} \neq Ix_{n+k}$ for all $k \in \mathbb{N}$ and for all distinct $n, m \in \mathbb{N}$ and $Ix_{n+k}, Ix_{m+k} \in X \setminus \{Ix_n, Ix_m\}$. Then

$$\begin{aligned}\omega_\lambda(Ix_m, Ix_n) &\leq \omega_{\lambda_1}(Ix_m, Ix_{m+n_0}) + \omega_{\lambda_2}(Ix_{m+n_0}, Ix_{n+n_0})\\ &\quad + \omega_{\lambda_3}(Ix_{n+n_0}, Ix_n),\end{aligned}$$

where $\lambda = (\lambda_1 + \lambda_2 + \lambda_3) > 0$ and $n_0 \in \mathbb{N}$ and each $\lambda_i > 0$ for $i = 1, 2, 3$. So using the result (2.4), we get

$$\begin{aligned}&\omega_\lambda(Ix_m, Ix_n)\\ \leq\ & \mu^m \omega_{\lambda_1}(Ix_0, Ix_{n_0}) + \mu^{n_0} \omega_{\lambda_2}(Ix_m, Ix_n) + \mu^n \omega_{\lambda_3}(Ix_0, Ix_{n_0})\\ \leq\ & \mu^m \omega_{\lambda_1}(Ix_0, Ix_{n_0}) + \mu^{m-1+n_0} \omega_{\lambda_2}(Ix_1, Ix_{n-m+1})\\ & + \mu^n \omega_{\lambda_3}(Ix_0, Ix_{n_0}).\end{aligned}$$

We then obtain $\omega_\lambda(Ix_m, Ix_n) \to 0$ as $m, n \to \infty$ and $\omega_\lambda(Ix, Iy) < \infty$ which implies $\{Ix_m\}$ is a ω-Cauchy sequence in X_ω. By completeness of X_ω there exists $p \in X_\omega$ such that

$$\lim_{n\to\infty} Ix_n = \lim_{n\to\infty} Tx_{n-1} = p.$$

Since I is continuous, (2.3) implies that both I and T are continuous. Since T and I commute, we obtain

$$Ip = I\left(\lim_{n\to\infty} Tx_n\right) = \lim_{n\to\infty} IT(x_n) = \lim_{n\to\infty} TI(x_n) = T\left(\lim_{n\to\infty} Ix_n\right) = Tp.$$

Let $Tp = Ip = q$. If possible, let $Tp \neq Tq$. Then we have $Tq = TIp = ITp = Iq$. Further, from (2.3) we obtain

$$\begin{aligned}\omega_\lambda(Tp, Tq) &\leq \mu\omega_\lambda(Ip, Iq)\\ &= \mu\omega_\lambda(Tp, Tq)\\ &< \omega_\lambda(Tp, Tq),\end{aligned}$$

which is a contradiction. So $Tp = Tq$ and hence we have $Tq = Iq = q$ and q is a common fixed point of T and I. Condition (2.3) implies that q is a unique common fixed point of T and I. ■

Corollary 2.16. *Let I and T be commuting mappings of a modular metric space X_ω such that ω is complete and*

$$\omega_\lambda(T^k(x), T^k(y)) \leq \mu\omega_\lambda(Ix, Iy),$$

for all $x, y \in X_\omega$ and $0 < \mu < 1$ and k be any positive integer. If $\omega_\lambda(Ix, Iy) < \infty$, I is continuous and $T(X_\omega) \subset I(X_\omega)$, Then T and I have a unique common fixed point in X_ω.

Proof. Clearly, T^k commutes with I and $T^k(X_\omega) \subset T(X_\omega) \subset I(X_\omega)$. Thus the theorem pertains to T^k and I. So there is a unique $p \in X_\omega$ such that $p = I(p) = T^k(p)$. Since I and T commute, we can write

$$T(p) = T(I(p)) = I(T(p)) = T^k(T(p)),$$

which says that $T(p)$ is a common fixed point of I and T^k. The uniqueness of p implies that $p = T(p) = I(p)$. ■

We give the following example in support of our theorem.

Example 2.17. Let $X = \mathbb{R}^7$. We define a mapping $\omega : (0, \infty) \times X \times X \to [0, \infty]$ by

$$\omega_\lambda(x, y) = \sum_{i=1}^{7} \frac{|x_i - y_i|}{\lambda},$$

where $x = (x_1, x_2, x_3, x_4, x_5, x_6, x_7), y = (y_1, y_2, y_3, y_4, y_5, y_6, y_7) \in \mathbb{R}^7$.

If we take $\lambda \to \infty$, then $X = X_\omega$ and also X_ω is a complete modular metric space.

Let us define $T, I : X_\omega \to X_\omega$ by

$$\begin{aligned} T(x) &= \left(\frac{5x_1 - 63}{14}, \frac{2x_2 - 15}{5}, \frac{x_3 - 30}{11}, \frac{5x_4 + 1}{4}, 2x_5 - 1, \frac{x_6 + 20}{5}, \frac{2x_7 + 21}{9}\right), \\ I(x) &= \left(\frac{6x_1 - 7}{7}, \frac{11x_2 + 30}{5}, \frac{7x_3 + 12}{3}, \frac{7x_4 + 4}{3}, 5x_5 - 4, 3x_6 - 10, \frac{11x_7 - 15}{6}\right). \end{aligned}$$

Then T and I are commuting. Also

$$\omega_\lambda(Tx, Ty) \leq k\omega_\lambda(Ix, Iy),$$

where $k = 15/28$.

It is clear that I is continuous and $T(X_\omega) \subset I(X_\omega)$. Therefore we can conclude that T and I have a unique common fixed point.

Here, $(-7, -5, -3, -1, 1, 5, 3) \in \mathbb{R}^7$ is a common fixed point of both T and I.

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