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Note on Generalized Distance Matrix of Graphs

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Abstract For any connected graph G with $\alpha \in [0,1]$, the generalized distance matrix $D_\alpha(G)$ is defined as, $D_\alpha(G) = \alpha T(G) + (1-\alpha)D(G)$, where $T(G)$ represents a transmission diagonal matrix of G and $D(G)$ is the distance matrix of G. The maximum eigenvalue of generalized distance matrix is called $D_\alpha(G)$ spectral radius. In this paper, we give some new results on generalized distance spectra of complete multipartite graphs. Later in the paper are given some sharp bounds for generalized distance spectral radius of connected graphs.

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1. Introduction

In this paper we consider finite undirected simple graphs. Let $G = (V(G), E(G))$ be a connected graph of order n with *vertex set* $V(G)$ and *edge set* $E(G)$. For any two vertices $u, v \in V(G)$, the *distance* between vertex u and v in G is defined as the length of a shortest path from vertex u to vertex v in G and denoted by $d_G(u,v)$ or simply d_{uv}. The maximum distance between any two distinct vertices in G is called the *diameter* of G, denoted by $d(G)$ or for simplicity just by d. For any $v \in V(G)$, the number of vertices adjacent to v is known as the *degree of vertex* v and denoted by d_v. The *distance matrix* of G denoted by $D(G)$ is the $n \times n$ matrix $D(G) = (d_G(u,v))_{u,v \in V(G)}$. For $u \in V(G)$, the *transmission* or degree distance of u is denoted by Tr_u, i.e., the sum of distances from u to all other vertices of G, $Tr_u = \sum\limits_{v \in V(G)} d_G(u,v)$. The *transmission matrix* $T(G)$ is a diagonal matrix, where diagonal entries are the transmissions of corresponding vertices of G. The *distance signless Laplacian matrix* of any graph G is defined as, $D_Q(G) = T(G) + D(G)$, where $D(G)$ denotes the distance matrix of G and $T(G)$ the transmission matrix of G. Throughout this paper, we denote by P_n the path, by C_n the cycle, by S_n the star and by K_n the complete graph, each on n vertices.

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The distance spectra of graphs have been extensively studied from last many years, see the recent survey [1] and references therein. The distance Laplacian and distance signless Laplacian spectrum of graphs have also received much attention in recent years especially the problems related to their spectral radius, see [2–8]. Aouchiche and Hansen [2] showed that deletion of an edge does not decrease the distance laplacian of the distance signless laplacian spectra of graphs. In [9], the same authors proved that the distance signless laplacian spectrum is minimized by the star graph. In [10], Alhevaz et al. gave some upper and lower bounds on distance signless Laplacian spectral radius and also determined the distance signless Laplacian spectrum of some graph operations. For more review about distance laplacian and distance signless laplacian see [2–8].

In [11], Nikiforov studied the convex linear combinations in relating to the adjacency matrix and diagonal degree matrix of G, which effectively reduce in merging the adjacency spectral and signless Laplacian spectral theories. Similarly [12], generalized the Nikiforov concept on distance spectral as *generalized distance matrix* or D_α-matrix denoted as $D_\alpha(G)$,

$$D_\alpha(G) = \alpha T(G) + (1-\alpha)D(G),\ \alpha \in [0,1],$$

where $D(G)$ represents the distance matrix of a graph G and $T(G)$ represents the transmission matrix of a graph G. Obviously, $D_0(G) = D(G)$ for $\alpha = 0$, which is actually the distance matrix of G and $2D_{1/2}(G) = D_Q(G)$ is the distance signless Laplacian matrix of a graph G. Let $\mu_1(G) \geqslant \mu_2(G) \geqslant \cdots \geqslant \mu_n(G)$ be the distance eigenvalues of $D(G)$. As $D_\alpha(G)$ is a symmetric matrix, the D_α-eigenvalues of G are all real and we denote them by $\rho_\alpha^{(1)}(G) \geqslant \cdots \geqslant \rho_\alpha^{(n)}(G)$, arranging them in non-increasing order, where $n = |V(G)|$. The $D_\alpha(G)$ spectral radius of G is the largest $D_\alpha(G)$ eigenvalue $\rho_\alpha^{(1)}(G)$ and denoted as $\rho_\alpha(G)$, the minimum $D_\alpha(G)$ eigenvalue $\rho_\alpha^n(G)$ as $\rho_{min}(G)$. Obviously, $\rho_0^{(1)}(G), \ldots, \rho_0^{(n)}(G)$ are the distance eigenvalues of G, and $2\rho_{1/2}^{(1)}(G), \ldots, 2\rho_{1/2}^{(n)}(G)$ are the distance signless Laplacian eigenvalues of G. Particularly, $\rho_0(G)$ is the distance spectral radius and $2\rho_{1/2}(G)$ is just the distance signless Laplacian spectral radius of G. Further results on generalized distance matrix can be seen in [13–17].

The paper is organized as follows. In the next section, we discuss the ground theory for generalized distance spectrum which will be helpful in proving lemmas and theorems in proceeding sections. In section 3, we give generalized distance spectrum of complete multipartite graph. In section 4, we discuss and prove some bounds for the generalized distance spectral radius of graphs, which is generalization of [10] and [18]. Finally, concluding remarks are given in section 5.

2. Preliminaries

In this section, we discuss some concepts and Perron theory for $D_\alpha(G)$-matrix that will be useful throughout the paper.

Let G be a connected graph with $V(G) = \{v_1, \ldots, v_n\}$ and a column vector $x = (x_{v_1}, \ldots, x_{v_n})^T \in \mathbb{R}^n$ can be considered as a function defined on $V(G)$, which maps vertex v_i to x_{v_i}, i.e., $x(v_i) = x_{v_i}$ for $i = 1, \ldots, n$. Then

$$x^T D_\alpha(G) x = \alpha \sum_{u \in V(G)} Tr_u(G) x_u^2 + 2 \sum_{u,v \in V(G)} (1-\alpha) d_{uv} x_u x_v, \tag{2.1}$$

or equivalently,

$$x^T D_\alpha(G)x = \sum_{u,v\in V(G)} d_{uv}(\alpha(x_u^2 + x_v^2) + 2(1-\alpha)x_u x_v). \tag{2.2}$$

From [15], we know that $D_\alpha(G)$ is a non-negative irreducible matrix, by Perron-Frobenius theorem, $\rho_\alpha(G)$ is simple and there is a unique positive unit eigenvector corresponding to $\rho_\alpha(G)$, which is called the $D_\alpha(G)$ Perron vector of G. If x is the $D_\alpha(G)$ Perron vector of G, then for each $u \in V(G)$,

$$\rho_\alpha(G)x_u = \alpha Tr_u(G)x_u + (1-\alpha)\sum_{v\in V(G)} d_{uv}x_v, \tag{2.3}$$

or equivalently,

$$\rho_\alpha(G)x_u = \sum_{v\in V(G)} d_{uv}(\alpha x_u + (1-\alpha)x_v),$$

which is called the α-eigenequation of G at vertex u. For a unit column vector $x \in \mathbb{R}^n$ with at least one non-negative entry, by Rayleigh's principle, we have $\rho_\alpha(G) \geqslant x^\top D_\alpha(G)x$ with equality if and only if x is the $D_\alpha(G)$ Perron vector of G.

We denote by $\xi(G)$, the sum of distances between all unordered pairs of vertices in G, i.e., $\xi(G) = \frac{1}{2}\sum_{v\in V(G)} Tr_v$. For a graph G with $V(G) = \{v_1, v_2, \ldots, v_n\}$, the second transmission of vertex v is denoted by $\hat{T}_v$, i.e., $\hat{T}_v = \sum_{u=1}^{n} d_{uv}(Tr_u)$. A graph is said to be *transmission regular* if Tr_v is a constant for each $v \in V(G)$. It is clear that any vertex-transitive graph (a graph G in which for every two vertices u and v, there exists an automorphism f on G such that $f(u) = f(v)$) is a transmission regular graph. Indeed, the graph on 9 vertices depicted in Figure 1 is 14-transmission regular graph as each vertex of the graph has transmission 14 but is not degree regular and therefore not vertex-transitive. For more examples of transmission regular but not degree regular graphs see [19]. We

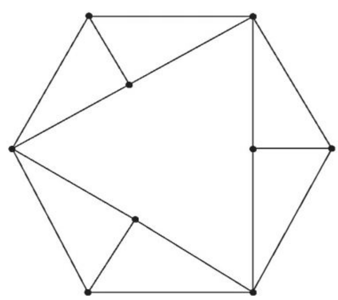

FIGURE 1. The transmission regular but not degree regular graph with the smallest order.

need the following important lemma to prove our main results.

Lemma 2.1 ([20]). **(Courant Weyl Inequality)** *For a real symmetric matrix M of order n, let $\lambda_1(M) \geqslant \lambda_2(M) \geqslant \cdots \geqslant \lambda_n(M)$ denote its eigenvalues. If A and B are two* *real symmetric matrices of order n and if $M = A + B$, then for every $i = 1, \dots, n$, we have*

$$\lambda_i(A) + \lambda_1(B) \geqslant \lambda_i(M) \geqslant \lambda_i(A) + \lambda_n(B).$$

3. The Generalized Distance Spectrum of Complete k-Partite Graphs

In this section, we focus on generalized distance spectrum of a complete k-partite graph. Guanglong Yu in [21], determined some graphs with the least distance eigenvalue. We begin with the following theorem which is the generalization of distance spectra of a complete multipartite graph and the proof is almost similar to Lemma 3.1 of [21].

Theorem 3.1. *The generalized distance spectrum of a complete k-partite graph $K_{n_1,n_2,\dots,n_k}$ ($\sum\limits_{i=1}^{k} n_i = n, n_i \geqslant 1$) consists of eigenvalues $\alpha(n+n_i)-2$ with multiplicity $n_i - 1$ where $i = 1, 2, \dots, k$ and k more eigenvalues which are the solution of the equation in ρ:*

$$\sum_{i=1}^{k} \frac{n_i}{\rho + 2 - n_i - \alpha(n - n_i)} - 1 = 0.$$

Proof. Let G be a complete k-partite graph. Then the D_α spectrum of a complete k-partite graph $K_{n_1,n_2,\dots,n_k}$ has the form $D_\alpha(G) = (1-\alpha)D(G) + \alpha T(G)$ with $\alpha \in [0, 1]$, where $D(G)$ is the distance matrix of graph and $T(G)$ is the transmission matrix of G. One can see that

$$D_\alpha(K_{n_1,\dots,n_k}) = \begin{pmatrix} S_1 & (1-\alpha)J_{n_1\times n_2} & (1-\alpha)J_{n_1\times n_3} & \cdots & (1-\alpha)J_{n_1\times n_k} \\ (1-\alpha)J_{n_2\times n_1} & S_2 & (1-\alpha)J_{n_2\times n_3} & \cdots & (1-\alpha)J_{n_2\times n_k} \\ (1-\alpha)J_{n_3\times n_1} & (1-\alpha)J_{n_3\times n_2} & S_3 & \cdots & (1-\alpha)J_{n_3\times n_k} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ (1-\alpha)J_{n_k\times n_1} & (1-\alpha)J_{n_k\times n_2} & (1-\alpha)J_{n_k\times n_3} & \cdots & S_k \end{pmatrix},$$

where $S_i = 2(1-\alpha)J_{n_i} + (\alpha(n+n_i)-2)I_{n_i}$, $i = 1, 2, \dots, k$ and J_{n_i} is the all-one square matrix of order n_i and I_{n_i} is the identity matrix of order n_i. For each $1 \leqslant i \leqslant k$, the spectrum of $2(1-\alpha)J_{n_i} + (\alpha(n+n_i)-2)I_{n_i}$ consists of $\alpha(n-n_i)+2(n_i-1)$ with multiplicity 1 and $\alpha(n+n_i)-2$ with multiplicity $n_i - 1$. If $n_i = 1$, then multiplicity of $\alpha(n+n_i)-2$ is 0 which implies that $2(1-\alpha)J_{n_i} + (\alpha(n+n_i)-2)I_{n_i}$ has unique eigenvalue 0. S_i has the all-one vector ξ_i as an eigenvector corresponding to the eigenvalue $\alpha(n-n_i)+2(n_i-1)$ while all other eigenvectors corresponding to eigenvalue $\alpha(n+n_i)-2$ are orthogonal to ξ_i.

Let us say X be an arbitrary eigenvector of the matrix S_i corresponding to an eigenvalue $\alpha(n+n_i)-2$ such that ${\xi_i}^T X = 0$. Then we can say that an eigenvector corresponding to $\alpha(n+n_i)-2$ is $\begin{pmatrix} 0_{1\times \sum\limits_{j=1}^{i-1} n_j} & X^T & 0_{1\times \sum\limits_{j=i+1}^{k} n_j} \end{pmatrix}^T$. So in this way we can totally construct $n-k$ mutually orthogonal eigenvectors of D_α. All of these eigenvectors correspond to eigenvalue $\alpha(n+n_i)-2$ and all of them are orthogonal to each vector $\begin{pmatrix} 0_{1\times \sum\limits_{j=1}^{i-1} n_j} & {\xi_i}^T & 0_{1\times \sum\limits_{j=i+1}^{k} n_j} \end{pmatrix}^T$, which means that remaining k eigenvectors of D_α are spanned by the vectors $\left(0_{1\times\sum_{j=1}^{i-1}n_j}\ {\xi_i}^T\ 0_{1\times\sum_{j=i+1}^{k}n_j}\right)^T$ where $(i=1,2,\ldots,k)$. So the remaining k eigenvectors of $D_\alpha(G)$ have the form $(\beta_1{\xi_1}^T\ \beta_2{\xi_2}^T\ \cdots\ \beta_i\xi_i^T\ \cdots\ \beta_k{\xi_k}^T)^T$, for a suitable choice of $\beta_1,\beta_2,\cdots,\beta_k$. Let us assume that ρ is an eigenvalue of $D_\alpha(G)$ and assume that an eigenvector corresponding to ρ is of the form $(\beta_1{\xi_1}^T\ \beta_2{\xi_2}^T\ \cdots\ \beta_i\xi_i^T\ \cdots\ \beta_k{\xi_k}^T)^T$. Then from eigenequation

$$D_\alpha\left(\beta_1{\xi_1}^T\ \beta_2{\xi_2}^T\ \cdots\ \beta_i\xi_i^T\ \cdots\ \beta_k{\xi_k}^T\right)^T=\rho\left(\beta_1{\xi_1}^T\ \beta_2{\xi_2}^T\ \cdots\ \beta_i\xi_i^T\ \cdots\ \beta_k{\xi_k}^T\right)^T,$$

using $(2(1-\alpha)J_{n_i}+(\alpha(n+n_i)-2)I_{n_i})\xi_i^T=(\alpha(n-n_i)+2(n_i-1))\xi_i^T$, for $i=1,2,\ldots,k$, we get that

$$\begin{aligned}
(\alpha(n-n_1)+2(n_1-1))\beta_1+n_2\beta_2+n_3\beta_3+\cdots+n_k\beta_k&=\rho\beta_1;\\
n_1\beta_1+(\alpha(n-n_2)+2(n_2-1))\beta_2+n_3\beta_3+\cdots+n_k\beta_k&=\rho\beta_2;\\
n_1\beta_1+n_2\beta_2+(\alpha(n-n_2)+2(n_2-1))\beta_3+\cdots+n_k\beta_k&=\rho\beta_3;\\
&\vdots\\
n_1\beta_1+n_2\beta_2+n_3\beta_3+\cdots+(\alpha(n-n_k)+2(n_k-1))\beta_k&=\rho\beta_k.
\end{aligned}$$

Then we have,

$$\begin{aligned}
n_1\beta_1+n_2\beta_2+n_3\beta_3+\ldots+n_k\beta_k&=(\rho+2-n_1-\alpha(n-n_1))\beta_1;\\
n_1\beta_1+n_2\beta_2+n_3\beta_3+\ldots+n_k\beta_k&=(\rho+2-n_2-\alpha(n-n_2))\beta_2;\\
n_1\beta_1+n_2\beta_2+n_3\beta_3+\ldots+n_k\beta_k&=(\rho+2-n_3-\alpha(n-n_3))\beta_3;\\
&\vdots\\
n_1\beta_1+n_2\beta_2+n_3\beta_3+\ldots+n_k\beta_k&=(\rho+2-n_k-\alpha(n-n_k))\beta_k;
\end{aligned}$$

Now let $n_1\beta_1+n_2\beta_2+n_3\beta_3+\ldots+n_k\beta_k=\varepsilon$. Then we have

$$\begin{aligned}
\frac{n_1\varepsilon}{\rho+2-n_1-\alpha(n-n_1)}&=n_1\beta_1\\
\frac{n_2\varepsilon}{\rho+2-n_2-\alpha(n-n_2)}&=n_2\beta_2\\
\frac{n_3\varepsilon}{\rho+2-n_3-\alpha(n-n_3)}&=n_3\beta_3\\
&\vdots\\
\frac{n_k\varepsilon}{\rho+2-n_k-\alpha(n-n_k)}&=n_k\beta_k
\end{aligned}$$

summing up all the above equations, we get

$$\varepsilon\sum_{i=1}^{k}\frac{n_i}{\rho+2-n_i-\alpha(n-n_i)}=\varepsilon,$$

and thus

$$\sum_{i=1}^{k}\frac{n_i}{\rho+2-n_i-\alpha(n-n_i)}-1=0.$$

Thus remaining k eigenvalues of $D_\alpha(K_{n_1,\ldots,n_k})$ are all the solutions of

$$\sum_{i=1}^{k} \frac{n_i}{\rho + 2 - n_i - \alpha(n - n_i)} - 1 = 0.$$

■

We arrive at our next result as follows.

Theorem 3.2. *Let $G = K_{n_1,n_2,\ldots,n_k}$ be a complete k-partite graph for $2 \leqslant k \leqslant n-1$ and $n_i \geqslant 1$, $i = 1,2,\ldots,k$. Then $\rho^n_\alpha(G) \geqslant \alpha(n + n_1) - 2$. Moreover, if $n_1 \geqslant n_2 \geqslant \cdots \geqslant n_k$, then $\rho^n_\alpha(G) = \alpha(n + n_k) - 2$ with multiplicity $n - k$.*

Proof. Let $G = K_{n_1,n_2,\ldots,n_k}$ be a complete k-partite graph for $2 \leqslant k \leqslant n-1$ and $n_i \geqslant 1$, $i = 1,2,\ldots,k$ and assume without loss of generality that $n_1 \geqslant n_2 \geqslant \cdots \geqslant n_k$. As G is a complete k-partite, it is obvious that $diam(G) = 2$. So for any $v \in V(G)$, we have $Tr_v = d_v + 2(n - d_v - 1) = 2n - 2 - d_v$. Therefore, the transmission matrix is

$$T(G) = (2n-2)I - Diag(d_v)I.$$

As G is a graph of diameter 2, so we can write the distance matrix of G as $D(G) = J - I + A^c$, where J is the all-one square matrix of order n, I is the identity matrix of order n and A^c is the adjacency matrix of G^c. So generally the distance alpha matrix of a graph G can be written as:

$$D_\alpha(G) = \alpha((2n-2)I - Diag(d_v)) + (1-\alpha)(J - I + A^c).$$

Now using Courant Weyl inequality from Lemma 2.1 we have

$$\rho^n_\alpha(G) \geqslant \alpha\rho^n_\alpha((2n-2)I - Diag(d_v)) + (1-\alpha)\rho^n_\alpha(J - I + A^c),$$
$$\rho^n_\alpha(G) \geqslant \alpha\rho^n_\alpha((2n-2)I) + \rho^n_\alpha(-Diag(d_v)) + (1-\alpha)\rho^n_\alpha(J-I) + (1-\alpha)\rho^n_\alpha(A^c).$$

By solving the above inequality for the minimum eigenvalue,

$$\rho^n_\alpha(G) \geqslant \alpha(n + n_k) - 2.$$

In the following, we will show that there exists a D_α-eigenvalue which is equal to $\alpha(n + n_k) - 2$. Let $V(G) = V_1 \cup V_2 \cup \cdots \cup V_k$, $2 \leqslant k \leqslant n-1$ where $n_1 \geqslant n_2 \geqslant \cdots \geqslant n_k$. Then the D_α-matrix of G can be written as:

$$D_\alpha(G) = \begin{pmatrix} S_1 & (1-\alpha)J_{n_1\times n_2} & (1-\alpha)J_{n_1\times n_3} & \cdots & (1-\alpha)J_{n_1\times n_k} \\ (1-\alpha)J_{n_2\times n_1} & S_2 & (1-\alpha)J_{n_2\times n_3} & \cdots & (1-\alpha)J_{n_2\times n_k} \\ (1-\alpha)J_{n_3\times n_1} & (1-\alpha)J_{n_3\times n_2} & S_3 & \cdots & (1-\alpha)J_{n_3\times n_k} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ (1-\alpha)J_{n_k\times n_1} & (1-\alpha)J_{n_k\times n_2} & (1-\alpha)J_{n_k\times n_3} & \cdots & S_k \end{pmatrix},$$

where $S_i = 2(1-\alpha)J_{n_i} + (\alpha(n + n_i) - 2)I_{n_i})$, $i = 1,2,\ldots,k$ and J_{n_i} is the all-one square matrix of order n_i and I_{n_i} is the identity matrix of order n_i. Hence $det(\rho I - D_\alpha(G))$

$$= \prod_{i=1}^{k}(\rho - (\alpha(n+n_i) - 2))^{n_i - 1} \begin{pmatrix} \hat{S_1} & (\alpha-1)n_2 & (\alpha-1)n_3 & \cdots & (\alpha-1)n_k \\ (\alpha-1)n_1 & \hat{S_2} & (\alpha-1)n_3 & \cdots & (\alpha-1)n_k \\ (\alpha-1)n_1 & (\alpha-1)n_2 & \hat{S_3} & \cdots & (\alpha-1)n_k \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ (\alpha-1)n_1 & (\alpha-1)n_2 & (\alpha-1)n_3 & \vdots & \hat{S_k} \end{pmatrix},$$

where $\hat{S}_i = \rho + 2 - \alpha(n+n_i) - 2n_i$ for $i = 1, 2, \ldots, k$. As $k \leqslant n-1$, so the multiplicity of $\alpha(n+n_i) - 2$ $(i = 1, 2, \ldots, k)$ is at least $n - k \geqslant 1$. As G is graph of diameter 2, so

$$D_\alpha(G) = \alpha((2n-2)I - Diag(d_v)) + (1-\alpha)(J - I + A^c).$$

Now by Lemma 2.1 using Courant Weyl inequality we have:

$$\rho^i_\alpha(G) \geqslant (1-\alpha)\rho^i_\alpha(J - I + A^c) + \alpha\rho^n_\alpha((2n-2)I - Diag(d_v)),$$

$$\rho^i_\alpha(G) \geqslant (1-\alpha)\rho^n_\alpha(J - I) + (1-\alpha)\rho^i_\alpha(A^c) + \alpha\rho^n_\alpha((2n-2)I - Diag(d_v)). \quad (3.1)$$

Also by Courant Weyl inequality,

$$\rho^i_\alpha(G) \leqslant (1-\alpha)\rho^i_\alpha(J - I + A^c) + \alpha\rho^1_\alpha((2n-2)I - Diag(d_v)),\ (2 \leqslant i \leqslant n),$$

$$\rho^i_\alpha(G) \leqslant (1-\alpha)\rho^2_\alpha(J-I) + (1-\alpha)\rho^{i-1}_\alpha(A^c) + \alpha\rho^1_\alpha((2n-2)I - Diag(d_v)), (2 \leqslant i \leqslant n). \quad (3.2)$$

Note that G^c is the union of complete graphs $K_{n_1}, K_{n_2}, \ldots, K_{n_k}$, which follows that $\rho^i_\alpha(A^c) = n_i - 1$, where $i = 1, 2, \ldots, k$. Therefore, from equations (3.1) and (3.2), we have

$$\alpha(n + n_k - n_i) + n_i - 2 \leqslant \rho^i_\alpha(G) \leqslant \alpha(n + n_k - n_{i-1}) + n_{i-1} - 2,\ \textit{for}\ 2 \leqslant i \leqslant k.$$

Thus, $\rho^i_\alpha(G) \geqslant \alpha(n + n_k) - (1+\alpha)$ for $i = 2, 3, \ldots, k$, since $n_i \geqslant 1$ and $\rho^1_\alpha(G) > n(1+\alpha) - 1$. Thus the multiplicity of $\alpha(n+n_1) - 2$ is $n - k$.

∎

Theorem 3.3. *Let $G = K_{n_1,\ldots,n_k}$ be a complete k-partite graph. Then the characteristic polynomial of $D_\alpha(G)$, i.e., $P_{D_\alpha(G)}(\rho)$*

$$= \prod_{i=1}^{k}(\rho - \alpha(n+n_i) - 2)^{n_i - 1}\left[\prod_{i=1}^{k}(\rho + 2 - n_i - \alpha(n + 2n_i))(1 - (1-\alpha)\sum_{j=1, j\neq i}^{k} n_j)\right].$$

Proof. Let G be a complete k-partite graph. Then the D_α-spectrum of a complete k-partite graph $K_{n_1,n_2,\ldots,n_k}$ has the form $D_\alpha(G) = (1-\alpha)D(G) + \alpha T(G)$ with $\alpha \in [0,1]$, where $D(G)$ is the distance matrix of G and $T(G)$ is the transmission matrix of G. We see that the $D_\alpha(K_{n_1,\ldots,n_k})$:

$$= \begin{pmatrix} S_1 & (1-\alpha)J_{n_1\times n_2} & (1-\alpha)J_{n_1\times n_3} & \ldots & (1-\alpha)J_{n_1\times n_k} \\ (1-\alpha)J_{n_2\times n_1} & S_2 & (1-\alpha)J_{n_2\times n_3} & \ldots & (1-\alpha)J_{n_2\times n_k} \\ (1-\alpha)J_{n_3\times n_1} & (1-\alpha)J_{n_3\times n_2} & S_3 & \ldots & (1-\alpha)J_{n_3\times n_k} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ (1-\alpha)J_{n_k\times n_1} & (1-\alpha)J_{n_k\times n_2} & (1-\alpha)J_{n_k\times n_3} & \ldots & S_k \end{pmatrix},$$

where $S_i = 2(1-\alpha)J_{n_i} + (\alpha(n+n_i) - 2)I_{n_i})$, $i = 1, 2, \ldots, k$ and J_{n_i} is the all-one square matrix of order n_i and I_{n_i} is the identity matrix of order n_i. By Theorem 3.2, we have $det(\rho I - D_\alpha(G))$

$$= \prod_{i=1}^{k}(\rho - \alpha(n+n_i) + 2)^{n_i - 1}\begin{pmatrix} \hat{S_1} & (\alpha-1)n_2 & (\alpha-1)n_3 & \ldots & (\alpha-1)n_k \\ (\alpha-1)n_1 & \hat{S_2} & (\alpha-1)n_3 & \ldots & (\alpha-1)n_k \\ (\alpha-1)n_1 & (\alpha-1)n_2 & \hat{S_3} & \ldots & (\alpha-1)n_k \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ (\alpha-1)n_1 & (\alpha-1)n_2 & (\alpha-1)n_3 & \vdots & \hat{S_k} \end{pmatrix},$$

where $\hat{S}_i = \rho + 2 - \alpha(n+n_i) - 2n_i$ for $i = 1, 2, \ldots, k$.

$$= \prod_{i=1}^{k}(\rho-\alpha(n+n_i)+2)^{n_i-1}\begin{pmatrix} \rho+2-\alpha(n+n_1)-2n_1 & (\alpha-1)n_2 & (\alpha-1)n_3 & \dots & (\alpha-1)n_k \\ \alpha(n+2n_1)+n_1-2-\rho & \bar{S_2} & 0 & \dots & 0 \\ \alpha(n+2n_1)+n_1-2-\rho & 0 & \bar{S_3} & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \alpha(n+2n_1)+n_1-2-\rho & 0 & 0 & \vdots & \bar{S_k} \end{pmatrix},$$

where $\bar{S_i} = \rho - \alpha(n+2n_i) - n_i + 2$ for $i = 2, 3, \dots, k$. Therefore,

$$P_{D_\alpha(G)}(\rho)$$
$$= \prod_{i=1}^{k}(\rho-\alpha(n+n_i)-2)^{n_i-1}\left[\prod_{i=1}^{k}(\rho+2-n_i-\alpha(n+2n_i))(1-(1-\alpha)\sum_{j=1,j\neq i}^{k} n_j)\right].$$

■

4. Some Bounds on Generalized Distance Spectral Radius

The line graph $L(G)$ of a graph G is such that the vertices of the line graph are the edges of the graph G and any two vertices in $L(G)$ are adjacent if and only if their corresponding edges in G share the common vertex [22].

Let F_1, F_2 and F_3 be graphs on 5 vertices as shown in Figure 2.

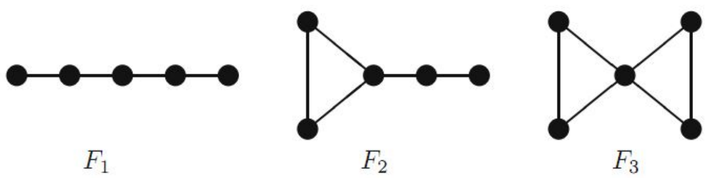

Figure 2. The forbidden induced subgraphs.

We use the following theorem to prove the next theorem.

Theorem 4.1 ([23])**.** *For a connected graph G, $d(L(G)) \leqslant 2$ if and only if none of the three graphs F_1, F_2 and F_3 (see Figure 2) is an induced subgraph of G.*

Theorem 4.2. *Let G be a connected graph on n vertices, m edges and $deg(v_i) = d_i$. If $d(G) \leqslant 2$ and G does not contain the induced subgraphs F_1, F_2 and F_3 (see Figure 2). Then,*

$$\rho_\alpha(L(G)) \geqslant \frac{2m^2 - \sum_{i=1}^{n} d_i^2}{m}.$$

Proof. Let G be a connected graph on n vertices with $d(G) \leqslant 2$ and G does not contain the induced subgraphs F_1, F_2 and F_3 (as shown in figure 2). Let $V(G) = \{v_1, v_2, \dots, v_n\}$ and d_i denote the degree of vertex v_i, $i = 1, 2, \dots, n$. As $d(G) \leqslant 2$, it can easily be seen that the i^{th} row of $D_\alpha(G)$ consists of $(1-\alpha)d_i$ ones, $(1-\alpha)(n-d_i-1)$ twos and the diagonal entry will be $\alpha(2n-d_i-2)$. Let $X=(1,1,\ldots,1)^t$ be the all one vector, then by using the Rayleigh Quotient, we have

$$\rho_\alpha(G)\geqslant\frac{X^TD_\alpha X}{X^TX}=\frac{1}{n}\sum_{i=1}^{n}(\alpha(2n-d_i-1)+2(1-\alpha)((n-d_i-1))+(1-\alpha)(d_i)).$$

Hence

$$\rho_\alpha(G)\geqslant\frac{1}{n}\sum_{i=1}^{n}(2n-d_i-2)=\frac{2n(n-1)-2m}{n}.$$

From graph G, let number of vertices of $L(G)$ be $n_1=m$ and number of edges of line graph $L(G)$ be $m_1=\frac{1}{2}\sum\limits_{i=1}^{n}d_i^2-m$. Now since G is connected with $d(G)\leqslant 2$, where G does not contain the induced subgraphs F_1, F_2 and F_3 (as shown in figure 2), then by Theorem 4.1, $d(L(G))\leqslant 2$. Therefore,

$$\rho_\alpha(G)\geqslant\frac{2n_1(n_1-1)-2m_1}{n_1}=\frac{2m(m-1)+2m-\sum\limits_{i=1}^{n}d_i^2}{m}$$

and thus

$$\rho_\alpha(L(G))\geqslant\frac{2m^2-\sum\limits_{i=1}^{n}d_i^2}{m}.$$

So we get the required bound for $L(G)$ with $d(L(G))\leqslant 2$. ∎

Corollary 4.3. *Let G be a connected $r-$regular graph on n vertices where G does not contain the induced subgraphs F_1, F_2 and F_3 (as shown in Figure 2). Then*

$$\rho_\alpha(L(G))\geqslant r(n-2).$$

Proof. Let G be $r-$regular graph of n vertices. Then the number of edges m in G is $m=\frac{nr}{2}$ and $d_i=r$. So by Theorem 4.2, we have

$$\rho_\alpha(L(G))\geqslant\frac{2m^2-\sum\limits_{i=1}^{n}d_i^2}{m}$$

$$=\frac{2(\frac{nr}{2})^2-\sum\limits_{i=1}^{n}r^2}{(\frac{nr}{2})}=r(n-2).$$

∎

The following lemma is useful to prove the next result.

Lemma 4.4 ([18])**.** *Let G be a connected graph with diameter $d\leqslant 2$. Then G is regular if and only if G is transmission regular.*

In any graph G, there is always a possibility that more than one vertex has the same degree. So, more than one vertex may also have the same maximum degree Δ_1. By second maximum degree, we mean the degree Δ_2 where $\Delta_1 > \Delta_2$. For illustration, consider the following figure where v_3, v_4 and v_5 have maximum degree 3 and v_1, v_2 have second maximum degree 2.

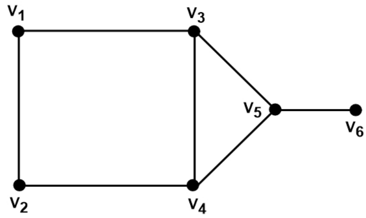

FIGURE 3. Graph with $\Delta_1 = 3$ and $\Delta_2 = 2$.

Theorem 4.5. *Let G be a connected graph on n vertices with maximum degree Δ_1 and second maximum degree Δ_2. Then $\rho_\alpha(G) \geqslant 4n - 4 - \Delta_1 - \Delta_2$ with equality if and only if G is a regular graph with diameter $d \leqslant 2$.*

Proof. Let G be a connected graph and $\rho_\alpha(G)$ be the D_α-spectral radius of G. Let $x = \{x_1, \dots, x_n\}$ be the Perron eigenvector of $D_\alpha(G)$, where each x_i corresponds to vertex v_i for $i = 1, \dots, n$, such that $x_i = \min\limits_{1 \leqslant k \leqslant n} x_k$ and $x_j = \min\limits_{1 \leqslant k \neq i \leqslant n} x_k$.

From eigenequation $\rho_\alpha(G)x_u = D_\alpha(G)x_u$, we have

$$\rho_\alpha(G)x_i = \alpha Tr_i x_i + (1-\alpha)\sum_{v=1, v\neq i}^{n} d_{iv}x_v \geqslant (\alpha x_i + (1-\alpha)x_j)Tr_i$$

and

$$\rho_\alpha(G)x_j = \alpha Tr_j x_j + (1-\alpha)\sum_{v=1, v\neq j}^{n} d_{jv}x_v \geqslant (\alpha x_j + (1-\alpha)x_i)Tr_j.$$

As we know,

$$Tr_i = \sum_{v=1, v\neq i}^{n} d_{iv} \geqslant d_i + 2(n-1-d_i) = 2n-2-d_i$$

and

$$Tr_j = \sum_{v=1, v\neq j}^{n} d_{jv} \geqslant d_j + 2(n-1-d_j) = 2n-2-d_j,$$

we can get,

$$\rho_\alpha(G)x_i \geqslant (2n-2-d_i)(\alpha x_i + (1-\alpha)x_j) \tag{4.1}$$

and

$$\rho_\alpha(G)x_j \geqslant (2n-2-d_j)(\alpha x_j + (1-\alpha)x_i) \tag{4.2}$$

Adding equations (4.1) and (4.2), we have

$$\rho_\alpha(G)(x_i + x_j) \geqslant (2n-2-d_i)(\alpha x_i + (1-\alpha)x_j) + (2n-2-d_j)(\alpha x_j + (1-\alpha)x_i)$$

which implies that

$$\rho_\alpha(G) \geqslant 2n-2-\alpha(d_i+d_j) = 2n-2-\alpha(\Delta_1+\Delta_2).$$

It can easily be deduced that equality holds in above inequalities only if G is a regular graph with diameter $d \leqslant 2$ and all x_i are equal. If $d=1$, then $G \cong K_n$. If $d=2$, then we get $\rho_\alpha(G) = 2n-2-d_i$, which implies that G is a regular graph. Conversely, from Lemma 4.4, we get that $\rho_\alpha(G) = 2n-2-\Delta_1$, if G is a regular graph with diameter $d \leqslant 2$. ■

In a similar way as explained before for maximum and second maximum degree in a graph, we can define minimum and second minimum degree in any graph. From Figure 3, we can see that minimum degree is $\delta_1 = 1$ and second minimum degree is $\delta_2 = 2$.

Theorem 4.6. *Let G be a simple connected graph on n vertices. Let δ_1 be the minimum degree of G and δ_2 be the second minimum degree. Then*

$$\rho_\alpha(G) \leqslant dn - \frac{d(d-1)}{2} - 1 - \alpha(d-1)(\delta_1+\delta_2).$$

With equality if and only if G is a regular graph with diameter $d \leqslant 2$.

Proof. Let G be a connected graph and $\rho_\alpha(G)$ be the D_α-spectral radius of G. Let $x = \{x_1, \ldots, x_n\}$ be the Perron eigenvector of $D_\alpha(G)$, where each x_i corresponds to vertex v_i for $i = 1, \ldots, n$, such that $x_i = \max\limits_{1 \leqslant k \leqslant n} x_k$ and $x_j = \max\limits_{1 \leqslant k \neq i \leqslant n} x_k$.

From eigenequation $\rho_\alpha(G)x_u = D_\alpha(G)x_u$ we have

$$\rho_\alpha(G)x_i = \alpha Tr_i x_i + \sum_{v=1, v\neq i}^{n} d_{iv}x_v \leqslant (\alpha x_i + (1-\alpha)x_j)Tr_i$$

and

$$\rho_\alpha(G)x_j = \alpha Tr_j x_j + \sum_{v=1, v\neq j}^{n} d_{jv}x_v \leqslant (\alpha x_j + (1-\alpha)x_i)Tr_j.$$

Now note that,

$$Tr_i = \sum_{v=1, v\neq i}^{n} d_{iv} \leqslant d_i + 2 + \cdots + (d-1) + d[n-1-d_i-(d-2)] = dn - d_i(d-1) - \frac{d(d-1)}{2} - 1,$$

$$Tr_j = \sum_{v=1, v\neq j}^{n} d_{jv} \leqslant d_i + 2 + \cdots + (d-1) + d[n-1-d_i-(d-2)] = dn - d_j(d-1) - \frac{d(d-1)}{2} - 1.$$

Then we can get

$$\rho_\alpha(G)x_i \leqslant (dn - d_i(d-1) - \frac{d(d-1)}{2} - 1)(\alpha x_i + (1-\alpha)x_j), \tag{4.3}$$

$$\rho_\alpha(G)x_j \leqslant (dn - d_j(d-1) - \frac{d(d-1)}{2} - 1)(\alpha x_j + (1-\alpha)x_i). \tag{4.4}$$

Adding equations (4.3) and (4.4), we have

$$\rho_\alpha(G)(x_i + x_j) \leqslant \left(dn - \frac{d(d-1)}{2} - 1 - \alpha(d-1)(d_i+d_j)\right)(x_i + x_j).$$

Then

$$\rho_\alpha(G) \leqslant dn - \frac{d(d-1)}{2} - 1 - \alpha(d-1)(d_i + d_j) = dn - \frac{d(d-1)}{2} - 1 - \alpha(d-1)(\delta_1 + \delta_2).$$

If the equality holds in the above inequalities, then we see that all x_i are equal, and hence $D_\alpha(G)$ has equal row sums. Therefore, we say that G is transmission regular. If diameter $d \geqslant 3$, then by Equation 4.3, we see that for each vertex v_i, there is exactly one vertex v_j such that $d_{ij} = 2$, and then $d \leqslant 4$. If the diameter of G is 3 and equality holds, then for a center vertex v_i, from $D_\alpha(G)x = \rho_\alpha(G)x$ and Equation (3), written for the component x_s, we have

$$\rho_\alpha(G)x_i = \alpha Tr_i x_i + (1-\alpha)d_i x_i + (1-\alpha)2(n-1-d_i)x_i = (2n-2-d_i)x_i.$$

Thus, $d_i = n-2$, which implies that $G = P_4$, but P_4 is not transmission regular, a contradiction. Therefore, G is a regular graph with diameter $d \leqslant 2$. Conversely, by Lemma 4.4, we can get that $\rho_\alpha(G) = 2n-2-\delta_1$, if G is a regular graph with diameter $d \leqslant 2$. ∎

Lemma 4.7 ([12]). *Let G be a connected graph with η being an automorphism of G, and x a distance α-Perron vector of G. Then for $u, v \in V(G)$, $\eta(u) = v$ implies that $x_u = x_v$.*

Theorem 4.8. *Let S_n be a star graph on $n \geqslant 4$ vertices. Then*

$$\rho_\alpha(S_n) = \frac{n(2+\alpha) - 4 + \sqrt{n^2(\alpha-2)^2 - 4(n-1)(2\alpha^2 - 6\alpha + 5)}}{2}.$$

Proof. Let S_n be a star graph on $n \geqslant 4$ and let $\rho_\alpha(S_n)$ be the D_α-spectral radius of S_n. Also let $x = \{x_1, \ldots, x_n\}$ be the Perron eigenvector of $D_\alpha(S_n)$, where each x_i corresponds to vertex v_i for $i = 1, \ldots, n$. Let v_1 be the central vertex of S_n. Then by Lemma 4.7, we see that $x_1 = \cdots = x_n$. Then by using eigenequation $\rho_\alpha(S_n)x_i = D_\alpha(S_n)x_i$, we have

$$\rho_\alpha(S_n)x_1 = \alpha(n-1)x_1 + (1-\alpha)(n-1)x_2 \tag{4.5}$$

and

$$\rho_\alpha(S_n)x_2 = (1-\alpha)x_1 + (2(n-2)+\alpha)x_2. \tag{4.6}$$

Thus, $\rho_\alpha(S_n)$ is the largest root of the equation obtained from equations (4.5) and (4.6),

$$\rho_\alpha^2 + (4 - n(2+\alpha)\rho_\alpha + (2\alpha^2+1)(n-1) + 2\alpha(n(n-4)+3) = 0,$$

$$\rho_\alpha(S_n) = \frac{n(2+\alpha) - 4 + \sqrt{n^2(\alpha-2)^2 - 4(n-1)(2\alpha^2 - 6\alpha + 5)}}{2}.$$

∎

We would like to pose the following problem.

Problem 1. *Let G be a tree on $n \geqslant 4$ vertices different from S_n. Then $\rho_\alpha(G) > \rho_\alpha(S_n)$.*

5. Concluding Remarks

Inspired by the work of [12] work on D_α-spectra, we consider some sharp bounds on generalized distance spectra of connected graphs. The results are the generalization of [24], [10] and [18]. Some results on D_α-spectrum of complete multipartite graphs are discussed, which basically lay the ground work for many other interesting results and problems. These ground results are helpful in many different graph theoretical problems, mainly the problems related to the graphs determined by their D_α-spectra. The bounds discussed and proved in the article are actually very effective in merging the distance Laplacian and distance signless Laplacian spectral theories.

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