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Empirical Measure of Multivariate Complete Dependence

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Abstract In this work, we constructed a kernel-based estimator for measure of complete dependence in the case of random response vectors. Asymptotic properties of this estimator were proved. Simulation studies have also been done to confirm the result.

MSC: 62G05

Keywords: measure of complete dependence; complete dependence; copula; statistical estimator; kernel estimator

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1. Introduction

How can we figure out whether a random variable is a function of other random variables. The answer is to use statistical indicators / measure of association. For example, to test whether a random variable is a linear function of another random variable, we can used Pearson correlation. To check for monotone functional relation, we may used either Hoeffding's phi square or Spearman's rank correlation. But what about the case of general functional relationship without any specific property. In this case, we can use measures of complete dependence. Measure of complete dependence is first defined for random variables [1] (see also [2, 3]). It was later extended to the case of random response vector in [4]. Unfortunately, [4] only studied mathematical properties of the measures. No statistical estimators were given. Over the years, several copula estimators are also given, see for example, [5–9]. However, their convergences are only considered under Chebyshev distance which is weaker than Sobolev distance. Thus, those results can not be used to provide estimators for measure of complete dependence. Therefore, we will need to construct explicitly an estimators for these type of measures.

In this work, we will provide statistical estimator for this measure of complete dependence using kernel-based estimators similarly to the one constructed for the case of random variables in [1] (Section 3). We will focus, however, only on the case where marginals are known which is usually the case for semi-parametric models. We will prove asymptotic

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behaviors of this estimator. Simulation study will also been done to confirm the result (Section 4). Related concepts and terminologies will be given in the next section.

2. Preliminaries

Henceforth, denote $\mathbb{R}$ for the set of real numbers, $\mathbb{I}$ the unit interval, $\vec{x}$ for a vector $(x_1, \ldots, x_d)$, $\vec{1}$ for the vector $(1, \ldots, 1)$, and $\vec{0}$ for the vector $(0, \ldots, 0)$. Sklar's theorem states that there is a one-to-one correspondence between a continuous joint distribution function $H : \mathbb{R}^d \to \mathbb{I}$ and a copula $C : \mathbb{I}^d \to \mathbb{I}$ via the identification

$$H(\vec{x}) = C(F_1(x_1), \ldots, F_d(x_d))$$

for all $\vec{x} \in \mathbb{R}^d$. Here, F_i denotes the ith marginal distribution function of H. Since F_i only contains information of the ith random variable, all dependence structure among these random variables is contained in the associated copula C. This implies that any measure of association should be written as a function of copula. For this reason, Dette et al. [1] defined a measure of regression dependence that quantify the level in which a random variable Y depends on a random variable X via

$$r(Y \mid X) = 6 \int \partial_u C(u, v)^2 \, du dv - 2$$

where C is the copula associated with the random vector (X, Y). Note that, we also have

$$r(Y \mid X) = \frac{\int (\partial_u C(u, v) - C(1, v))^2 \, du dv}{\int C(1, v) - C(1, v)^2 \, dv}.$$

Similar measures are also defined in [2, 3]. Thus, it is natural to extend this measure to the case random vector $\left(X, \vec{Y}\right)$ via

$$\delta\left(\vec{Y} \mid X\right) = \frac{\int (\partial_u C(u, \vec{v}) - C(1, \vec{v}))^2 \, du d\vec{v}}{\int C(1, \vec{v}) - C(1, \vec{v})^2 \, d\vec{v}}.$$

The above measure is denoted by $\delta_1(C)$ in [4]. Clearly, $r(Y \mid X)$ is a special case of $\delta\left(\vec{Y} \mid X\right)$ when the response vector $\vec{Y}$ is actually a random variable Y. Basically, the following properties hold.

(1) $0 \leq \delta\left(\vec{Y} \mid X\right) \leq 1$.
(2) $\delta\left(\vec{Y} \mid X\right) = 0$ if and only if $\vec{Y}$ and X are independent.
(3) $\delta\left(\vec{Y} \mid X\right) = 1$ if and only if $\vec{Y}$ is completely dependent on X, that is, $\vec{Y}$ is a (measurable) function of X.

See [4] for further properties of this measure. Note that measures of complete dependence are also defined in various setting using various concepts, see [10–13] for example. See also [14] for a recent survey of the topic.

Similar to other statistical indicators, these measures have to be estimated since the joint distribution function H and, hence, the copula C is unknown in practice. For this, Dette et al. [1] used kernel estimators defined below as an estimator of $r(Y \mid X)$.

Let $(X_1, Y_1), \ldots, (X_n, Y_n)$ be an i.i.d. sample from a distribution H with associated copula C and marginal distribution functions F and G, respectively. Denote

$$F_n(x) = \frac{1}{n} \sum_{i=1}^{n} 1_{(-\infty, x]}(X_i)$$

and

$$G_n(y) = \frac{1}{n} \sum_{i=1}^{n} 1_{(-\infty, y]}(Y_i)$$

the empirical marginal distribution functions. Let ϕ denote a symmetric kernel with support lying in $[-1, 1]$ and let

$$\Phi(t) = \int_{-\infty}^{t} \phi(s)\, ds$$

for all $t \in \mathbb{R}$. Define

$$\tau_n(u, v) = \frac{1}{nh_1} \sum_{i=1}^{n} \phi\left(\frac{u - F(X_i)}{h_1}\right) \Phi\left(\frac{v - G(Y_i)}{h_2}\right)$$

$$\hat{\tau}_n(u, v) = \frac{1}{nh_1} \sum_{i=1}^{n} \phi\left(\frac{u - F_n(X_i)}{h_1}\right) \Phi\left(\frac{v - G_n(Y_i)}{h_2}\right)$$

for all $u, v \in \mathbb{I}$.

Denote $N(0, \sigma^2)$ the normal distribution with mean zero and variance $\sigma^2 \geq 0$. The case $N(0, 0)$ simply refers to the degenerate distribution concentrated at zero.

Theorem 2.1. [1, Theorem 5.1] *Assume that the bivariate copula C is three times differentiable with respect to the first variable and two times differentiable with respect to the second variable and the kernel ϕ is two times continuously differentiable. If the bandwidth $h_j \to 0$ with*

$$nh_1^3 \to \infty; \qquad nh_1h_2 \to \infty; \qquad nh_1^4 \to 0; \qquad nh_2^4 \to 0,$$

then

$$\sqrt{n}(\hat{r}_n - r) \xrightarrow{D} N(0, 144\sigma^2)$$

for some $\sigma^2 \geq 0$ where $\hat{r}_n = 6\int \hat{\tau}_n^2 - 2$, $r = r(Y \mid X)$, $\tau = \partial_u C$.

In their proof, they actually show that $\hat{r}_n$ can be approximated with $r_n = 6\int \tau_n^2 - 2$ and that the latter converges to a normal distribution by further approximate it as a sum of i.i.d. bounded random variables. They also given an explicit expression of σ^2 which is fairly complicated. Therefore, we decided to left it out of the statement.

In the next section, we will prove similar results for $\delta\left(\vec{Y} \mid X\right)$.

3. Main Results

Henceforth, let $\phi : \mathbb{R} \to \mathbb{R}$ be a nonnegative symmetric smooth function such that $\int_{\mathbb{R}} \phi(t) = 1$ and $\phi(t) = 0$ whenever $|t| \geq 1$. Define $\phi_n(t) = \frac{1}{h_n}\phi\left(\frac{t}{h_n}\right)$, $\Phi(t) = \int_{-\infty}^{t} \phi$ for all $t \in \mathbb{R}$,

$$\omega\left(u, \vec{v}\right) = \phi\left(u\right)\Phi\left(v_1\right)\cdots\Phi\left(v_d\right)$$

and

$$\Omega\left(\vec{v}\right) = \Phi\left(v_1\right)\cdots\Phi\left(v_d\right)$$

for all $u \in \mathbb{R}$ and $\vec{v} \in \mathbb{R}^d$. Notice that $\Phi\left(t\right) = 0$ when $t \leq -1$, $\Phi\left(t\right) = 1$ when $t \geq 1$, and $\int_{-1}^{1} \Phi\left(t\right) dt = 1$ which follows from Fubini's theorem and the fact that ϕ is symmetric and bounded with compact support.

Let $\left(X, \vec{Y}\right)$ be a continuous random vector with associated copula C, $\tau = \partial_1 C$, and $\upsilon\left(\vec{v}\right) = C\left(1, \vec{v}\right)$ for all $\vec{v} \in \mathbb{I}^d$. Denote F the distribution function of X, G_i the distribution function of Y_i, and F_n, G_{in} their empirical counterparts. For convenience, also denote

$$\vec{G}\left(\vec{y}\right) = \left(G_1\left(y_1\right), \ldots, G_d\left(y_d\right)\right)$$

and

$$\vec{G}_n\left(\vec{y}\right) = \left(G_{1n}\left(y_1\right), \ldots, G_{dn}\left(y_d\right)\right)$$

for all $\vec{y} \in \mathbb{R}^d$.

Let $\left(X_j, \vec{Y}_j\right)$ be i.i.d. random vectors with the same distribution as $\left(X, \vec{Y}\right)$. Denote

$$\alpha_{nj}\left(u, \vec{v}\right) = \frac{1}{h_n}\omega\left(\frac{u - F\left(X_j\right)}{h_n}, \frac{1}{h_n}\left(\vec{v} - \vec{G}\left(\vec{Y}_j\right)\right)\right)$$

and

$$\beta_{nj}\left(\vec{v}\right) = \Omega\left(\frac{1}{h_n}\left(\vec{v} - \vec{G}\left(\vec{Y}_j\right)\right)\right)$$

for all $\left(u, \vec{v}\right) \in \mathbb{I}^{d+1}$. Then $\alpha_{nj}\left(u, \vec{v}\right)$ are also i.i.d random variables for each fixed n so we can define $\bar{\alpha}_n\left(u, \vec{v}\right) = \mathbb{E}\alpha_{nj}\left(u, \vec{v}\right) \geq 0$. Note that

$$\begin{aligned}
\bar{\alpha}_n\left(u, v\right) &= \int \frac{1}{h_n}\omega\left(\frac{u - s}{h_n}, \frac{\vec{v} - \vec{t}}{h_n}\right) dC\left(s, \vec{t}\right) \\
&= \int \frac{1}{h_n}\phi\left(\frac{u - s}{h_n}\right)\prod_{i=1}^{d}\Phi\left(\frac{v_i - t_i}{h_n}\right) dC\left(s, \vec{t}\right) \\
&\leq \int \frac{1}{h_n}\phi\left(\frac{u - s}{h_n}\right) dC\left(s, \vec{t}\right) \\
&= \int \frac{1}{h_n}\phi\left(\frac{u - s}{h_n}\right) ds \\
&= 1
\end{aligned}$$

for all $\left(u, \vec{v}\right) \in \mathbb{I}^{d+1}$. Similarly, $\beta_{nj}\left(\vec{v}\right) \in \mathbb{I}$ are also i.i.d random variables for each fixed n so we can define $\bar{\beta}_n\left(\vec{v}\right) = \mathbb{E}\beta_{nj}\left(\vec{v}\right) \in \mathbb{I}$ for all $\vec{v} \in \mathbb{I}^d$.

Now, we will state our first main result.

Theorem 3.1. *Assume $nh_n^2 \to \infty$ while $nh_n^4 \to 0$ and that all first partial derivatives of τ are continuous and bounded. Then*

$$\sqrt{n}\left(\int\int\left(\frac{1}{n}\sum_{j=1}^{n}\left(\alpha_{nj}\left(u,\vec{v}\right)-\beta_{nj}\left(\vec{v}\right)\right)\right)^2 dud\vec{v}-\int\int\left(\tau\left(u,\vec{v}\right)-\upsilon\left(\vec{v}\right)\right)^2 dud\vec{v}\right)$$
$$\xrightarrow{D} \mathrm{N}\left(0,4\sigma^2\right)$$

as $n\to\infty$ where

$$\sigma^2=\int\int\int\left(\tau\left(s,\vec{v}\right)-\upsilon\left(\vec{v}\right)\right)\left(\tau\left(s,\vec{q}\right)-\upsilon\left(\vec{q}\right)\right)\left(\tau\left(s,\vec{v}\wedge\vec{q}\right)-\upsilon\left(\vec{v}\wedge\vec{q}\right)\right)dsd\vec{v}d\vec{q}$$
$$-\left(\int\int\tau\left(u,\vec{v}\right)\left(\tau\left(u,\vec{v}\right)-\upsilon\left(\vec{v}\right)\right)dud\vec{v}\right)^2.$$

Proof. For this, denote

$$A_n=\int\int\left(\frac{1}{n}\sum_{j=1}^{n}\left(\alpha_{nj}\left(u,\vec{v}\right)-\beta_{nj}\left(\vec{v}\right)\right)\right)^2 dud\vec{v}$$

and

$$T=\int\int\left(\tau\left(u,\vec{v}\right)-\upsilon\left(\vec{v}\right)\right)^2 dud\vec{v}.$$

Notice that

$$\begin{aligned}
&A_n-T\\
&=\int\int\left(\frac{1}{n}\left(\alpha_{nj}\left(u,\vec{v}\right)-\beta_{nj}\left(\vec{v}\right)\right)\right)^2 dud\vec{v}-\int\int\left(\tau\left(u,\vec{v}\right)-\upsilon\left(\vec{v}\right)\right)^2 dud\vec{v}\\
&=\int\int\left(\frac{1}{n}\sum_{j=1}^{n}\left(\alpha_{nj}\left(u,\vec{v}\right)-\beta_{nj}\left(\vec{v}\right)\right)-\tau\left(u,\vec{v}\right)+\upsilon\left(\vec{v}\right)\right)^2 dud\vec{v}\\
&\quad+2\int\int\left(\tau\left(u,\vec{v}\right)-\upsilon\left(\vec{v}\right)\right)\left(\frac{1}{n}\sum_{j=1}^{n}\left(\alpha_{nj}\left(u,\vec{v}\right)-\beta_{nj}\left(\vec{v}\right)\right)-\tau\left(u,\vec{v}\right)+\upsilon\left(\vec{v}\right)\right)dud\vec{v}.
\end{aligned}$$

Let

$$Q_n=\int\int\left(\frac{1}{n}\sum_{j=1}^{n}\left(\alpha_{nj}\left(u,\vec{v}\right)-\beta_{nj}\left(\vec{v}\right)\right)-\tau\left(u,\vec{v}\right)+\upsilon\left(\vec{v}\right)\right)^2 dud\vec{v},$$

$$\begin{aligned}
L_{jn}&=\int\int\left(\tau\left(u,\vec{v}\right)-\upsilon\left(\vec{v}\right)\right)\left(\alpha_{nj}\left(u,\vec{v}\right)-\bar{\alpha}_n\left(u,\vec{v}\right)-\beta_{nj}\left(\vec{v}\right)+\bar{\beta}_n\left(\vec{v}\right)\right)dud\vec{v}\\
&=\int\int\left(\tau\left(u,\vec{v}\right)-\upsilon\left(\vec{v}\right)\right)\left(\alpha_{nj}\left(u,\vec{v}\right)-\bar{\alpha}_n\left(u,\vec{v}\right)\right)dud\vec{v},
\end{aligned}$$

and

$$\begin{aligned}
R_n&=\int\int\left(\tau\left(u,\vec{v}\right)-\upsilon\left(\vec{v}\right)\right)\left(\bar{\alpha}_n\left(u,\vec{v}\right)-\bar{\beta}_n\left(\vec{v}\right)-\tau\left(u,\vec{v}\right)+\upsilon\left(\vec{v}\right)\right)dud\vec{v}\\
&=\int\int\left(\tau\left(u,\vec{v}\right)-\upsilon\left(\vec{v}\right)\right)\left(\bar{\alpha}_n\left(u,\vec{v}\right)-\tau\left(u,\vec{v}\right)\right)dud\vec{v}
\end{aligned}$$

so that

$$A_n - T = Q_n + 2\frac{1}{n}\sum_{j=1}^{n} L_{jn} + 2R_n.$$

Here, simplifications of L_{jn} and R_n follows from the fact that $\int (\tau(u,\vec{v}) - v(\vec{v}))\, du = 0$.

We claim that $\sqrt{n}Q_n \to 0$ in probability and $\sqrt{n}R_n \to 0$ which implies

$$\sqrt{n}(A_n - T) = \sqrt{n}Q_n + 2\frac{1}{\sqrt{n}}\sum_{j=1}^{n} L_{jn} + 2\sqrt{n}R_n$$

has the same limiting distribution as $2\frac{1}{\sqrt{n}}\sum_{j=1}^{n} L_{jn}$. The proofs of these two claims are given in the appendix under Claim 1 and Claim 2. Now, L_{jn} are i.i.d. samples with mean zero. Its variance converges to σ^2 as shown in Claim 3. If $\sigma^2 = 0$, we are done. Otherwise, the fact that L_{jn} are bounded (Claim 3) implies

$$\lim_{n\to\infty} \frac{1}{\sqrt{n}\sigma_n^3}\mathbb{E}\left|L_{jn}\right|^3 \leq \frac{8}{\sigma^3}\lim_{n\to\infty}\frac{1}{\sqrt{n}} = 0.$$

In any case, the result must follow. ■

Using similar arguments, we can also show that

$$\mathrm{Var}\left(\int \beta_n(\vec{v}) - \beta_n(\vec{v})^2 \, d\vec{v}\right) = O\left(\frac{1}{n}\right).$$

Therefore, we have

$$\sqrt{n}\left(\delta_n - \delta\left(\vec{Y} \mid X\right)\right)$$

converges to a normal distribution where

$$\delta_n = \frac{\int\int \left(\frac{1}{n}\sum_{j=1}^{n}(\alpha_{nj}(u,\vec{v}) - \beta_{nj}(\vec{v}))\right)^2 du d\vec{v}}{\int \beta_n(\vec{v}) - \beta_n(\vec{v})^2 \, d\vec{v}}.$$

In other words, δ_n can be used as an estimator of $\delta\left(\vec{Y} \mid X\right)$.

Next, we will discuss simulation result.

4. Simulation Study

In this section, we will provide a simulation study for the estimators

$$\delta_n = \frac{\int\int \left(\frac{1}{n}\sum_{j=1}^{n}(\alpha_{nj}(u,\vec{v}) - \beta_{nj}(\vec{v}))\right)^2 du d\vec{v}}{\int \beta_n(\vec{v}) - \beta_n(\vec{v})^2 \, d\vec{v}}.$$

The sample will be drawn from the Clayton copula C_θ defined by

$$C_\theta(\vec{v}) = \left(\sum_i \left(v_i^{-\theta} - 1\right) + 1\right)^{-1/\theta}$$

when $\theta > 0$ and $C_0 = \lim_{\theta\to 0^+} C_\theta$ is the product copula. The true value $\delta(C_\theta)$ is numerically computed using the cubature method with the maximum tolerance sets to 10^{-3} for better performance. Samples are simulated using multivariate quantile transform.

A simple kernel $\phi(t) = \left(1-t^2\right)^2 1_{\{-1\le t\le 1\}}$ is used. To reduce the boundary effects, we adapted the method presented in [15, 16] where $\phi\left(\frac{u-s}{h}\right)$ is replaced by $\phi\left(\frac{u-s}{b(u)h}\right) k(u,s,h)$ where $b(u) = \min\left(\sqrt{u}, \sqrt{1-u}\right)$ and

$$k(u,s,h) = \frac{a_2(u,h) - a_1(u,h)s}{a_0(u,h)a_2(u,h) - a_1^2(u,h)} 1_{\left\{\frac{u-1}{h}\le s\le \frac{u}{h}\right\}}$$

with

$$a_i(u,h) = \int_{(u-1)/h}^{u/h} t^i \phi(t)\, dt.$$

The function Φ is also redefined analogously. To improve on the speed of computation, $\delta_n(C_\theta)$ will be computed using Riemann sums over a grid of 50 points in each dimension.

To be precise, samples (X, Y_1, Y_2) of sizes $n = 50, 100, 200$ will be drawn from a trivariate Clayton copula C_θ to compute $\delta_n(C_\theta)$. The simulation was repeated 1000 times for each $\theta = 0, 0.5, 1.0, 1.5, 2.0$. We observed, in this case, that estimators provide a reasonable precision for δ_n when the sample size is at least 100. The results of simulation are reported in the following figures.

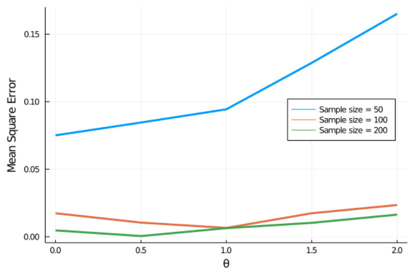

FIGURE 1. Mean square errors for δ_n.

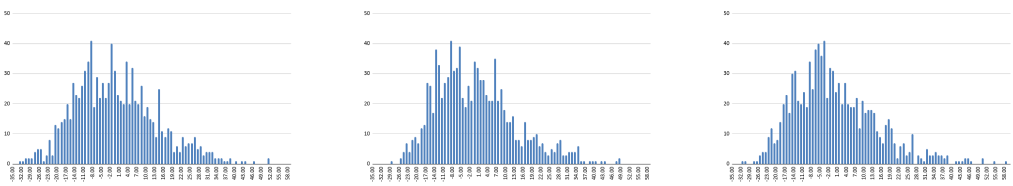

FIGURE 2. Histograms for $\delta_n(C_\theta)$ with sample size $n = 200$ and $\theta = 0, 1, 2$ (left to right).

5. Conclusion and Discussion

In this work, we constructed a kernel-based estimator for measure of complete dependence defined in [4] in the case that the marginal distributions are known which is usually the case for semi-parametric models. The estimator is proved to be consistent and its asymptotic variance has been computed. Its limiting distribution is confirmed to be normal. A simulation study have also been done to verify the result.

It should also be mentioned that measures of complete dependence has also been defined for random vectors [10, 11]. The formulation, however, does not rely on copulas. Thus, different strategies for estimators are required in those cases.

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References

[1] H. Dette, K.F. Siburg, P.A. Stoimenov, A copula-based nonparametric measure of regression dependence, Scandinavian Journal of Statistics 40 (1) (2013) 21–41.

[2] W. Trutschnig, On a strong metric on the space of copulas and its induced dependence measure, Journal of Mathematical Analysis and Applications 384 (2) (2011) 690–705.

[3] K. Siburg, P. Stoimenov, A measure of mutual complete dependence, Metrika 71 (2) (2010) 239–251.

[4] S. Tasena, S. Dhompongsa, A measure of multivariate mutual complete dependence, International Journal of Approximate Reasoning 54 (6) (2013) 748–761.

[5] J. Segers, Asymptotics of empirical copula processes under non-restrictive smoothness assumptions, Bernoulli 18 (3) (2012) 764–782.

[6] J. Rachasingho, S. Tasena, Metric space of subcopulas, Thai Journal of Mathematics (2018) 35–44.

[7] J. Rachasingho, S. Tasena, A metric space of subcopulas - an approach via hausdorff distance, Fuzzy Sets and Systems 378 (2020) 144–156.

[8] S. Tasena, On metric spaces of subcopulas, Fuzzy Sets and Systems 415 (2021) 76–88.

[9] S. Tasena, On a distribution form of subcopulas, International Journal of Approximate Reasoning 128 (2021) 1–19.

[10] S. Tasena, S. Dhompongsa, Measures of the functional dependence of random vectors, International Journal of Approximate Reasoning 68 (2016) 15–26.

[11] T. Boonmee, S. Tasena, Measure of complete dependence of random vectors, Journal of Mathematical Analysis and Applications 443 (1) (2016) 585–595.

[12] Q. Shan, T. Wongyang, T. Wang, S. Tasena, A measure of mutual complete dependence in discrete variables through subcopula, International Journal of Approximate Reasoning 65 (2015) 11–23, SI: Modeling Dependence in Econometrics.

[13] Z. Wei, D. Kim, Subcopula-based measure of asymmetric association for contingency tables, Statistics in medicine 36 (24) (2017) 3875–3894.

[14] S. Tasena, Complete dependence, Direction Dependence in Statistical Modeling: Methods of Analysis (2020) 167.

[15] S.X. Chen, T.M. Huang, Nonparametric estimation of copula functions for dependence modelling, Canadian Journal of Statistics 35 (2) (2007) 265–282.

[16] M. Omelka, I. Gijbels, N. Veraverbeke, Improved kernel estimation of copulas: weak convergence and goodness-of-fit testing, The Annals of Statistics 37 (5B) (2009) 3023–3058.

Appendix

Claim 1. $\sqrt{n}Q_n \to 0$ *in probability.*

Proof. For convenience, denote

$$\begin{aligned}
\tilde{Q}_n &= \int \left(\frac{1}{n} \sum_{j=1}^{n} \left(\alpha_{nj}\left(u, \vec{v}\right) - \beta_{nj}\left(\vec{v}\right)\right) - \bar{\alpha}_n\left(u, \vec{v}\right) + \bar{\beta}_n\left(\vec{v}\right) \right)^2 du d\vec{v}, \\
\bar{Q}_n &= \int \left(\bar{\alpha}_n\left(u, \vec{v}\right) - \bar{\beta}_n\left(\vec{v}\right) - \tau\left(u, \vec{v}\right) + \upsilon\left(\vec{v}\right) \right)^2 du d\vec{v}, \text{ and} \\
\hat{Q}_n &= \int \left(\frac{1}{n} \sum_{j=1}^{n} \left(\alpha_{nj}\left(u, \vec{v}\right) - \beta_{nj}\left(\vec{v}\right)\right) - \bar{\alpha}_n\left(u, \vec{v}\right) + \bar{\beta}_n\left(\vec{v}\right) \right) \\
&\qquad \times \left(\bar{\alpha}_n\left(u, \vec{v}\right) - \bar{\beta}_n\left(\vec{v}\right) - \tau\left(u, \vec{v}\right) + \upsilon\left(\vec{v}\right) \right) du d\vec{v}.
\end{aligned}$$

Then

$$Q_n = \tilde{Q}_n + \bar{Q}_n + 2\hat{Q}_n.$$

By Cauchy–Schwarz inequality,

$$\left|\hat{Q}_n\right| \leq \sqrt{\tilde{Q}_n \bar{Q}_n}$$

which means $\sqrt{n}\hat{Q}_n \to 0$ in probability whenever $\sqrt{n}\tilde{Q}_n \to 0$ in probability and $\sqrt{n}\bar{Q}_n \to 0$. Therefore, the proof is finished when we show these two facts.

For the first fact, we will actually prove that $\sqrt{n}\mathbb{E}\tilde{Q}_n \to 0$ which is a stronger result. Notice that

$$\begin{aligned}
\mathbb{E}\tilde{Q}_n &= \int \frac{1}{n^2} \sum_{j=1}^{n} \operatorname{Var}\left(\alpha_{nj}\left(u, \vec{v}\right) - \beta_{nj}\left(\vec{v}\right)\right) du d\vec{v} \\
&\leq \frac{2}{n} \int \left(\mathbb{E}\alpha_{n1}^2\left(u, \vec{v}\right) + \mathbb{E}\beta_{n1}^2\left(\vec{v}\right) \right) du d\vec{v} \\
&= \frac{2}{nh_n^2} \int \int \phi^2 \left(\frac{u-t}{h_n} \right) dt du + \frac{2}{n} \\
&\leq \frac{2}{nh_n} \left\|\phi\right\| + \frac{2}{n}
\end{aligned}$$

which implies $\sqrt{n}\mathbb{E}\tilde{Q}_n \leq \frac{2}{\sqrt{n}h_n} \left\|\phi\right\| + \frac{2}{\sqrt{n}} \to 0$.

For the second fact, set $\gamma_n(u,\vec{v}) = \int \frac{1}{h_n}\phi\left(\frac{u-s}{h_n}\right) 1_{\left\{\vec{t}\le\vec{1}\wedge\left(\vec{v}+h_n\vec{1}\right)\right\}}\, dC\left(s,\vec{t}\right)$. Using the fact that $\Phi(x)=1$ whenever $x\ge 1$, we have

$$
\begin{aligned}
&\left|\bar{\alpha}_n(u,\vec{v})-\gamma_n(u,\vec{v})\right| \\
&\le \int \frac{1}{h_n}\phi\left(\frac{u-s}{h_n}\right)\left(\int 1_{\left\{\vec{t}\le\vec{1}\wedge\left(\vec{v}+h_n\vec{1}\right)\right\}}1_{\left\{\exists i, t_i\ge 0\vee(v_i-h_n)\right\}}\,\tau\left(s,d\vec{t}\right)\right) ds \\
&\le \int \frac{1}{h_n}\phi\left(\frac{u-s}{h_n}\right)\left(\sum_{i=1}^{d}\int 1_{\left\{0\vee(v_i-h_n)\le t_i\le 1\wedge(v_i+h_n), 0\le t_j\le 1\forall j\ne i\right\}}\,\tau\left(s,d\vec{t}\right)\right) ds \\
&\le \sum_{i=2}^{d+1}\int \frac{1}{h_n}\phi\left(\frac{u-s}{h_n}\right)\left(\tau\left(s,\vec{1}\wedge(\vec{v}+h_n\vec{e}_i)\right)-\tau\left(s,\vec{0}\vee(\vec{v}-h_n\vec{e}_i)\right)\right) ds \\
&\le \sum_{i=2}^{d+1}\int \frac{1}{h_n}\phi\left(\frac{u-s}{h_n}\right)\left(2h_n\left\|\partial_i\tau\right\|\right) ds \\
&= O(h_n)
\end{aligned}
$$

and

$$
\begin{aligned}
&\gamma_n(u,\vec{v})-\tau(u,\vec{v}) \\
&= \int \frac{1}{h_n}\phi\left(\frac{u-s}{h_n}\right)\left(1_{\left\{\vec{t}\le\vec{1}\wedge\left(\vec{v}+h_n\vec{1}\right)\right\}}-1_{\left\{\vec{t}\le\vec{v}\right\}}\right) dC\left(s,\vec{t}\right) \\
&\quad+\int \frac{1}{h_n}\phi\left(\frac{u-s}{h_n}\right)1_{\left\{\vec{t}\le\vec{v}\right\}}\, dC\left(s,\vec{t}\right)-\int \frac{1}{h_n}\phi\left(\frac{u-s}{h_n}\right)\tau(u,\vec{v})\, ds \\
&= \int \frac{1}{h_n}\phi\left(\frac{u-s}{h_n}\right)\left(1_{\cup_{i=1}^{d}\left\{v_i\le t_i\le 1\wedge(v_i+h_n), 0\le t_j\le 1\wedge(v_j+h_n)\forall j\ne i\right\}}\right) dC\left(s,\vec{t}\right) \\
&\quad+\int \phi(w)\left(\tau(u-wh_n,\vec{v})-\tau(u,\vec{v})\right) dw
\end{aligned}
$$

so that $\left|\gamma_n(u,\vec{v})-\tau(u,\vec{v})\right| = O(h_n)$. It follows that $\left|\bar{\alpha}_n(u,\vec{v})-\tau(u,\vec{v})\right| = O(h_n)$ also. This leads to

$$\sqrt{n}\int\int\left(\bar{\alpha}_n(u,\vec{v})-\tau(u,\vec{v})\right)^2 du d\vec{v} = O\left(\sqrt{n}h_n^2\right)\to 0.$$

Similarly,

$$
\begin{aligned}
\left|\bar{\beta}_n(\vec{v})-\upsilon(\vec{v})\right| &= \left|\int\left(\prod_{i=1}^{d}\Phi\left(\frac{v_i-t_i}{h_n}\right)-1_{\left\{\vec{t}\le\vec{v}\right\}}\right) d\upsilon\left(\vec{t}\right)\right| \\
&= O(h_n)
\end{aligned}
$$

so that

$$\sqrt{n}\int\left(\bar{\beta}_n(\vec{v})-\upsilon(\vec{v})\right)^2 d\vec{v} = O\left(\sqrt{n}h_n^2\right)\to 0.$$

Therefore,

$$
\begin{aligned}
0\le\sqrt{n}\bar{Q}_n &\le 2\sqrt{n}\int\int\left(\bar{\alpha}_n(u,\vec{v})-\tau(u,\vec{v})\right)^2 du d\vec{v}+2\sqrt{n}\int\left(\bar{\beta}_n(\vec{v})-\upsilon(\vec{v})\right)^2 d\vec{v} \\
&\to 0
\end{aligned}
$$

as desired. ■

Claim 2. $\sqrt{n}R_n \to 0$.

Proof. We will separate R_n into two parts:

$$\hat{R}_n = \int\int \tau(u,\vec{v})\left(\bar{\alpha}_n(u,\vec{v}) - \tau(u,\vec{v})\right) du d\vec{v}$$

and

$$\tilde{R}_n = \int\int \upsilon(\vec{v})\left(\bar{\alpha}_n(u,\vec{v}) - \tau(u,\vec{v})\right) du d\vec{v}$$

so that $R_n = \hat{R}_n - \tilde{R}_n$.

We will first focus on $\tilde{R}_n$ which is easier to deal with. Since $|\bar{\alpha}_n(u,\vec{v}) - \tau(u,\vec{v})| = O(h_n)$, we have

$$\begin{aligned}
\tilde{R}_n &= \int\int 1_{\{-h_n \le u \le 1+h_n\}} \upsilon(\vec{v})\left(\bar{\alpha}_n(u,\vec{v}) - \tau(u,\vec{v})\right) du d\vec{v} + O\left(h_n^2\right)\\
&= \int \upsilon(\vec{v})\left(\int 1_{\{-h_n \le u \le 1+h_n\}} \bar{\alpha}_n(u,\vec{v})\, du - \upsilon(\vec{v})\right) d\vec{v} + O\left(h_n^2\right)\\
&= \int \upsilon(\vec{v})\left(\int \prod_{i=1}^{d} \Phi\left(\frac{v_i - t_i}{h_n}\right) d\upsilon\left(\vec{t}\right) - \upsilon(\vec{v})\right) d\vec{v} + O\left(h_n^2\right)\\
&= \int \upsilon(\vec{v}) \int \left(\prod_{i=1}^{d} \Phi\left(\frac{v_i - t_i}{h_n}\right) - 1_{\{\vec{t} \le \vec{v}\}}\right) d\upsilon\left(\vec{t}\right) d\vec{v} + O\left(h_n^2\right)\\
&= \int \upsilon(\vec{v}) \int 1_{\{\vec{t} \le \vec{v} + h_n \vec{1}, t_i \ge v_i - h_n\ \exists i\}} \left(\prod_{i=1}^{d} \Phi\left(\frac{v_i - t_i}{h_n}\right) - 1_{\{\vec{t} \le \vec{v}\}}\right) d\upsilon\left(\vec{t}\right) d\vec{v}\\
&\quad + O\left(h_n^2\right)\\
&= \sum_{i=1}^{d} \int \upsilon(\vec{v}) \int \Bigg[1_{\{v_i + h_n \ge t_i \ge v_i - h_n, t_j \le v_j - h_n\ \forall j \ne i\}}\\
&\qquad \times \left(\Phi\left(\frac{v_i - t_i}{h_n}\right) - 1_{\{t_i \le v_i\}}\right)\Bigg] d\upsilon\left(\vec{t}\right) d\vec{v} + O\left(h_n^2\right)\\
&= \sum_{i=1}^{d} \int \Bigg[\upsilon(\vec{v})\, d\left(v_1, \ldots, v_{i-1}, v_{i+1}, \ldots, v_d\right)\\
&\qquad \times \int 1_{\{v_i + h_n \ge t_i \ge v_i - h_n\}} \left(\Phi\left(\frac{v_i - t_i}{h_n}\right) - 1_{\{t_i \le v_i\}}\right) dt_i\Bigg] dv_i + O\left(h_n^2\right)\\
&= \sum_{i=1}^{d} \int \Bigg[\upsilon(\vec{v})\, d\left(v_1, \ldots, v_{i-1}, v_{i+1}, \ldots, v_d\right)\\
&\qquad \times \left(\int 1_{\{-1 \le w \le 1\}} h_n \Phi(w)\, dw - h_n\right)\Bigg] dv + O\left(h_n^2\right)
\end{aligned}$$

which implies

$$\left|\tilde{R}_n\right| \le d\left|\int 1_{\{-1\le w\le 1\}} h_n \Phi\left(w\right)\, dw - h_n\right| + O\left(h_n^2\right) = O\left(h_n^2\right).$$

Next, write

$$\begin{aligned}
\hat{R}_n = & \int\int \tau\left(u,\vec{v}\right)\int\Bigg[\frac{1}{h_n}\phi\left(\frac{u-s}{h_n}\right) \\
& \qquad \times \int\prod_{i=1}^{d}\Phi\left(\frac{v_i-t_i}{h_n}\right)\,\left(\tau\left(s,d\vec{t}\right)-\tau\left(u,d\vec{t}\right)\right)\,ds\Bigg]dud\vec{v} \\
& + \int\int \tau\left(u,\vec{v}\right)\int\Bigg[\frac{1}{h_n}\phi\left(\frac{u-s}{h_n}\right) \\
& \qquad \times \int\left(\prod_{i=1}^{d}\Phi\left(\frac{v_i-t_i}{h_n}\right)-1_{\{\vec{t}\le\vec{v}\}}\right)\tau\left(u,d\vec{t}\right)\,dw\Bigg]dud\vec{v} \\
= & \hat{R}_{1n}+\hat{R}_{2n}.
\end{aligned}$$

For the first term, we have

$$\begin{aligned}
\hat{R}_{1n} = & \int\int \tau\left(u,\vec{v}\right)\int\Bigg[\frac{1}{h_n}\phi\left(\frac{u-s}{h_n}\right) \\
& \qquad \times 1_{\{u\le s\}}\int\prod_{i=1}^{d}\Phi\left(\frac{v_i-t_i}{h_n}\right)\,\left(\tau\left(s,d\vec{t}\right)-\tau\left(u,d\vec{t}\right)\right)\,ds\Bigg]dud\vec{v} \\
& + \int\int \tau\left(u,\vec{v}\right)\int\Bigg[\frac{1}{h_n}\phi\left(\frac{u-s}{h_n}\right)1_{\{s\le u\}} \\
& \qquad \times \int\prod_{i=1}^{d}\Phi\left(\frac{v_i-t_i}{h_n}\right)\,\left(\tau\left(s,d\vec{t}\right)-\tau\left(u,d\vec{t}\right)\right)\,ds\Bigg]dud\vec{v} \\
= & \int\int\int\Bigg[\frac{1}{h_n}\phi\left(\frac{u-s}{h_n}\right)\left(\tau\left(u,\vec{v}\right)-\tau\left(s,\vec{v}\right)\right)1_{\{u\le s\}} \\
& \qquad \times \int\prod_{i=1}^{d}\Phi\left(\frac{v_i-t_i}{h_n}\right)\,\left(\tau\left(s,d\vec{t}\right)-\tau\left(u,d\vec{t}\right)\right)\,ds\Bigg]dud\vec{v} \\
= & \int\int\int\Bigg[\phi\left(w\right)\left(\tau\left(u,\vec{v}\right)-\tau\left(u-wh_n,\vec{v}\right)\right)1_{\{0\le w\le 1\}} \\
& \qquad \times \int\prod_{i=1}^{d}\Phi\left(\frac{v_i-t_i}{h_n}\right)\,\left(\tau\left(u-wh_n,d\vec{t}\right)-\tau\left(u,d\vec{t}\right)\right)\,dw\Bigg]dud\vec{v} \\
= & O\left(h_n^2\right)
\end{aligned}$$

where the second equality is done via interchanging u and s and the last equality follows from the fact that τ has continuous and bounded first order derivatives.

For the second term, we have

$$\hat{R}_{2n} = \sum_{i=1}^{d}\int\int \tau\left(u,\vec{v}\right)\int\frac{1}{h_n}\phi\left(\frac{u-s}{h_n}\right)\hat{R}_{3n}\tau\left(u,d\vec{t}\right)\,dwdu + O\left(h_n^2\right)$$

where

$$\begin{aligned}\hat{R}_{3n} &= \int 1_{\{v_i+h_n\geq t_i\geq v_i-h_n, t_j\leq v_j-h_n\ \forall j\neq i\}}\left(\Phi\left(\frac{v_i-t_i}{h_n}\right)-1_{\{t_i\leq v_i\}}\right)d\vec{v}\\ &= \int 1_{\{t_j\leq v_j-h_n\ \forall j\neq i\}}d\vec{v}\left(h_n\int\Phi(z)\,dz-h_n\right)\\ &= 0.\end{aligned}$$

Therefore, $\hat{R}_n = O\left(h_n^2\right)$ as desired. ∎

Claim 3. *For any fix n, L_{jn} are i.i.d. with $\mathbb{E}L_{jn}=0$ and*

$$\begin{aligned}&\operatorname{Var}(L_{jn})\\ &\to \int\int\int\left(\tau(s,\vec{v})-\upsilon(\vec{v})\right)\left(\tau(s,\vec{q})-\upsilon(\vec{q})\right)\left(\tau(s,\vec{v}\wedge\vec{q})-\upsilon(\vec{v}\wedge\vec{q})\right)ds\,d\vec{v}\,d\vec{q}\\ &\quad-\left(\int\int\tau(u,\vec{v})\left(\tau(u,\vec{v})-\upsilon(\vec{v})\right)du\,d\vec{v}\right)^2.\end{aligned}$$

Moreover, $|L_{jn}|\leq 2$ almost surely.

Proof. The fact that L_{jn} are i.i.d. simply follows from the fact that $\left(X_j,\vec{Y}_j\right)$ are i.i.d. Also,

$$\begin{aligned}\mathbb{E}L_{jn} &= \mathbb{E}\int\int\left(\tau(u,\vec{v})-\upsilon(\vec{v})\right)\left(\alpha_{nj}(u,\vec{v})-\bar{\alpha}_n(u,\vec{v})\right)du\,d\vec{v}\\ &= \int\int\left(\tau(u,\vec{v})-\upsilon(\vec{v})\right)\left(\mathbb{E}\alpha_{nj}(u,\vec{v})-\bar{\alpha}_n(u,\vec{v})\right)du\,d\vec{v}\\ &= 0\end{aligned}$$

by Fubini's Theorem.

Denote

$$\begin{aligned}K_n(s,\vec{v},\vec{q}) = \int\int\Bigg[&\frac{1}{h_n^2}\phi\left(\frac{u-s}{h_n}\right)\phi\left(\frac{p-s}{h_n}\right)\\ &\times\left(\tau(u,\vec{v})-\upsilon(\vec{v})\right)\left(\tau(p,\vec{q})-\upsilon(\vec{q})\right)\Bigg]du\,dp\end{aligned}$$

and

$$K(s,\vec{v},\vec{q}) = \left(\tau(s,\vec{v})-\upsilon(\vec{v})\right)\left(\tau(s,\vec{q})-\upsilon(\vec{q})\right).$$

The fact that τ is continuous implies that $K_n\to K$ pointwisely.

Similarly, let

$$M_n(s,\vec{v},\vec{q}) = \int\prod_{j=1}^{d}\Phi\left(\frac{v_j-t_j}{h_n}\right)\prod_{j=1}^{d}\Phi\left(\frac{q_j-t_j}{h_n}\right)\left(\tau\left(s,d\vec{t}\right)-d\upsilon\left(\vec{t}\right)\right)$$

and

$$M(s,\vec{v},\vec{q}) = \int 1_{\{\vec{t}\leq\vec{v}\}}1_{\{\vec{t}\leq\vec{q}\}}\left(\tau\left(s,d\vec{t}\right)-d\upsilon\left(\vec{t}\right)\right) = \tau(s,\vec{v}\wedge\vec{q})-\upsilon(\vec{v}\wedge\vec{q}).$$

Then $M_n \to M$ pointwisely by the Bounded Convergence Theorem and the fact that

$$\lim_{n\to\infty} \Phi\left(\frac{v-t}{h_n}\right) = \begin{cases} 1, & t<v; \\ \frac{1}{2}, & t=v; \\ 0, & t>v. \end{cases}$$

Now,

$$\begin{aligned}
&\mathrm{Var}\left(L_{jn}\right) \\
&= \mathbb{E}\left(\int\int \alpha_{nj}\left(u,\vec{v}\right)\left(\tau\left(u,\vec{v}\right) - \upsilon\left(\vec{v}\right)\right)\, dud\vec{v}\right)^2 \\
&\quad - \left(\int\int \bar{\alpha}_n\left(u,\vec{v}\right)\left(\tau\left(u,\vec{v}\right) - \upsilon\left(\vec{v}\right)\right)\, dud\vec{v}\right)^2 \\
&= \int\int\int\int \mathbb{E}\left(\alpha_{nj}\left(u,\vec{v}\right)\alpha_{nj}\left(p,\vec{q}\right)\right)\left(\tau\left(u,\vec{v}\right) - \upsilon\left(\vec{v}\right)\right)\left(\tau\left(p,\vec{q}\right) - \upsilon\left(\vec{q}\right)\right)\, dud\vec{v}dpd\vec{q} \\
&\quad - \left(\int\int \bar{\alpha}_n\left(u,\vec{v}\right)\left(\tau\left(u,\vec{v}\right) - \upsilon\left(\vec{v}\right)\right)\, dud\vec{v}\right)^2 \\
&= \int\int\int K_n\left(s,\vec{v},\vec{q}\right) M_n\left(s,\vec{v},\vec{q}\right)\, dsd\vec{v}d\vec{q} \\
&\quad - \left(\int\int \bar{\alpha}_n\left(u,\vec{v}\right)\left(\tau\left(u,\vec{v}\right) - \upsilon\left(\vec{v}\right)\right)\, dud\vec{v}\right)^2 \\
&\to \int\int\int K\left(s,\vec{v},\vec{q}\right) M\left(s,\vec{v},\vec{q}\right)\, dsd\vec{v}d\vec{q} - \left(\int\int \tau\left(u,\vec{v}\right)\left(\tau\left(u,\vec{v}\right) - \upsilon\left(\vec{v}\right)\right)\, dud\vec{v}\right)^2
\end{aligned}$$

where the last step follows again from the Bounded Convergence Theorem.

Last, notice that $\bar{\alpha}_n, \beta_{nj}, \bar{\beta}_n, \tau, \upsilon \in [0,1]$ and $\alpha_{nj} \geq 0$ so that

$$\begin{aligned}
L_{jn} &\leq \int\int \tau\left(u,\vec{v}\right)\alpha_{nj}\left(u,\vec{v}\right)\, dud\vec{v} + \int\int \upsilon\left(\vec{v}\right)\bar{\alpha}_n\left(u,\vec{v}\right)\, dud\vec{v} \\
&\leq \int\int \alpha_{nj}\left(u,\vec{v}\right)\, dud\vec{v} + \int\int \upsilon\left(\vec{v}\right)\, d\vec{v} \\
&\leq 1 + \frac{1}{2} \\
&\leq 2
\end{aligned}$$

and

$$\begin{aligned}
-L_{jn} &\leq \int\int \tau\left(u,\vec{v}\right)\bar{\alpha}_n\left(u,\vec{v}\right)\, dud\vec{v} + \int\int \upsilon\left(\vec{v}\right)\alpha_{nj}\left(u,\vec{v}\right)\, dud\vec{v} \\
&\leq \int\int \tau\left(u,\vec{v}\right)\, dud\vec{v} + \int\int \upsilon\left(\vec{v}\right)\, d\vec{v} \\
&\leq \frac{1}{2} + \frac{1}{2} \\
&\leq 1
\end{aligned}$$

as desired. ∎