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Applicability of Representations of F_4 to Solutions of a System of PDEs

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Abstract This paper is concerned with the applications of representations of the Lie group of class F_4 to PDEs. A realization of all irreducible finite-dimensional representations of F_4 is found and their application to a study of solutions of some systems of partial differential equations is given.

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1. Introduction

The theory of group representations is important mathematical objects with broad applications. It has been known for many years that the F_4 Lie algebra and group are closely related to atomic physics (see [1]). Wadzinski [2] considered the group F_4 in the classification of the states of an N-electron configuration $(s+d+g+h)^N$. Judd [3] has considered the applicability of the Lie group F_4 to the atomic f-shell.

Xu [4] considered partial differential equations approach to F_4 and used partial differential equations to explicitly find all the singular vectors of the polynomial representation of the simple Lie algebra of type F_4 over its 26-dimensional basic irreducible module, which also supplements a proof of the completeness of Brion's abstractly described generators. In this paper, we consider partial differential equations approach to F_4 in other ways. We utilize the 26×26 matrix generators for Lie algebra F_4 in [5] for constructing representations of F_4. A realization of all irreducible finite-dimensional representations of F_4 is found in section 3. In the last section, we study solutions of a system of PDEs through the representations.

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Consider the system of four partial differential equations as follows:

$$
\begin{aligned}
\Big[-\frac{\partial}{\partial a} - p\frac{\partial}{\partial b} - q\frac{\partial}{\partial c} - r\frac{\partial}{\partial d} - s\frac{\partial}{\partial e} - t\frac{\partial}{\partial f} + (e - as - pc + bq)\frac{\partial}{\partial i} - u\frac{\partial}{\partial g} & \\
+(br - at + f - pd)\frac{\partial}{\partial j} + (-pu + qt - rs)\frac{\partial}{\partial h} + (-dq + cr - au + g)\frac{\partial}{\partial k} & \\
+(-2pau + 2er + bu + ds + 2pg - 2fq - ct - h - 2asr + 2atq)\frac{\partial}{\partial l} & \\
+(bqt - pbu - brs + et + ph - fs)\frac{\partial}{\partial m} + (eu + hq - gs + ctq - crs - pcu)\frac{\partial}{\partial n} & \\
+(dqt - pdu + hr + fu - gt - dsr)\frac{\partial}{\partial o}\Big]^{n_1+1}\varphi &= 0, \qquad (1.1)\\
\Big[-\frac{\partial}{\partial p} - v\frac{\partial}{\partial q} - w\frac{\partial}{\partial r} + (q - pv)\frac{\partial}{\partial s} + (r - pw)\frac{\partial}{\partial t} + (vr - qw)\frac{\partial}{\partial u}\Big]^{n_2+1}\varphi &= 0,\\
\Big[-\frac{\partial}{\partial v} - x\frac{\partial}{\partial w}\Big]^{n_3+1}\varphi &= 0,\\
\Big[-\frac{\partial}{\partial x}\Big]^{n_4+1}\varphi &= 0,
\end{aligned}
$$

where n_1, n_2, n_3, and n_4 are non-negative integers. We will find all solutions of the system by examining the Lie algebra of differential operators generated by the linear differential operators

$$
\begin{aligned}
A &= -\frac{\partial}{\partial a} - p\frac{\partial}{\partial b} - q\frac{\partial}{\partial c} - r\frac{\partial}{\partial d} - s\frac{\partial}{\partial e} - t\frac{\partial}{\partial f} + (e - as - pc + bq)\frac{\partial}{\partial i} - u\frac{\partial}{\partial g} + \\
&\quad (br - at + f - pd)\frac{\partial}{\partial j} + (-pu + qt - rs)\frac{\partial}{\partial h} + (-dq + cr - au + g)\frac{\partial}{\partial k} + \\
&\quad (-2pau + 2er + bu + ds + 2pg - 2fq - ct - h - 2asr + 2atq)\frac{\partial}{\partial l} + \\
&\quad (bqt - pbu - brs + et + ph - fs)\frac{\partial}{\partial m} + (eu + hq - gs + ctq - crs - pcu)\frac{\partial}{\partial n} + \\
&\quad (dqt - pdu + hr + fu - gt - dsr)\frac{\partial}{\partial o},\\
B &= -\frac{\partial}{\partial p} - v\frac{\partial}{\partial q} - w\frac{\partial}{\partial r} + (q - pv)\frac{\partial}{\partial s} + (r - pw)\frac{\partial}{\partial t} + (vr - qw)\frac{\partial}{\partial u},\\
C &= -\frac{\partial}{\partial v} - x\frac{\partial}{\partial w},\\
D &= -\frac{\partial}{\partial x}.
\end{aligned}
$$

The system can be written as

$$
\begin{aligned}
A^{n_1+1}\varphi &= 0,\\
B^{n_2+1}\varphi &= 0,\\
C^{n_3+1}\varphi &= 0,\\
D^{n_4+1}\varphi &= 0.
\end{aligned}
$$

The Lie algebra $\mathbb{Z}$ of differential operators generated by A, B, C, and D is the Lie algebra of a 24-dimensional nilpotent Lie algebra, which turns out to be isomorphic to a maximal nilpotent subalgebra of the exceptional simple Lie algebra of class F_4. This property of the differential operators A, B, C, and D is useful for studying the solutions of the system of PDEs (1.1).

2. Matrix Generators for the Lie Algebra F_4

Howlett et al. [5] gives a uniform method of constructing matrix generators for algebras of Lie type with particular emphasis on the exceptional Lie algebras. The constructions have been implemented in a computer algebra system Magma. We consider the 26×26 matrix generators for Lie algebra F_4 in [5] as follows:

$$\begin{aligned}
e_{\alpha_1} &= E_{1,2}+E_{6,8}+E_{7,10}+E_{9,12}+2E_{11,13}+E_{11,14}+E_{13,15}+E_{16,17}+E_{18,19}+\\
&\quad E_{20,21}+E_{25,26},\\
e_{\alpha_2} &= E_{2,3}+E_{4,6}+E_{5,7}+E_{9,11}+E_{12,13}+2E_{12,14}+E_{14,16}+E_{15,17}+E_{19,22}+\\
&\quad E_{21,23}+E_{24,25},\\
e_{\alpha_3} &= E_{3,4}+E_{7,9}+E_{10,12}+E_{16,18}+E_{17,19}+E_{23,24},\\
e_{\alpha_4} &= E_{4,5}+E_{6,7}+E_{8,10}+E_{18,20}+E_{19,21}+E_{22,23},\\
e_{-\alpha_1} &= E_{2,1}+E_{8,6}+E_{10,7}+E_{12,9}+E_{13,11}+2E_{15,13}+E_{15,14}+E_{17,16}+E_{19,18}+\\
&\quad E_{21,20}+E_{26,25},\\
e_{-\alpha_2} &= E_{3,2}+E_{6,4}+E_{7,5}+E_{11,9}+E_{14,12}+E_{16,13}+2E_{16,14}+E_{17,15}+E_{22,19}+\\
&\quad E_{23,21}+E_{25,24},\\
e_{-\alpha_3} &= e_{\alpha_3}^t,\\
e_{-\alpha_4} &= e_{\alpha_4}^t,
\end{aligned}$$

where $E_{i,j}$ is the 26×26 matrix with 1 in the ijth position and zeros elsewhere.

Let

$$\begin{aligned}
h_1 &= [e_{\alpha_1}, e_{-\alpha_1}],\\
h_2 &= [e_{\alpha_2}, e_{-\alpha_2}],\\
h_3 &= [e_{\alpha_3}, e_{-\alpha_3}],\\
h_4 &= [e_{\alpha_4}, e_{-\alpha_4}],\\
e_{\alpha_1+\alpha_2} &= [e_{\alpha_1}, e_{\alpha_2}],\\
e_{\alpha_1+\alpha_2+\alpha_3} &= [e_{\alpha_1+\alpha_2}, e_{\alpha_3}],\\
e_{\alpha_1+2\alpha_2+\alpha_3} &= [e_{\alpha_1+\alpha_2+\alpha_3}, e_{\alpha_2}],\\
e_{\alpha_1+\alpha_2+\alpha_3+\alpha_4} &= [e_{\alpha_1+\alpha_2+\alpha_3}, e_{\alpha_4}],\\
e_{\alpha_1+2\alpha_2+\alpha_3+\alpha_4} &= [e_{\alpha_1+\alpha_2+\alpha_3+\alpha_4}, e_{\alpha_2}],\\
e_{\alpha_1+2\alpha_2+2\alpha_3+\alpha_4} &= [e_{\alpha_1+2\alpha_2+\alpha_3+\alpha_4}, e_{\alpha_3}],\\
e_{\alpha_1+3\alpha_2+2\alpha_3+\alpha_4} &= [e_{\alpha_1+2\alpha_2+2\alpha_3+\alpha_4}, e_{\alpha_2}],\\
e_{\alpha_2+\alpha_3} &= [e_{\alpha_2}, e_{\alpha_3}],\\
e_{\alpha_2+\alpha_3+\alpha_4} &= [e_{\alpha_2+\alpha_3}, e_{\alpha_4}],\\
e_{2\alpha_1+3\alpha_2+2\alpha_3+\alpha_4} &= [e_{\alpha_1+3\alpha_2+2\alpha_3+\alpha_4}, e_{\alpha_1}],\\
e_{2\alpha_1+2\alpha_2+\alpha_3} &= \frac{1}{2}[e_{\alpha_1+2\alpha_2+2\alpha_3}, e_{\alpha_1}],\\
e_{2\alpha_1+2\alpha_2+\alpha_3+\alpha_4} &= [e_{2\alpha_1+2\alpha_2+\alpha_3}, e_{\alpha_4}],\\
e_{2\alpha_1+2\alpha_2+2\alpha_3+\alpha_4} &= [e_{2\alpha_1+2\alpha_2+\alpha_3+\alpha_4}, e_{\alpha_3}],
\end{aligned}$$

$$
\begin{aligned}
e_{\alpha_3+\alpha_4} &= [e_{\alpha_3}, e_{\alpha_4}], \\
e_{2\alpha_2+\alpha_3} &= \frac{1}{2}[e_{\alpha_2+\alpha_3}, e_{\alpha_2}], \\
e_{2\alpha_2+\alpha_3+\alpha_4} &= [e_{2\alpha_2+\alpha_3}, e_{\alpha_4}], \\
e_{2\alpha_2+2\alpha_3+\alpha_4} &= [e_{2\alpha_2+\alpha_3+\alpha_4}, e_{\alpha_3}], \\
e_{2\alpha_1+4\alpha_2+2\alpha_3+\alpha_4} &= \frac{1}{2}[e_{\alpha_1+3\alpha_2+2\alpha_3+\alpha_4}, e_{\alpha_1+\alpha_2}], \\
e_{2\alpha_1+4\alpha_2+3\alpha_3+\alpha_4} &= [e_{2\alpha_1+4\alpha_2+2\alpha_3+\alpha_4}, e_{\alpha_3}], \\
e_{2\alpha_1+4\alpha_2+3\alpha_3+2\alpha_4} &= [e_{2\alpha_1+4\alpha_2+3\alpha_3+\alpha_4}, e_{\alpha_4}], \\
e_{-\alpha_1-\alpha_2} &= [e_{-\alpha_1}, e_{-\alpha_2}], \\
e_{-\alpha_1-\alpha_2-\alpha_3} &= [e_{-\alpha_1-\alpha_2}, e_{-\alpha_3}], \\
e_{-\alpha_1-2\alpha_2-\alpha_3} &= [e_{-\alpha_1-\alpha_2-\alpha_3}, e_{-\alpha_2}], \\
e_{-\alpha_1-\alpha_2-\alpha_3-\alpha_4} &= [e_{-\alpha_1-\alpha_2-\alpha_3}, e_{-\alpha_4}], \\
e_{-\alpha_1-2\alpha_2-\alpha_3-\alpha_4} &= [e_{-\alpha_1-\alpha_2-\alpha_3-\alpha_4}, e_{-\alpha_2}], \\
e_{-\alpha_1-2\alpha_2-2\alpha_3-\alpha_4} &= [e_{-\alpha_1-2\alpha_2-\alpha_3-\alpha_4}, e_{-\alpha_3}], \\
e_{-\alpha_1-3\alpha_2-2\alpha_3-\alpha_4} &= [e_{-\alpha_1-2\alpha_2-2\alpha_3-\alpha_4}, e_{-\alpha_2}], \\
e_{-\alpha_2-\alpha_3} &= [e_{-\alpha_2}, e_{-\alpha_3}], \\
e_{-\alpha_2-\alpha_3-\alpha_4} &= [e_{-\alpha_2-\alpha_3}, e_{-\alpha_4}], \\
e_{-2\alpha_1-3\alpha_2-2\alpha_3-\alpha_4} &= [e_{-\alpha_1-3\alpha_2-2\alpha_3-\alpha_4}, e_{-\alpha_1}], \\
e_{-2\alpha_1-2\alpha_2-\alpha_3} &= \frac{1}{2}[e_{-\alpha_1-2\alpha_2-2\alpha_3}, e_{-\alpha_1}], \\
e_{-2\alpha_1-2\alpha_2-\alpha_3-\alpha_4} &= [e_{-2\alpha_1-2\alpha_2-\alpha_3}, e_{-\alpha_4}], \\
e_{-2\alpha_1-2\alpha_2-2\alpha_3-\alpha_4} &= [e_{-2\alpha_1-2\alpha_2-\alpha_3-\alpha_4}, e_{-\alpha_3}], \\
e_{-\alpha_3-\alpha_4} &= [e_{-\alpha_3}, e_{-\alpha_4}], \\
e_{-2\alpha_2-\alpha_3} &= \frac{1}{2}[e_{-\alpha_2-\alpha_3}, e_{-\alpha_2}], \\
e_{-2\alpha_2-\alpha_3-\alpha_4} &= [e_{-2\alpha_2-\alpha_3}, e_{-\alpha_4}], \\
e_{-2\alpha_2-2\alpha_3-\alpha_4} &= [e_{-2\alpha_2-\alpha_3-\alpha_4}, e_{-\alpha_3}], \\
e_{-2\alpha_1-4\alpha_2-2\alpha_3-\alpha_4} &= \frac{1}{2}[e_{-\alpha_1-3\alpha_2-2\alpha_3-\alpha_4}, e_{-\alpha_1-\alpha_2}], \\
e_{-2\alpha_1-4\alpha_2-3\alpha_3-\alpha_4} &= [e_{-2\alpha_1-4\alpha_2-2\alpha_3-\alpha_4}, e_{-\alpha_3}], \\
e_{-2\alpha_1-4\alpha_2-3\alpha_3-2\alpha_4} &= [e_{-2\alpha_1-4\alpha_2-3\alpha_3-\alpha_4}, e_{-\alpha_4}],
\end{aligned}
$$

where the commutator $[a, b] = ab - ba$.

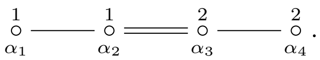
We shall denote by $\mathfrak{g}$ the Lie algebra of class F_4 spanned by h_1, h_2, h_3, h_4 and the root vectors e_i, which correspond to the roots i, where that positive roots are α_1, α_2, α_3, α_4, $\alpha_1+\alpha_2$, $\alpha_1+\alpha_2+\alpha_3$, $\alpha_1+2\alpha_2+\alpha_3$, $\alpha_1+\alpha_2+\alpha_3+\alpha_4$, $\alpha_1+2\alpha_2+\alpha_3+\alpha_4$, $\alpha_1+2\alpha_2+2\alpha_3+\alpha_4$, $\alpha_1+3\alpha_2+2\alpha_3+\alpha_4$, $\alpha_2+\alpha_3$, $\alpha_2+\alpha_3+\alpha_4$, $2\alpha_1+3\alpha_2+2\alpha_3+\alpha_4$, $2\alpha_1+2\alpha_2+\alpha_3$, $2\alpha_1+2\alpha_2+\alpha_3+\alpha_4$, $2\alpha_1+2\alpha_2+2\alpha_3+\alpha_4$, $\alpha_3+\alpha_4$, $2\alpha_2+\alpha_3$, $2\alpha_2+\alpha_3+\alpha_4$, $2\alpha_2+2\alpha_3+\alpha_4$, $2\alpha_1+4\alpha_2+2\alpha_3+\alpha_4$, $2\alpha_1+4\alpha_2+3\alpha_3+\alpha_4$, $2\alpha_1+4\alpha_2+3\alpha_3+2\alpha_4$, and negative roots are $-\alpha_1$, $-\alpha_2$, $-\alpha_3$, $-\alpha_4$, $-\alpha_1-\alpha_2$, $-\alpha_1-\alpha_2-\alpha_3$, $-\alpha_1-2\alpha_2-\alpha_3$, $-\alpha_1-\alpha_2-\alpha_3-\alpha_4$, $-\alpha_1-2\alpha_2-\alpha_3-\alpha_4$, $-\alpha_1-2\alpha_2-2\alpha_3-\alpha_4$, $-\alpha_1-3\alpha_2-2\alpha_3-\alpha_4$,

$-\alpha_2-\alpha_3$, $-\alpha_2-\alpha_3-\alpha_4$, $-2\alpha_1-3\alpha_2-2\alpha_3-\alpha_4$, $-2\alpha_1-2\alpha_2-\alpha_3$, $-2\alpha_1-2\alpha_2-\alpha_3-\alpha_4$, $-2\alpha_1-2\alpha_2-2\alpha_3-\alpha_4$, $-\alpha_3-\alpha_4$, $2\alpha_2-\alpha_3$, $2\alpha_2-\alpha_3-\alpha_4$, $-2\alpha_2-2\alpha_3-\alpha_4$, $-2\alpha_1-4\alpha_2-2\alpha_3-\alpha_4$, $-2\alpha_1-4\alpha_2-3\alpha_3-\alpha_4$, $-2\alpha_1-4\alpha_2-3\alpha_3-2\alpha_4$, and a simple system of roots $(\alpha_1,\alpha_2,\alpha_3,\alpha_4)$.

The Cartan matrix is

$$\begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -2 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{pmatrix}$$

and the corresponding Dynkin diagram is



3. Constructing Representations of F_4

Let G be the Lie group with Lie algebra $\mathfrak{g}$. Then $H_i(t_i)=e^{t_ih_i}$, $E_i(t_i)=e^{t_ie_i}$ are one parameter subgroups of Lie group G.

Using these one-parameter subgroups of group G, we shall now construct some of its subgroups that will be utilized for constructing some representations of G. The first of these subgroups is a maximal nilpotent subgroup of group G and is constructed as follows:

$$Z_+ = \left\{ \prod_{i\in\triangle^+} E_i(t_i) | t_i\in\mathbb{R} \right\},$$

where $\triangle^+$ is a set of all positive roots.

We also construct another maximal nilpotent subgroup as follows:

$$Z_- = \left\{ \prod_{i\in\triangle^+} E_{-i}(t_{-i}) | t_{-i}\in\mathbb{R} \right\}.$$

The subgroup denoted by H, which is a maximal abelian subgroup of G, is defined as follows:

$$H = \{H_1(t_1)H_2(t_2)H_3(t_3)H_4(t_4)|t_1,t_2,t_3,t_4\in\mathbb{R}\},$$

and the subgroup denoted by B_-, which is a maximal solvable subgroup of G, is defined as $B_-=Z_-H=\{z_-h|z_-\in Z_-, h\in H\}$.

For $(P,Q,R,S)\in\mathbb{C}^4$, we define a mapping $\alpha_{P,Q,R,S}: H\to\mathbb{C}$ by

$$H_1(t_1)H_2(t_2)H_3(t_3)H_4(t_4)\mapsto e^{Pt_1}e^{Qt_2}e^{Rt_3}e^{St_4}.$$

By the basic property of the exponential function, $\alpha_{P,Q,R,S}$ is a character of group H.

We extend $\alpha_{P,Q,R,S}$ from H to B_- by the rule: for $b_-=z_-h\in B_-$, $z_-\in Z_-$, $h\in H$,

$$\alpha_{P,Q,R,S}(b_-)=\alpha_{P,Q,R,S}(z_-h)=\alpha_{P,Q,R,S}(h).$$

For $(P,Q,R,S)\in\mathbb{C}^4$, we define an induced representation $T^{\alpha_{P,Q,R,S}}=\mathrm{ind}_{B_-}^G\alpha_{P,Q,R,S}$. It operates in the space

$$F_{\alpha_{P,Q,R,S}}(G)=\{f\in C^\infty(G)|f(b_-g)=\alpha_{P,Q,R,S}(b_-)f(g), b_-\in B_-, g\in G\},$$

where $C^\infty(G)$ is the set of all complex-valued smooth functions on G, by

$$T_g^{\alpha_{P,Q,R,S}} f(h) \quad = \quad f(hg), h, g \in G.$$

Because the subset B_-Z_+ is dense in G (see [6]), the functions from the space $F_{\alpha_{P,Q,R,S}}(G)$ are completely determined by their restrictions to the subgroup Z_+. This allows to realize the representations $T^{\alpha_{P,Q,R,S}}$ of G in the space $C^\infty(Z_+)$. The respective representation of the Lie algebra $\mathfrak{g}$ in the space $C^\infty(Z_+)$ is realized via the differential operators as follows: It will be convenient to introduce new parameters. Then we obtain

$$\begin{aligned}
T_{h_1} &= -2a\frac{\partial}{\partial a} - b\frac{\partial}{\partial b} - c\frac{\partial}{\partial c} - d\frac{\partial}{\partial d} - 2i\frac{\partial}{\partial i} - 2j\frac{\partial}{\partial j} + h\frac{\partial}{\partial h} - 2k\frac{\partial}{\partial k} - l\frac{\partial}{\partial l} + p\frac{\partial}{\partial p} + q\frac{\partial}{\partial q} \\
&\quad + r\frac{\partial}{\partial r} + 2s\frac{\partial}{\partial s} + 2t\frac{\partial}{\partial t} + 2u\frac{\partial}{\partial u} + P, \\
T_{h_2} &= a\frac{\partial}{\partial a} - b\frac{\partial}{\partial b} + c\frac{\partial}{\partial c} + d\frac{\partial}{\partial d} - e\frac{\partial}{\partial e} - f\frac{\partial}{\partial f} + g\frac{\partial}{\partial g} - h\frac{\partial}{\partial h} + 2k\frac{\partial}{\partial k} - 2m\frac{\partial}{\partial m} - 2p\frac{\partial}{\partial p} \\
&\quad - 2s\frac{\partial}{\partial s} - 2t\frac{\partial}{\partial t} + 2v\frac{\partial}{\partial v} + 2w\frac{\partial}{\partial w} + Q, \\
T_{h_3} &= b\frac{\partial}{\partial b} - c\frac{\partial}{\partial c} + f\frac{\partial}{\partial f} - g\frac{\partial}{\partial g} + j\frac{\partial}{\partial j} - k\frac{\partial}{\partial k} + m\frac{\partial}{\partial m} - n\frac{\partial}{\partial n} + p\frac{\partial}{\partial p} - q\frac{\partial}{\partial q} + t\frac{\partial}{\partial t} \\
&\quad - u\frac{\partial}{\partial u} - 2v\frac{\partial}{\partial v} - w\frac{\partial}{\partial w} + x\frac{\partial}{\partial x} + R, \\
T_{h_4} &= c\frac{\partial}{\partial c} - d\frac{\partial}{\partial d} + e\frac{\partial}{\partial e} - f\frac{\partial}{\partial f} + i\frac{\partial}{\partial i} - j\frac{\partial}{\partial j} + n\frac{\partial}{\partial n} - o\frac{\partial}{\partial o} + q\frac{\partial}{\partial q} - r\frac{\partial}{\partial r} + s\frac{\partial}{\partial s} \\
&\quad - t\frac{\partial}{\partial t} + v\frac{\partial}{\partial v} - w\frac{\partial}{\partial w} - 2x\frac{\partial}{\partial x} + S, \\
T_{\alpha_1} &= \frac{\partial}{\partial a} + e\frac{\partial}{\partial i} + f\frac{\partial}{\partial j} + g\frac{\partial}{\partial k} + 2h\frac{\partial}{\partial l}, \\
T_{\alpha_2} &= a\frac{\partial}{\partial b} + c\frac{\partial}{\partial e} + d\frac{\partial}{\partial f} + g\frac{\partial}{\partial h} + k\frac{\partial}{\partial l} + (2l - 3ah + 3bg + 3de - 3cf)\frac{\partial}{\partial m} + \frac{\partial}{\partial p} \\
&\quad + q\frac{\partial}{\partial s} + r\frac{\partial}{\partial t}, \\
T_{\alpha_3} &= b\frac{\partial}{\partial c} + f\frac{\partial}{\partial g} + j\frac{\partial}{\partial k} + m\frac{\partial}{\partial n} + p\frac{\partial}{\partial q} + t\frac{\partial}{\partial u} + \frac{\partial}{\partial v}, \\
T_{\alpha_4} &= c\frac{\partial}{\partial d} + e\frac{\partial}{\partial f} + i\frac{\partial}{\partial j} + n\frac{\partial}{\partial o} + q\frac{\partial}{\partial r} + s\frac{\partial}{\partial t} + v\frac{\partial}{\partial w} + \frac{\partial}{\partial x}, \\
T_{-\alpha_1} &= -a^2\frac{\partial}{\partial a} - ba\frac{\partial}{\partial b} - ac\frac{\partial}{\partial c} - da\frac{\partial}{\partial d} + (i - ae)\frac{\partial}{\partial e} + (j - af)\frac{\partial}{\partial f} + (k - ga)\frac{\partial}{\partial g} \\
&\quad + (l - ah)\frac{\partial}{\partial h} - ia\frac{\partial}{\partial i} - ja\frac{\partial}{\partial j} - ka\frac{\partial}{\partial k} \\
&\quad + (al - 2id - 2bk - 2a^2h + 2cj + 2dae + 2bag - 2acf)\frac{\partial}{\partial l} \\
&\quad + (ej - 2b^2g - bl + 2bah - 2bde + 2bcf - if)\frac{\partial}{\partial m} \\
&\quad + (2c^2f - cl - ig + ek + 2ach - 2bcg - 2cde)\frac{\partial}{\partial n} \\
&\quad + (fk - gj - 2d^2e - dl + 2dah + 2cdf - 2bgd)\frac{\partial}{\partial o} \\
&\quad + (pa - b)\frac{\partial}{\partial p} + (aq - c)\frac{\partial}{\partial q} + (ar - d)\frac{\partial}{\partial r} + (pc - bq - 2e + 2as)\frac{\partial}{\partial s} \\
&\quad + (2at - br - 2f + py)\frac{\partial}{\partial t} + (2au + dq - cr - 2g)\frac{\partial}{\partial u} + Pa,
\end{aligned}$$

$$
\begin{aligned}
T_{-\alpha_2} &= b\frac{\partial}{\partial a}+e\frac{\partial}{\partial c}+f\frac{\partial}{\partial d}+h\frac{\partial}{\partial g}+(2l-3ah+3bg+3de-3cf)\frac{\partial}{\partial k}+m\frac{\partial}{\partial l}-p^2\frac{\partial}{\partial p}\\
&\quad+(s-pq)\frac{\partial}{\partial q}+(t-pr)\frac{\partial}{\partial r}-sp\frac{\partial}{\partial s}-tp\frac{\partial}{\partial t}+(rs-qt)\frac{\partial}{\partial u}+(2pv-2q)\frac{\partial}{\partial v}\\
&\quad+(2pw-2r)\frac{\partial}{\partial w}+Qp,\\
T_{-\alpha_3} &= c\frac{\partial}{\partial b}+g\frac{\partial}{\partial f}+k\frac{\partial}{\partial j}+n\frac{\partial}{\partial m}+q\frac{\partial}{\partial p}+u\frac{\partial}{\partial t}-v^2\frac{\partial}{\partial v}-wv\frac{\partial}{\partial w}+(vx-w)\frac{\partial}{\partial x}+Rv,\\
T_{-\alpha_4} &= d\frac{\partial}{\partial c}+f\frac{\partial}{\partial e}+j\frac{\partial}{\partial i}+o\frac{\partial}{\partial n}+r\frac{\partial}{\partial q}+t\frac{\partial}{\partial s}+w\frac{\partial}{\partial v}-x^2\frac{\partial}{\partial x}+Sx.
\end{aligned}
$$

Since Z_+ is a group with 24-dimensional nilpotent Lie algebra over $\mathbb{R}$, we obtain Z_+ is diffeomorphic to $\mathbb{R}^{24}$. Thus we can consider the space of all complex-valued smooth functions on $\mathbb{R}^{24}$ for the representation space of this representation of $\mathfrak{g}$. Furthermore, we can consider the space of all complex-valued polynomial functions of twenty-four variables for the representation space of a representation of $\mathfrak{g}$, because it is invariant under the action of all operators T.

Denote this representation by $\varphi^{P,Q,R,S}$. Note that only 1 is annulled by the above positive root vectors, so 1, whose weight is $P\omega_1+Q\omega_2+R\omega_3+S\omega_4$, where $\omega_1,\omega_2,\omega_3,\omega_4$ are fundamental weights, is the only highest weight vector in the space of the polynomials. Thus, for $\varphi^{P,Q,R,S}$, where P,Q,R,S are non-negative integers, applying products of $T_{-\alpha_1}$, $T_{-\alpha_2}$, $T_{-\alpha_3}$, and $T_{-\alpha_4}$ to 1, we obtain an invariant subspace of the irreducible representation of $\mathfrak{g}$.

For $P=1,Q=0,R=0,S=0$, the subspace is spanned by 26 polynomials: $1,A_2,\ldots,A_{26}$.

For $P=0,Q=1,R=0,S=0$, the subspace is spanned by 273 polynomials: $1,B_2,\ldots,B_{273}$.

For $P=0,Q=0,R=1,S=0$, the subspace is spanned by $1,274$ polynomials: $1,C_2,\ldots,C_{1,274}$.

Finally, for $P=0,Q=0,R=0,S=1$, it is spanned by 52 polynomials: $1,D_2,\ldots,D_{52}$.

4. Irreducible Representations of F_4

Let $V^{P,Q,R,S}$ be the real vector space spanned by

$$A_2^{a_2}\ldots A_{26}^{a_{26}}B_2^{b_2}\ldots B_{273}^{b_{273}}C_2^{c_2}\ldots C_{1,274}^{c_{1,274}}D_2^{d_2}\ldots D_{52}^{d_{52}},$$

where $a_2+\ldots+a_{26}\le P$, $b_2+\ldots+b_{273}\le Q$, $c_2+\ldots+c_{1,274}\le R$, and $d_2+\ldots+d_{52}\le S$.

By direct computation, we obtain $V^{P,Q,R,S}$ is invariant under $\varphi^{P,Q,R,S}$. Thus $\varphi^{P,Q,R,S}$ is a finite-dimensional representation of $\mathfrak{g}$ in $V^{P,Q,R,S}$. Since 1, whose weight is $P\omega_1+Q\omega_2+R\omega_3+S\omega_4$, is the only highest weight vector of $\varphi^{P,Q,R,S}$ in $V^{P,Q,R,S}$, and $\varphi^{P,Q,R,S}$ is completely reducible, so $\varphi^{P,Q,R,S}$ is an irreducible representation of $\mathfrak{g}$ in $V^{P,Q,R,S}$ (see [6]).

5. Solutions of the PDEs

Consider in the space $C^\infty(G)$, two representations of the group G, $S_gf(h)=f(g^{-1}h)$ and $T_gf(h)=f(hg)$, where $h,g\in G$. Observe that S_g and T_h commute in $C^\infty(G)$, for all $g,h\in G$. Let $\mathfrak{X}$ be the dual space for the space $C^\infty(G)$, that is the space of all distributions with compact support on G. Consider in $\mathfrak{X}$, two representations of the group G conjugate to S_g and T_g that we shall denote by $\widetilde{S}_g$ and $\widetilde{T}_g$, where $\widetilde{S}_g = S^*_{g^{-1}}$ and $\widetilde{T}_g = T^*_{g^{-1}}$. Here $*$ denotes the adjoint operator, that is, if $\langle f, F\rangle$ is the canonical bilinear form for the pair $C^\infty(G)$ and $\mathfrak{X}$, then $\langle Af, F\rangle=\langle f, A^*F\rangle$, for any linear operator A in $C^\infty(G)$, $f \in C^\infty(G)$, and $F \in \mathfrak{X}$. Let 1_e be the δ-function on G with support at the identity e of the group G. Then it is easy to see that $\widetilde{S}_g 1_e = 1_g$ and $\widetilde{T}_g 1_e = 1_{g^{-1}}$, where 1_g is the δ-function with support at the point g of G and $1_{g^{-1}}$ is the same for g^{-1}. This implies that $\widetilde{S}_g 1_e = \widetilde{T}_{g^{-1}} 1_e$.

Because the representations S and T are C^∞-differentiable, we may consider their differentials, that is the representations of the universal enveloping algebra $U(\mathfrak{L})$, which we shall also denote by S and T, and the conjugate representations will be again denoted by $\widetilde{S}$ and $\widetilde{T}$. In algebra $U(\mathfrak{L})$, we consider the principal anti-automorphism $u \longrightarrow u'$, where $u' = -u$, for $u \in \mathfrak{L}$, and $(u_1u_2\dots u_k)' = u'_k u'_{k-1}\dots u'_1$. Then we obtain $\widetilde{S}(u)1_e = \widetilde{T}(u')1_e$, for all $u \in U(\mathfrak{L})$.

The space $E_\sigma = F_{\alpha_{p,q}}(G) = \{f \in C^\infty(G) | f(b_- g) = \alpha_{p,q}(b_-)f(g), b_- \in B, g \in G\}$ can be identified as the space of the solutions to the system of PDE.

$S(x_\gamma)f = 0$, for all $\gamma \in \Pi_0$, where Π_0 is the set of simple roots,

$S(x - \langle\sigma - \rho, x\rangle)f = 0$, for all $x \in \mathfrak{H}$, where σ is the signature of the inducing representation and ρ the half-sum of all positive roots (see [7]).

Denote by I_σ the cyclic submodule in $U(\mathfrak{L})$-module $\mathfrak{X}$ generated by the elements $\widetilde{T}(x_\gamma)1_e$, for all $\gamma \in \Pi_0$, $\widetilde{T}(x - \langle\sigma - \rho, x\rangle)1_e$, for all $x \in \mathfrak{H}$.

Proposition 5.1. *E_σ is the orthogonal complement for I_σ with respect to the canonical bilinear form $\langle .,.\rangle$.*

Proof. Let $f \in E_\sigma$. Then
$\left\langle f, \widetilde{T}_g\widetilde{T}_{x_\beta}1_e\right\rangle = \left\langle T_{g^{-1}}f, \widetilde{S}_{x'_\beta}1_e\right\rangle = \left\langle S_{x_\beta}T_{g^{-1}}\ f, 1_e\right\rangle = \left\langle T_{g^{-1}}S_{x_\beta}f, 1_e\right\rangle = 0,$ for all $\beta \in \Pi_0$, and
$\left\langle f, \widetilde{T}_g\widetilde{T}_{x-\langle\sigma-\rho,x\rangle}1_e\right\rangle = \left\langle T_{g^{-1}}f, \widetilde{S}_{-x-\langle\sigma-\rho,x\rangle}1_e\right\rangle = \left\langle T_{g^{-1}}S_{x-\langle\sigma-\rho,x\rangle}f, 1_e\right\rangle = 0,$ for all $x \in h$. So that $E_\sigma \subset (I_\sigma)^\perp$.

Reversely, let $\varphi \in (I_\sigma)^\perp$. Then $T_g\varphi \in (I_\sigma)^\perp$, and
$0 = \left\langle T_{g^{-1}}\varphi, \widetilde{T}_{x_\beta}1_e\right\rangle = \left\langle T_{g^{-1}}S_{x_\beta}\varphi, 1_e\right\rangle = \left\langle S_{x_\beta}\varphi, \widetilde{T}_g 1_e\right\rangle = \left\langle T_{x_\beta}\varphi, 1_g\right\rangle,$ for all $g \in G$.
But this is equivalent to $S_{x_\beta}\varphi = 0$. Similarly for $\langle\sigma - \rho, x\rangle$, $x \in h$. Thus $E_\sigma = (I_\sigma)^\perp$. This is the end of the proof. ∎

Following [8], let $\sigma \in \mathfrak{H}^*$ and χ be a positive root such that $\sigma(\chi) = N$, where N is a positive integer, and let M_σ be the Verma module corresponding to σ, and 1_σ be a highest weight vector of weight $\sigma - \rho$ in the module M_σ. Then $M_{\sigma-N\chi}$ is imbeddable into M_σ, and so there exist $T^N_{\sigma,\chi}$ in a universal enveloping algebra of a maximal nilpotent subalgebra of $\mathfrak{L}$ spanned by all negative root vectors such that $\widetilde{1}_{\sigma-N\chi} = T^N_{\sigma,\chi}1_\sigma$, where $\widetilde{1}_{\sigma-N\chi}$ is the image of $1_{\sigma-N\chi}$ under the imbedding.

Proposition 5.2. *$S(T^N_{\sigma,\chi})$ is an intertwining operator for E_σ and $E_{\sigma'}$, where $\sigma' = \sigma - N\chi$.*

Proof. It is sufficient to show that $S(T^N_{\sigma,\chi})E_\sigma \subset E_{\sigma'}$. Let $f \in E_\sigma$ and $\gamma \in \prod_0$. Then
$\left\langle S_{T^N_{\sigma,\chi}}f, \widetilde{T}_g\widetilde{T}_{x_\gamma}1_e\right\rangle = \left\langle T_{g^{-1}}f, \widetilde{T}_{x_\gamma}\widetilde{S}_{(T^N_{\sigma,\chi})'}1_e\right\rangle = \left\langle T_{g^{-1}}f, \widetilde{T}_{x_\gamma}\widetilde{T}_{T^N_{\sigma,\chi}}1_e\right\rangle = \left\langle T_{g^{-1}}f, \widetilde{T}_{x_\gamma T^N_{\sigma,\chi}}1_e\right\rangle.$
But $x_\gamma T^N_{\sigma,\chi} \in I_\sigma$. So $\widetilde{T}_{x_\gamma T^N_{\sigma,\chi}}1_e \in I_\sigma$. Since $T_{g^{-1}}f \in E_\sigma$, the last expression is equal to 0.

Let $x \in \mathfrak{H}$. Then

$$\begin{aligned} \left\langle S_{T^N_{\sigma,\chi}} f, \widetilde{T}_g \widetilde{T}_{(x-\langle \sigma - N\chi - \rho, x\rangle)} 1_e \right\rangle &= \left\langle T_{g^{-1}} f, \widetilde{S}_{(T^N_{\sigma,\chi})'} \widetilde{T}_{(x-\langle \sigma - N\chi - \rho, x\rangle)} 1_e \right\rangle \\ &= \left\langle T_{g^{-1}} f,\ \widetilde{T}_{(x-\langle \sigma - N\chi - \rho, x\rangle) T^N_{\sigma,\chi}} 1_e \right\rangle . \end{aligned}$$

But $(x - \langle \sigma - N\chi - \rho, x\rangle) T^N_{\sigma,\chi} \in I_\sigma$. So $\widetilde{T}_{(x-\langle \sigma - N\alpha - \rho, x\rangle) T^N_{\sigma,\chi}} 1_e \in I_\sigma$. Therefore the last expression equals 0, because $T_{g^{-1}} f \in E_\sigma \subset I_\sigma^\perp$. This is the end of the proof. ■

When γ is a simple root, then $T^N_{\sigma,\gamma} = z^N_{-\gamma}$, where for any root λ, z_λ is a root vector for λ.

For the simple root α_1,

$$\begin{aligned} I_1 f(h) &= \frac{d}{dr} S_{E^{P+1}_{\alpha_1}(t)} f(h)|_{t=0} \\ &= (-\frac{\partial}{\partial a} - p\frac{\partial}{\partial b} - q\frac{\partial}{\partial c} - r\frac{\partial}{\partial d} - s\frac{\partial}{\partial e} - t\frac{\partial}{\partial f} + (e - as - pc + bq)\frac{\partial}{\partial i} - u\frac{\partial}{\partial g} + \\ &\quad (br - at + f - pd)\frac{\partial}{\partial j} + (-pu + qt - rs)\frac{\partial}{\partial h} + (-dq + cr - au + g)\frac{\partial}{\partial k} + \\ &\quad (-2pau + 2er + bu + ds + 2pg - 2fq - ct - h - 2asr + 2atq)\frac{\partial}{\partial l} + \\ &\quad (bqt - pbu - brs + et + ph - fs)\frac{\partial}{\partial m} + (eu + hq - gs + ctq - crs - pcu)\frac{\partial}{\partial n} + \\ &\quad (dqt - pdu + hr + fu - gt - dsr)\frac{\partial}{\partial o})^{P+1} f(h) \\ &= A^{P+1} f(h) \end{aligned}$$

is an intertwining operator for $E_{P\omega_1+Q\omega_2+R\omega_3+S\omega_4}$ and $E_{-(P+2)\omega_1+(P+Q+1)\omega_2+R\omega_3+S\omega_4}$. So, $V^{P,Q,R,S} \subseteq Ker\ I_1$.

For the simple root α_2,

$$\begin{aligned} I_2 f(h) &= \frac{d}{dt} S_{E^{Q+1}_{\alpha_2}(t)} f(h)|_{t=0} \\ &= (-\frac{\partial}{\partial p} - v\frac{\partial}{\partial q} - w\frac{\partial}{\partial r} + (q - pv)\frac{\partial}{\partial s} + (r - pw)\frac{\partial}{\partial t} + (vr - qw)\frac{\partial}{\partial u})^{Q+1} f(h) \\ &= B^{Q+1} f(h) \end{aligned}$$

is an intertwining operator for

$$E_{P\omega_1+Q\omega_2+R\omega_3+S\omega_4} \text{ and } E_{(P+Q+1)\omega_1-(Q+2)\omega_2+(Q+R+1)\omega_3+S\omega_4}.$$

So $V^{P,Q,R,S} \subseteq Ker\ I_2$.

For the simple root α_3,

$$I_3 f(h) = \frac{d}{dt} S_{E^{s+1}_{\alpha_3}(t)} f(h)|_{t=0} = (-\frac{\partial}{\partial v} - x\frac{\partial}{\partial w})^{R+1} f(h) = C^{R+1} f(h)$$

is an intertwining operator for

$$E_{P\omega_1+Q\omega_2+R\omega_3+S\omega_4} \text{ and } E_{P\omega_1+(2R+Q+2)\omega_2-(R+2)\omega_3+(R+S+1)\omega_4}.$$

So $V^{P,Q,R,S} \subseteq Ker\ I_3$.

For the simple root α_4,

$$I_4 f(h) = \frac{d}{dt} S_{E^{t+1}_{\alpha_4}(t)} f(h)|_{t=0} = (-\frac{\partial}{\partial x})^{S+1} f(h) = D^{S+1} f(h)$$

is an intertwining operator for $E_{P\omega_1+Q\omega_2+R\omega_3+S\omega_4}$ and $E_{P\omega_1+Q\omega_2+(R+S+1)\omega_3-(S+2)\omega_4}$. So $V^{P,Q,R,S} \subseteq Ker\ I_4$.

Then $V^{P,Q,R,S} = (Ker\ I_1) \cap (Ker\ I_2) \cap (Ker\ I_3) \cap (Ker\ I_4)$. That is,

$$A_2^{a_2} \ldots A_{26}^{a_{26}} B_2^{b_2} \ldots B_{273}^{b_{273}} C_2^{c_2} \ldots C_{1,274}^{c_{1,274}}\ D_2^{d_2} \ldots D_{52}^{d_{52}},$$

where $a_2 + \ldots + a_{26} \leq P$, $b_2 + \ldots + b_{273} \leq Q$, $c_2 + \ldots + c_{1,274} \leq R$, and $d_2 + \ldots + d_{52} \leq S$, are all solutions of the system (1.1), and they are the only solutions of this system of partial differential equations.

6. Conclusion

In this paper, we study a method to find all solutions of a certain system of PDEs. by considering representations of a Lie group. That is, the space of solutions of the systems of partial differential equations

$$\begin{aligned}
\Big[-\frac{\partial}{\partial a} - p\frac{\partial}{\partial b} - q\frac{\partial}{\partial c} - r\frac{\partial}{\partial d} - s\frac{\partial}{\partial e} - t\frac{\partial}{\partial f} + (e - as - pc + bq)\frac{\partial}{\partial i} - u\frac{\partial}{\partial g} \\
+(br - at + f - pd)\frac{\partial}{\partial j} + (-pu + qt - rs)\frac{\partial}{\partial h} + (-dq + cr - au + g)\frac{\partial}{\partial k} \\
+(-2pau + 2er + bu + ds + 2pg - 2fq - ct - h - 2asr + 2atq)\frac{\partial}{\partial l} \\
+(bqt - pbu - brs + et + ph - fs)\frac{\partial}{\partial m} + (eu + hq - gs + ctq - crs - pcu)\frac{\partial}{\partial n} \\
+(dqt - pdu + hr + fu - gt - dsr)\frac{\partial}{\partial o}\Big]^{n_1+1} \varphi &= 0, \\
\Big[-\frac{\partial}{\partial p} - v\frac{\partial}{\partial q} - w\frac{\partial}{\partial r} + (q - pv)\frac{\partial}{\partial s} + (r - pw)\frac{\partial}{\partial t} + (vr - qw)\frac{\partial}{\partial u}\Big]^{n_2+1} \varphi &= 0, \\
\Big[-\frac{\partial}{\partial v} - x\frac{\partial}{\partial w}\Big]^{n_3+1} \varphi &= 0, \\
\Big[-\frac{\partial}{\partial x}\Big]^{n_4+1} \varphi &= 0,
\end{aligned}$$

where n_1, n_2, n_3, and n_4 are non-negative integers, and the operators $-\frac{\partial}{\partial a} - p\frac{\partial}{\partial b} - q\frac{\partial}{\partial c} - r\frac{\partial}{\partial d} - s\frac{\partial}{\partial e} - t\frac{\partial}{\partial f} + (e - as - pc + bq)\frac{\partial}{\partial i} - u\frac{\partial}{\partial g} + (br - at + f - pd)\frac{\partial}{\partial j} + (-pu + qt - rs)\frac{\partial}{\partial h} + (-dq + cr - au + g)\frac{\partial}{\partial k} + (-2pau + 2er + bu + ds + 2pg - 2fq - ct - h - 2asr + 2atq)\frac{\partial}{\partial l} + (bqt - pbu - brs + et + ph - fs)\frac{\partial}{\partial m} + (eu + hq - gs + ctq - crs - pcu)\frac{\partial}{\partial n} + (dqt - pdu + hr + fu - gt - dsr)\frac{\partial}{\partial o}$, $-\frac{\partial}{\partial p} - v\frac{\partial}{\partial q} - w\frac{\partial}{\partial r} + (q - pv)\frac{\partial}{\partial s} + (r - pw)\frac{\partial}{\partial t} + (vr - qw)\frac{\partial}{\partial u}$, $-\frac{\partial}{\partial v} - x\frac{\partial}{\partial w}$, $-\frac{\partial}{\partial x}$ generate a Lie algebra of differential operators, that is isomorphic to a maximal nilpotent subalgebra of exceptional Lie algebra of class F_4, is the space $V^{P,Q,R,S}$ of an irreducible representation spanned by the vectors $A_2^{a_2} \ldots A_{26}^{a_{26}} B_2^{b_2} \ldots B_{273}^{b_{273}} C_2^{c_2} \ldots C_{1,274}^{c_{1,274}}$ $D_2^{d_2} \ldots D_{52}^{d_{52}}$, where $a_2 + \ldots + a_{26} \leq P$, $b_2 + \ldots + b_{273} \leq Q$, $c_2 + \ldots + c_{1,274} \leq R$, and $d_2 + \ldots + d_{52} \leq S$.

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