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A Modification of Simulation Function and Its Applications to Fixed Point Theory

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Abstract In this paper a modification of a simulation function is introduced. Further, it is shown that the class of simulation functions is a proper subset of it. Finally some fixed point theorems are established.

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1. Introduction and Preliminaries

In [1] the simulation function introduced as follows.
Let $\zeta : [0,\infty) \times [0,\infty) \to \mathbb{R}$ be a mapping, then ζ is called a simulation function if satisfies the following conditions:

(ζ1) $\zeta(0,0) = 0$;
(ζ2) $\zeta(t,s) < s - t$ for all $t, s > 0$;
(ζ3) if $\{t_n\}$, $\{s_n\}$ are sequences in $(0,\infty)$ such that $\lim_{n\to\infty} t_n = \lim_{n\to\infty} s_n > 0$ then

$$\limsup_{n\to\infty} \zeta(t_n, s_n) < 0.$$

In order to extend Banach contraction principle [2]. In this note we are going to replace condition (ζ3) by the following condition:
($\zeta 3'$) if $\{t_n\}$, $\{s_n\}$ are sequences in $(0,\infty)$ such that $\lim_{n\to\infty} t_n = \lim_{n\to\infty} s_n > 0$ then

$$\liminf_{n\to\infty} \zeta(t_n, s_n) < 0.$$

It is clear that condition (ζ3) implies ($\zeta 3'$) while the following example shows that converse may fail.

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Example 1.1. Let $\zeta : [0,\infty) \times [0,\infty) \to \mathbb{R}$ be defined by

$$\zeta(s,t) = \begin{cases} (-1)^n - 1 & (t,s) \in A = \{(e^{-n},1) : n \in \mathbb{N}\} \\ -|s-t| - 1 & (t,s) \in [0,\infty) \times [0,\infty) \setminus A \cup \{(0,0)\} \\ 0 & (t,s) = (0,0) \end{cases} .$$

Then $\liminf_{n\to\infty} \zeta(t_n, s_n) < 0$. But $\limsup_{n\to\infty} \zeta(t_n, s_n) = 0 \not< 0$. Thus ζ is extended simulation function.

We say that the mapping $\zeta : [0,\infty) \times [0,\infty) \to \mathbb{R}$ is extended simulation function if ζ satisfies the conditions $(\zeta 1), (\zeta 2)$ and $(\zeta 3')$.

Hence the class of simulation functions is a proper subset of the class of all functions whose satisfy in $(\zeta 1), (\zeta 2)$ and $(\zeta 3')$. We denote the set of all simulation functions by $\mathcal{Z}_e$.

The rest of this section we recall standard definitions.
Let (X,d) be a metric space and $T : X \to X$ be a mapping, then T is called a contraction (Banach contraction) on X if

$$d(Tx,Ty) \le \lambda d(x,y) \text{ for all } x, y \in X,$$

where λ is a real such that $\lambda \in [0,1)$. A point $x \in X$ is called a fixed point of T if $Tx = x$. The well known Banach contraction principle [2] ensures the existence and uniqueness of fixed point of a contraction on a complete metric space. After this interesting principle, several authors generalized this principle by introducing the various contractions on metric spaces (see, e.g., [3–12]). Rhoades [3], in his work we compare several contractions defined on metric spaces.

2. Main Results

In this section we present a fixed point which extends Banach contraction principle by using extended simulation function. This result improves the corresponding results given in this area, specially Theorem 2.8 of [1].

The next definition has a crucial role in this section.

Definition 2.1. Let (X,d) be a metric space, $T : X \to X$ a mapping and $\zeta \in \mathcal{Z}_e$. Then T is called a $\mathcal{Z}_e$-contraction with respect to ζ if the following condition is satisfied

$$\zeta(d(Tx,Ty), d(x,y)) \ge 0 \text{ for all } x, y \in X. \tag{2.1}$$

A simple example of $\mathcal{Z}_e$-contraction is the Banach contraction which can be obtained by taking $\lambda \in [0,1)$ and $\zeta(t,s) = \lambda s - t$ for all $s, t \in [0,\infty)$ in the above definition.

We now prove some properties of $\mathcal{Z}_e$-contractions defined on a metric space.

Remark 2.2. It is clear from the definition simulation function that $\zeta(t,s) < 0$ for all $t \ge s > 0$. Therefore, if T is a $\mathcal{Z}_e$-contraction with respect to $\zeta \in \mathcal{Z}_e$ then

$$d(Tx,Ty) < d(x,y) \text{ for all distinct } x, y \in X.$$

This shows that every $\mathcal{Z}_e$-contraction mapping is contractive, therefore it is continuous.

In the following lemma the uniqueness of fixed point of a $\mathcal{Z}_e$-contraction is proved.

Lemma 2.3. *Let (X,d) be a metric space and $T : X \to X$ be a $\mathcal{Z}_e$-contraction with respect to $\zeta \in \mathcal{Z}_e$. Then the fixed point of T in X is unique, provided it exists.*

Proof. Suppose $u \in X$ be a fixed point of T. If possible, let $v \in X$ be another fixed point of T and it is distinct from u, that is, $Tv = v$ and $u \neq v$. Now it follows from Remark 2.2, that

$$d(u, v) = d(Tu, Tv) < d(u, v).$$

above inequality yields a contradiction and proves result. ∎

A self map T of a metric space (X, d) is said to be asymptotically regular at point $x \in X$ if $\lim_{n\to\infty} d(T^n x, T^{n+1}x) = 0$ (see [13]).
The next lemma shows that a $\mathcal{Z}_e$-contraction is asympotically regular at every point of X.

Lemma 2.4. *Let (X, d) be a metric space and $T : X \to X$ be a $\mathcal{Z}_e$-contraction with respect to $\zeta \in \mathcal{Z}_e$. Then T is asymptotically regular at every $x \in X$.*

Proof. Let $x \in X$ be arbitrary. If for some $p \in \mathbb{N}$ we have $T^p x = T^{p-1}x$, that is, $Ty = y$, where $y = T^{p-1}x$, then $T^n y = T^{n-1}Ty = T^{n-1}y = \ldots = Ty = y$ for all $n \in \mathbb{N}$. Now for sufficient large $n \in \mathbb{N}$ we have

$$\begin{aligned} d(T^n x, T^{n+1}x) =& d(T^{n-p+1}T^{p-1}x, T^{n-p+1}T^{p-1}x) \\ =& d(T^{n-p+1}y, T^{n-p+2}y) = d(y, y) = 0, \end{aligned}$$

therefore, $\lim_{n\to\infty} d(T^n x, T^{n+1}x) = 0$.

Suppose $T^n x \neq T^{n-1}x$, for all $n \in \mathbb{N}$, then it followes from (2.1) that

$$\begin{aligned} 0 \leq& \zeta(d(T^{n+1}x, T^n x), d(T^n x, T^{n-1}x)) \\ \leq& d(T^n x, T^{n-1}x) - d(T^{n+1}x, T^n). \end{aligned}$$

The above inequality shows that $\{d(T^n x, T^{n-1}x)\}$ is a monotonically decreasing sequence of nonnegative reals and so it must be convergent. Let $\lim_{n\to\infty} d(T^n x, T^{n+1}x) = r \geq 0$. If $r > 0$ then since T is $\mathcal{Z}_e$-contraction with respect to $\zeta \in \mathcal{Z}_e$ therefore by (ζ3), we have

$$0 \leq \liminf_{n\to\infty} \zeta(d(T^{n+1}x, T^n x), d(T^n x, T^{n-1}x)) < 0.$$

This contraction shows that $r = 0$, that is, $\lim_{n\to\infty} d(T^n x, T^{n+1}x) = 0$. Thus T is an asymptotically regular mapping at x. ∎

The next lemma shows that the Picard sequence $\{x_n\}$ generated by a $\mathcal{Z}_e$-contraction is always bounded.

Lemma 2.5. *Let (X, d) be a metric space and $T : X \to X$ be a $\mathcal{Z}_e$-contraction with respect to ζ. Then the Picard sequence $\{x_n\}$ generated by T with initial value $x_0 \in X$ is a bounded sequence, where $x_n = Tx_{n-1}$ for all $n \in \mathbb{N}$.*

Proof. Let $x_0 \in X$ be arbitrary and $\{x_n\}$ be the Picard sequence, that is, $x_n = Tx_{n-1}$ for all $n \in \mathbb{N}$. On the contrary, assume that $\{x_n\}$ is not bounded. Without loss of generality we can assume that $x_{n+p} \neq x_n$ for all $n, p \in \mathbb{N}$. Since $\{x_n\}$ is not bounded, there exists a subsequence $\{x_{n_k}\}$ such that $n_1 = 1$ and for each $k \in \mathbb{N}$, n_{k+1} is the minimum iteger such that $d(x_{n_k+1}, x_{n_k}) > 1$ and $d(x_m, x_{n_k}) < 1$ for $n_k \leq m \leq n_{k+1} - 1$. Therefore by the triangular inequality we have

$$\begin{aligned} 1 <& d(x_{n_{k+1}}, x_{n_k}) \leq d(x_{n_{k+1}}, x_{n_{k+1}-1}) + d(x_{n_{k+1}-1}, x_{n_k}) \\ \leq& d(x_{n_{k+1}}, x_{n_{k+1}-1}) + 1. \end{aligned}$$

Letting $k \to \infty$ and using Lemma 2.4, we obtain $\lim_{k\to\infty} d(x_{n_{k+1}}, x_{n_k}) = 1$. By (2.1) we have $d(x_{n_{k+1}}, x_{n_k}) \le d(x_{n_{k+1}-1}, x_{n_k-1})$, therefore using the triangular inequality we obtain

$$\begin{aligned} 1 <& d(x_{n_{k+1}}, x_{n_k}) \le d(x_{n_{k+1}-1}, x_{n_k-1}) \\ \le& d(x_{n_{k+1}-1}, x_{n_k}) + d(x_{n_k}, x_{n_k-1}) \\ \le& 1 + d(x_{n_k}, x_{n_k-1}). \end{aligned}$$

Letting $k \to \infty$ and using Lemma 2.4, we obtain $\lim_{k\to\infty} d(x_{n_{k+1}-1}, x_{n_k-1}) = 1$. Now since T is a $\mathcal{Z}_e$-contraction with respect to $\zeta \in \mathcal{Z}_e$ therefore by ($\zeta3$), we have

$$\begin{aligned} 0 \le& \liminf_{k\to\infty} \zeta(d(Tx_{n_{k+1}-1}, Tx_{n_k-1}), d(x_{n_{k+1}-1}, x_{n_k-1})) \\ =& \liminf_{k\to\infty} \zeta(d(x_{n_{k+1}}, x_{n_k}), d(x_{n_{k+1}-1}, x_{n_k-1})) < 0. \end{aligned}$$

This contraction completes the proof. ■

In the next theorem we prove the existence of fixed point of a $\mathcal{Z}_e$-contraction.

Theorem 2.6. *Let (X, d) be a complete metric space and $T : X \to X$ be a $\mathcal{Z}_e$-contraction with respect to ζ. Then T has a unique fixed point u in X and for every $x_0 \in X$ the Picard sequence $\{x_n\}$, where $x_n = Tx_{n-1}$ for all $n \in \mathbb{N}$ converges to the fixed point of T.*

Proof. Let $x_0 \in X$ be arbitrary and $\{x_n\}$ be the Picard sequence, that is, $x_n = Tx_{n-1}$ for all $n \in \mathbb{N}$. We shall show that this sequence is a Cauchy sequence. For this, let $C_n = \inf\{d(x_i, x_j) : i, j \ge n\}$. Note that the sequence $\{C_n\}$ is a monotonically increasing sequence of positive reals and by Lemma 2.5, the sequence $\{x_n\}$ is bounded. Thus $\{C_n\}$ is monotonic bounded sequence, therefore convergent, that is, there exists $C \ge 0$ such that $\lim_{n\to\infty} C_n = C$. We shall show that $C = 0$. If $C > 0$, then by the definition of C_n, for every $k \in \mathbb{N}$ there exists n_k, m_k such that $m_k > n_k \ge k$ and $C_k \le d(x_{m_k}, x_{n_k}) < C_k + \dfrac{1}{k}$. Hence

$$\lim_{k\to\infty} d(x_{m_k}, x_{n_k}) = C. \tag{2.2}$$

Using (2.1) and the triangular inequality we have

$$d(x_{m_k}, x_{n_k}) \le d(x_{m_k-1}, x_{n_k-1}) \le d(x_{m_k-1}, x_{m_k}) + d(x_{m_k}, x_{n_k}) + d(x_{n_k}, x_{n_k-1}).$$

Using (2.2) and letting $k \to \infty$ in the above inequality we obtain

$$\lim_{k\to\infty} d(x_{m_k-1}, x_{n_k-1}) = C. \tag{2.3}$$

Since T is a $\mathcal{Z}_e$-contraction with respect to $\zeta \in \mathcal{Z}_e$ therefore using (2.1), (2.2), (2.3) and ($\zeta3$), we have

$$0 \le \liminf_{k\to\infty} \zeta(d(x_{m_k-1}, x_{n_k-1}), d(x_{m_k}, x_{n_k})) < 0.$$

This contradiction proves that $C = 0$ and so $\{x_n\}$ is a Cauchy sequence. Since X is a complete space, there exists $u \in X$ such that $\lim_{n\to\infty} x_n = u$. We shall show that the point u is a fixed point of T. Suppose $Tu \ne u$ then $d(u, Tu) > 0$. Again, using (2.1), ($\zeta2$) and

(ζ3), we have

$$\begin{aligned}0 &\leq \liminf_{n\to\infty} \zeta(d(Tx_n, Tu), d(x_n, u)) \\ &\leq \liminf_{n\to\infty} [d(x_n, u) - d(x_{n+1}, Tu)] \\ &= -\, d(u, Tu).\end{aligned}$$

This contradiction shows that $d(u, Tu) = 0$, that is, $Tu = u$. Thus u is a fixed point of T. Uniqueness of the fixed point follows from Lemma 2.3. ■

Following example shows that the above theorem is a proper generalization of Banach contraction principle.

Example 2.7. Let $X = [0, 1]$ and $d : X \times X \to \mathbb{R}$ be defined by $d(x, y) = |x - y|$. Then (X, d) is a complete metric space. Define a mapping $T : X \to X$ as $Tx = \dfrac{x}{x+1}$ for all $x \in X$. T is a continuous function but it is not a Banach contraction. But it is a $\mathcal{Z}_e$-contraction with respect to $\zeta \in \mathcal{Z}_e$, where

$$\zeta = \frac{s}{s+1} - t \text{ for all } t, s \in [0, \infty).$$

Indeed, if $x, y \in X$, then

$$\begin{aligned}\zeta(d(Tx, Ty), d(x, y)) =& \frac{d(x, y)}{1 + d(x, y)} - d(Tx, Ty) \\ =& \frac{|x - y|}{1 + |x - y|} - |\frac{x}{x+1} - \frac{y}{y+1}| \\ =& \frac{|x - y|}{1 + |x - y|} - \frac{|x - y|}{(x+1)(y+1)}| \geq 0.\end{aligned}$$

Note that, all the conditions of Theorem 2.6, are satisfied and T has a unique fixed point $u = 0 \in X$.

In the following corollaries we obtain some known and some new results in fixed point theory via the simulation function.

Corollary 2.8. (Banach contraction principle [2]). *Let (X, d) be a complete metric space and $T : X \to X$ be a mapping satisfying the following condition:*

$$d(Tx, Ty) \leq \lambda d(x, y) \text{ for all } x, y \in X,$$

where $\lambda \in [0, 1)$. Then T has a unique fixed point in X.

Proof. Define $\zeta_B : [0, \infty) \times [0, \infty) \to \mathbb{R}$ by $\zeta_B(t, s) = \lambda s - t$ for all $s, t \in [0, \infty)$. Note that, the mapping T is a $\mathcal{Z}_e$-contraction with respect to $\zeta_B \in \mathcal{Z}_e$. Therefore the result follows by taking $\zeta = \zeta_B$ in Theorem 2.6. ■

Corollary 2.9. (Rhoades type). *Let (X, d) be a complete metric space and $T : X \to X$ be a mapping satisfying the following condition:*

$$d(Tx, Ty) \leq d(x, y) - \varphi(d(x, y)) \text{ for all } x, y \in X,$$

where $\varphi : [0, \infty) \to [0, \infty)$ is upper semi continuos function and $\varphi^{-1}(0) = \{0\}$. Then T has a unique fixed point in X.

Proof. Define $\zeta_R : [0, \infty) \times [0, \infty) \to \mathbb{R}$ by $\zeta_R(t, s) = s - \varphi(s) - t$ for all $s, t \in [0, \infty)$. Note that, the mapping T is a $\mathcal{Z}_e$-contraction with respect to $\zeta_R \in \mathcal{Z}_e$. Therefore the result follows by taking $\zeta = \zeta_R$ in Theorem 2.6. ∎

Remark 2.10. Note that, in the [11] the function φ is assumed to be continuous and nondecreasing and $\lim_{t\to\infty} \psi(t) = \infty$. In Corollary 2.9 we replace these conditions by lower semi continuity of φ. Therefore our result is stronger than the original version of Rhoades [11].

Corollary 2.11. *Let (X, d) be a complete metric space and $T : X \to X$ be a mapping. Suppose that for every $x, y \in X$,*

$$d(Tx, Ty) \leq \varphi(d(x, y))d(x, y) \text{ for all } x, y \in X,$$

where $\varphi : [0, \infty) \to [0, 1)$ be a mapping such that $\liminf_{t\to r^+} \varphi(t) < 1$, for all $r > 0$. Then T has a unique fixed point in X.

Proof. Define $\zeta_T : [0, \infty) \times [0, \infty) \to \mathbb{R}$ by $\zeta_T(t, s) = s\varphi(s) - t$ for all $s, t \in [0, \infty)$. Note that, the mapping T is a $\mathcal{A}$-contraction with respect to $\zeta_T \in \mathcal{A}$. Therefore the result follows by taking $\zeta = \zeta_T$ in Theorem 2.6. ∎

Corollary 2.12. (Rhoades type). *Let (X, d) be a complete metric space and $T : X \to X$ be a mapping. Suppose that $d(Tx, Ty) \leq \eta(d(x, y))$ for all $x, y \in X$, where $\eta : [0, \infty) \to [0, \infty)$ is upper semi continuos mapping such that $\eta(t) < t$ for all $t > 0$ and $\eta(0) = 0$. Then T has a unique fixed point in X.*

Proof. Define $\zeta_{BW} : [0, \infty) \times [0, \infty) \to \mathbb{R}$ by $\zeta_{BW}(t, s) = \eta(s) - t$ for all $s, t \in [0, \infty)$. Note that, the mapping T is a $\mathcal{Z}_e$-contraction with respect to $\zeta_{BW} \in \mathcal{Z}_e$. Therefore the result follows by taking $\zeta = \zeta_{BW}$ in Theorem 2.6. ∎

Corollary 2.13. *Let (X, d) be a complete metric space and $T : X \to X$ be a mapping satisfying the following condition:*

$$\int_0^{d(Tx,Ty)} \phi(t)\, dt \leq d(x, y) \text{ for all } x, y \in X,$$

where $\phi : [0, \infty) \to [0, \infty)$ is function such that $\int_0^{\epsilon} \phi(t)dt$ exists and $\int_0^{\epsilon} \phi(t)dt > \epsilon$, for each $\epsilon > 0$. Then T has a unique fixed point in X.

Proof. Define $\zeta_K : [0, \infty) \times [0, \infty) \to \mathbb{R}$ by $\zeta_K(t, s) = s - \int_0^t \phi(u)du$ for all $s, t \in [0, \infty)$. Then $\zeta_K \in \mathcal{Z}_e$. Therefore the result follows by taking $\zeta = \zeta_K$ in Theorem 2.6. ∎

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