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Generators for Group Homology and a Vanishing Conjecture

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Abstract Letting $G = F/R$ be a finitely-presented group, Hopf's formula expresses the second integral homology of G in terms of F and R. Expanding on previous work, we explain how to find generators of $H_2(G; \mathbb{F}_p)$. The context of the problem, which is related to a conjecture of Quillen, is presented, as well as example calculations.

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1. Introduction

Exploiting a classical theorem due to Hopf, we presented a series of algorithms in [1] that give upper bounds on group homology in homological dimensions one and two, provided coefficients are taken in a finite field. In particular, examples confirmed the results in [2], as well a new result, concerning the rank two special linear group over rings of number theoretic interest. This paper can be viewed as both a sequel and expansion of the results in [1].

The initial motivation for constructing the algorithms was to gain insight into special cases of a conjecture originally given by Quillen in 1971, which we briefly discuss in Section 2. However, since the algorithms in [1] depend only upon Hopf's formula for H_2, the usefulness of these algorithms extends to groups beyond the scope of Quillen's Conjecture. Moreover, the algorithms are distinct from existing methods of calculating low dimensional group homology in that they give an upper bound on the homology of any finitely-presented group, though the upper bound is, at times, very large.

The main contribution of this paper is in Section 3 wherein we present a technique that expounds on the algorithms in [1] to find explicit generators of these homology groups. The technique relies heavily upon the above mentioned Hopf's formula for the second homology group of a finitely-presented group; the calculations are carried out with the computational algebra program GAP [3].

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As a byproduct of the calculations related to Quillen's Conjecture we are involved in a long term project of preparing a database for low dimensional group homology of linear groups over number fields and their rings of integers. This work will be extended to other classes of finitely-presented groups of interest to computational group theory and algebraic topology. The first set of these calculations is found in Section 4.

We note that when it is clear from the context, we occasionally omit explicitly writing the ground ring of linear groups as well as homology coefficients.

2. A Vanishing Conjecture

One motivational problem for low dimensional group homology, which is related to algebraic K-theory, is the study of homology for groups $GL_j(R)$, where GL_j is a finite rank j general linear group and R is the ring of integers in a number field. An approach to this problem is to consider the diagonal matrices inside GL_j. Let D_j denote the subgroup formed by these matrices. Then the canonical inclusions $D_j \subset GL_j$ for $j = 0, 1, ...$ induce homomorphisms on group homology with k-coefficients

$$\rho : H_i(D_j(R); k) \to H_i(GL_j(R); k). \tag{2.1}$$

In [4] Quillen conjectured:

Conjecture 2.1. The homomorphism ρ, as given above, is an epimorphism for $R = \mathbb{Z}[\zeta_p, 1/p]$, p a regular odd prime, ζ_p a primitive pth root of unity, $k = \mathbb{F}_p$ and any values of i and j.

Conjecture 2.1 has been proved in a few cases and disproved in infinitely many other cases. For $R = \mathbb{Z}[1/2]$ it was proved by Mitchel in [5] for $j = 2$ and by Henn in [6] for $j = 3$. Anton gave a proof for $R = \mathbb{Z}[1/3, \zeta_3]$ and $j = 2$ in [7].

Dwyer gave a disproof for the conjecture for $R = \mathbb{Z}[1/2]$ and $j = 32$ in [8] which Henn and Lannes improved to $j = 14$ in [9]; this is an improvement in light of Henn's result in [10] that states that if Conjecture 2.1 is false for j_0 then it is false for all $j \geq j_0$. Anton disproved the conjecture for $R = \mathbb{Z}[1/3, \zeta_3]$ and $j \geq 27$ also in [7]. The interested reader should consult [11] for more details.

This conjecture was reformulated and, in a sense, corrected by Anton:

Conjecture 2.2. [12] Given p, k and R as above, the determinant map induces an isomorphism:

$$H_2(GL_2(R); k) \cong H_2(D_1(R); k). \tag{2.2}$$

Anton's conjecture led to a proof of Conjecture 2.1 for $\mathbb{Z}[1/5, \zeta_5]$ and $i = j = 2$. For a survey on the current status of conjectures 2.1 and 2.2 we cite [2].

2.1. Reduction via a Spectral Sequence

Given a group extension

$$1 \to N \to G \to Q \to 1$$

there is the Hochschild-Serre Spectral Sequence [13, p. 341] with

$$E^2_{p,q} \cong H_p(Q; H_q(N; k)) \Longrightarrow H_{p+q}(G; k), \tag{2.3}$$

where we take coefficients in a field k regarded as a trivial G-module. We use this spectral sequence to reduce a special case of Quillen's conjecture to an exercise in linear algebra.

Lemma 2.3. *Fix R a Euclidean Ring and field of coefficients $k = \mathbb{F}_p$,*

$$H_2(GL_2(R); k) \cong (H_2(SL_2(R); k)_{GL_1(R)}/Im(\tau) \oplus H_2(GL_1(R); k), \tag{2.4}$$

where, for a group G and a G-module M, M_G is the group of co-invariants and τ is the transgression map $E^3_{3,0} \to E^3_{0,2}$.

Proof. Since R is a Euclidean ring, $SL_2(R)$ is a perfect group by Lemma 7.2 in [12], and thus $H_1(SL_2(R)) = 0$. Then applying the spectral sequence 2.3 to the extension defined by the determinant map,

$$1 \to SL_2(R) \to GL_2(R) \to GL_1(R) \to 1, \tag{2.5}$$

we see that the entries $E^2_{p,1}$ are all 0. Thus for $q < 3$ the $E^3_{p,q}$ page is equal to the $E^2_{p,q}$ page.

We also note that

$$GL_1(R) \cong D_1(R) \cong R^{\times}, \tag{2.6}$$

where $R^{\times}$ is the group of units of R.

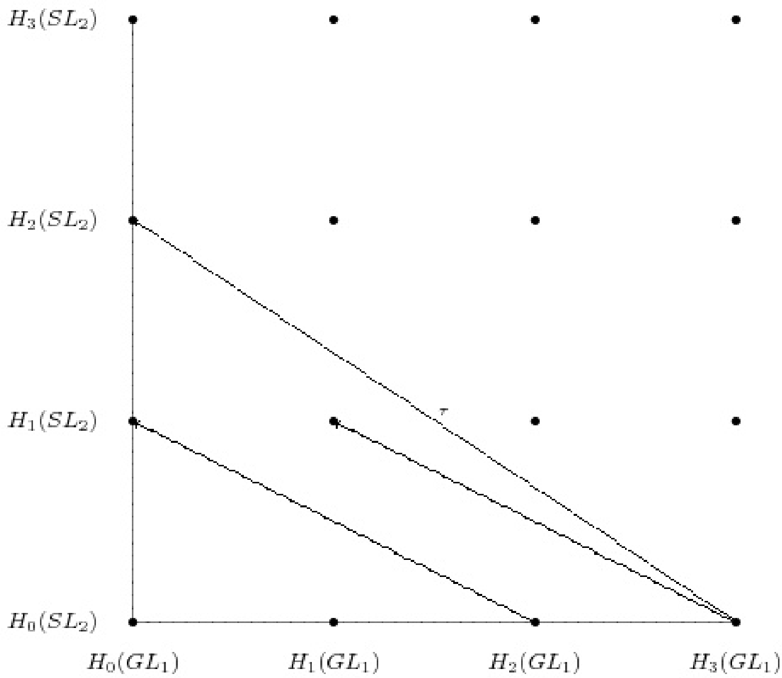


FIGURE 1. E^2 page with $\tau : E^3_{3,0} \to E^3_{0,2}$ displayed

Figure 1 displays the E^2 page of this spectral sequence, and we have included the transgression $\tau : E^3_{3,0} \to E^3_{0,2}$ for reference. Note that since $E^2_{p,q} \cong E^3_{p,q}$ for all p and for all $q < 3$ then $E^2_{p,1} \cong E^{\infty}_{p,1}$. Moreover, $E^4_{p,q} \cong E^{\infty}_{p,q}$ for $p, q+1 < 4$ and $E^4_{0,2} \cong H_2(SL_2(R))_{GL_1(R)}/Im(\tau)$. Since we have chosen field coefficients, any extension problems are trivial. Thus we have the following decomposition.

$$H_2(GL_2(R)) \cong E^4_{2,0} \oplus E^4_{1,1} \oplus E^4_{0,2} \tag{2.7}$$
$$\cong H_2(SL_2(R))_{GL_1(R)}/Im(\tau) \oplus H_2(GL_1(R)). \tag{2.8}$$

■

This immediately implies the following corollary.

Corollary 2.4. *As vector spaces over k,*

$$dim\ H_2(GL_2(R); k) \geq dim\ H_2(GL_1(R); k) \tag{2.9}$$

According to equation 2.6 and Lemma 2.3, the Conjecture 2.2 for R a Euclidean ring is equivalent to the vanishing of the cokernel of the transgression map

$$\tau : H_3(GL_1(R)) \to H_2(SL_2(R))_{GL_1(R)}.$$

In this context, the purpose of [1] was to give a series of algorithms that estimated the second homology group of any finitely-presented group. More precisely, given a finitely-presented group G and a finite field k, the second homology group $H_2(G; k)$ with coefficients in k is a finite dimensional vector space over k. Our algorithms give an upper bound for the dimension of $H_2(G; k)$ and, in particular cases, the algorithms calculate precisely this dimension.

The algorithms confirmed results by Anton that Conjecture 2.1 holds for $R = SL_2(\mathbb{Z}[1/p, \zeta_p])$ and $k = \mathbb{F}_p$ for $p = 3$ and $p = 5$ ([12] and [2]).

3. Generators of Homology Groups

Let $1 \to K \xrightarrow{i} F \xrightarrow{q} G \to 1$ be an exact sequence of groups where F is a finitely generated free group and K is finitely generated as an F-module with the F-action given by conjugation, i and q denote inclusion and quotient homomorphism, respectively. That is, G has finite presentation given by the generators of F modulo the normal closure of K in F.

Theorem 3.1 (Hopf)**.** *Given G, F, K as above, there is an exact sequence*

$$1 \to [F, R] \to [F, F] \to H_2(G, \mathbb{Z}) \to 1.$$

This gives an exact sequence

$$1 \to H_2(G, \mathbb{Z}) \to \frac{R}{[F, R]} \to \frac{F}{[F, F]} \to \frac{F}{R[F, F]} \to 1.$$

The last two terms are finitely generated abelian groups and algorithms exist to give their structure. Also in [1], we explain how to use this exact sequence to find an upper bound on the dimension of $H_2(G; k)$, where k is the finite field of prime characteristic p.

The inclusion homomorphism $i : K \to F$ induces a homomorphism $i_* : A \to B$ where we have denoted $K/K^p[F, K]$ by A and $F/K^p[F, F]$ by B. Note that for $k \in K$ and $f \in F$ we have that $[k, f] = k^f k^{-1} = 1$ in A. Thus $k^f = k$ in A which gives that A is a trivial F-module. Let S_K be the set of generators of K as an F-module.

We note that the image of i_* is generated by the set of all $i_*(k)$ for $k \in S_K$. Then since B is a vector space over k, there is a subset $S'_K \subset S_K$ such that $i_*(k')$ with $k \in S'_K$ is a basis for the image of i_*.

The primary interest is on the kernel of i_*, which is isomorphic to $H_2(G;k)$. As stated above, a previous paper gives an upper bound n on the dimension of this vector space. We seek an explicit description of these n elements of S_K. To this end, we restate two facts:

1. A is a vector space that is spanned by S_K
2. $S'_K \subset S_K$ is a subset with $i_*(S'_K)$ a basis for the image of i_* in B.

Let $v \in A$, then $v = \sum_{\lambda \in S_K} c_\lambda \lambda$, where $c_\lambda \in k$, by (1). Note that $i_*(v) = 0$ is equivalent to $\sum_{\lambda \in S_K} c_\lambda i_*(\lambda) = 0$ in B.

Moreover, each $i_*(\lambda) = \sum_{\mu \in S'_K} a_{\lambda,\mu} i_*(\mu)$, where $a_{\lambda,\mu} \in k$, by (2). Therefore, $i_*(v) = 0$ in B if and only if

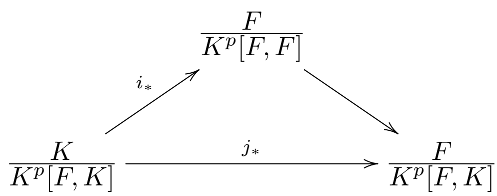
$$\sum_{\mu \in S'_K} \left(\sum_{\lambda \in S_K} c_\lambda a_{\lambda,\mu} \right) i_*(\mu) = 0 \text{ in } B,$$

which is true if and only if

$$\sum_{\lambda \in S_K} c_\lambda a_{\lambda,\mu} = 0$$

for all $\mu \in S'_K$. We must solve for the c_λ coefficients to find a basis for the solutions.

If $a \in A$ then $a = k_1^{f_1} k_2^{f_2} \cdots = k_1 k_2 \cdots = k_1 + k_2 + \cdots$ in $\mathbb{F}_p$. We use linear algebra in B to find a basis for the image of i_* $\{i_*(k) : k \in K\}$. The following commutative diagram illustrates the above discussion.



3.1. At the Prime 7

We consider the group $G = SL_2(\mathbb{Z}[1/7, \zeta_7])$, where ζ_p is a primitive pth root of unity. In [2] it is proven that this group is generated by

$$S = \{z, u_1, u_2, u_3, a, b, b_0, b_1, b_2, b_3, b_4, b_5, b_6, w\}$$

modulo the relators

$$\begin{aligned} R \;=\; \{ & b_t^{-1} z^{3t} b z^{3t} a, w^{-1} z^4 u_1 u_2 u_3, z^7, [z, u_i], [u_i, u_j], a^4, [a^2, z], [a^2, u_i], \\ & a^{-1} z a z, a^{-1} u_i a u_i, [b_s, b_t], b^{-3} a^2, b^{-3} b_0 b_1 b_2 b_3 b_4 b_5 b_6, b_t^{-7} w^{-1} b_t^{-1} w, \\ & (b_0 b_1^{-1} a^{-1} u_1)^3, (b_0 b_2^{-1} a^{-1} u_2)^3, (b_0 b_3^{-1} a^{-1} u_3)^3, \\ & (b_0 b_1^{-1} b_2^{-1} b_3 a^{-1} u_1 u_2)^3, (b_0 b_1^{-1} b_3^{-1} b_4 a^{-1} u_1 u_3)^3, (b_0 b_2^{-1} b_3^{-1} b_5 a^{-1} u_2 u_3)^3, \\ & (b_0 b_1^{-1} b_2^{-1} b_3 b_4 b_5 b_6^{-1} a^{-1} u_1 u_2 u_3)^3, a^{-2} b^{-1} u_i b z^{-3i} b^{-1} b_0^{-1} z^{3i} b z^{-i} u_i \} \end{aligned}$$

where $i, j \in \{1, 2, 3\}$ and $s, t \in \{1, 2, 3, 4, 5, 6\}$.

That is, there is a short exact sequence $1 \to N(R) \to F(S) \to G \to 1$ with the set S generating the free group $F(S)$ and the set R normally generating the subgroup $N(R) \subset F(S)$.

We begin by reducing the number of generators and relators in $F(S)/N(R)$ in order to simplify the final calculations. Via GAP, it is easy to verify the following.

Proposition 3.2. *There is an isomorphism of finitely-presented groups that maps the generators of the free group $F(S)$ to the free group generated by $S' = \{z, u_1, u_2, u_3, a, b_1\}$ in the following way:*

$$\begin{array}{lcl} z & \mapsto & z \\ u_1 & \mapsto & u_1 \\ u_2 & \mapsto & u_2 \\ u_3 & \mapsto & u_3 \\ a & \mapsto & a \\ b & \mapsto & z^{-3}b_1z^3a^{-1} \\ b_0 & \mapsto & z^{-3}b_1z^3 \\ b_1 & \mapsto & b_1 \\ b_2 & \mapsto & z^3b_1z^{-3} \\ b_3 & \mapsto & z^{-1}b_1z \\ b_4 & \mapsto & z^2b_1z^{-2} \\ b_5 & \mapsto & z^{-2}b_1z^2 \\ b_6 & \mapsto & zb_1z^{-1} \\ w & \mapsto & z^{-2}u_1z^{-1}u_2u_3. \end{array}$$

Moreover, the isomorphic finitely-presented group has set of 32 relators

$$\begin{aligned}
R' = \{&zu_3z^{-1}u_3^{-1}, u_2u_3u_2^{-1}u_3^{-1},\\
&u_1u_2u_1^{-1}u_2^{-1},\\
&u_3au_3a^{-1},\\
&u_1au_1a^{-1},\\
&zu_2z^{-1}u_2^{-1},\\
&a^4,\\
&u_1u_3u_1^{-1}u_3^{-1},\\
&zaza^{-1},\\
&zu_1z^{-1}u_1^{-1},\\
&u_2au_2a^{-1},\\
&z^7,\\
&b_1z^{-1}b_1zb_1^{-1}z^{-1}b_1^{-1}z,\\
&b_1z^{-2}b_1z^2b_1^{-1}z^{-2}b_1^{-1}z^2,\\
&z^{-3}b_1z^{-1}a^{-1}b_1z^{-1}a^{-1}b_1z^3a,\\
&b_1z^{-3}b_1z^{-2}b_1^{-2}z^{-1}a^{-1}u_3a^{-1}z^{-1}b_1^{-1}u_3,\\
&b_1z^{-1}b_1^{-2}z^{-1}b_1a^{-1}z^3u_2a^{-1}z^{-2}b_1^{-1}u_2,\\
&b_1z^{-1}b_1z^{-3}b_1^{-2}z^{-1}u_1^{-1}a^{-2}z^{-2}b_1^{-1}u_1,\\
&b_1z^{-3}b_1z^3b_1^{-1}z^{-3}b_1^{-1}z^3,\\
&z^{-1}b_1^-7zu_2^{-1}zu_3^{-1}u_1^{-1}zb_1^{-1}z^{-1}u_1z^{-1}u_2u_3,\\
&b_1^-7u_2^{-1}zu_3^{-1}u_1^{-1}z^2b_1^{-1}z^{-3}u_1u_2u_3,\\
&z^{-3}b_1^-7z^{-1}u_2^{-1}z^{-2}u_3^{-1}u_1^{-1}z^{-1}b_1^{-1}u_1u_2u_3,\\
&\quad zb_1^-7u_2^{-1}u_3^{-1}u_1^{-1}z^3b_1^{-1}z^{-3}u_1z^{-1}u_2u_3,
\end{aligned}$$

$z^3b_1^{-}7u_2^{-1}z^{-2}u_3^{-1}u_1^{-1}z^{-2}b_1^{-1}zu_1u_2u_3,$
$z^2b_1z^{-3}b_1z^{-3}b_1zb_1zb_1z^3b_1z^{-1}b_1a^{-2},$
$z^{-3}b_1z^{-1}b_1^{-1}u_2^{-1}a^{-1}b_1z^{-1}b_1^{-1}z^{-3}u_2^{-1}$
$\quad a^{-1}z^{-3}b_1z^{-1}b_1^{-1}z^{-3}u_2^{-1}a^{-1},$
$z^{-1}b_1^{-1}z^{-2}b_1z^3u_3^{-1}a^{-1}z^{-1}b_1^{-1}z^{-2}b_1z^3$
$\quad u_3^{-1}a^{-1}z^{-1}b_1^{-1}z^{-2}b_1z^3u_3^{-1}a^{-1},$
$b_1^{-1}z^{-3}b_1za^{-1}z^{-2}u_1b_1^{-1}z^{-3}b_1z^3u_1^{-1}a^{-1}$
$\quad b_1^{-1}z^{-3}b_1z^3u_1^{-1}a^{-1},$
$b_1^{-1}z^{-1}b_1z^{-2}b_1z^{-1}b_1^{-1}a^{-1}z^3u_1u_2z^3b_1^{-1}$
$\quad zb_1z^2b_1zb_1^{-1}u_1^{-1}a^{-1}u_2z^3b_1^{-1}zb_1z^2b_1zb_1^{-1}$
$\quad u_1^{-1}a^{-1}u_2,$
$b_1^{-1}z^{-1}b_1^{-1}z^{-2}b_1z^{-2}b_1z^{-1}u_1^{-1}z^{-1}a^{-1}u_3$
$\quad z^{-3}b_1z^2b_1^{-1}zb_1^{-1}z^2b_1u_1^{-1}z^{-2}a^{-1}u_3z^{-1}b_1^{-1}z^{-2}$
$\quad b_1z^3b_1^{-1}z^2b_1u_1^{-1}z^{-2}a^{-1}u_3,$
$b_1^{-1}z^3b_1^{-1}z^{-1}b_1z^{-1}b_1za^{-1}$
$\quad z^{-2}u_3u_2z^{-3}b_1z^{-1}b_1^{-1}z^2b_1zb_1^{-1}z$
$\quad u_3^{-1}a^{-1}u_2z^{-3}b_1z^{-1}b_1^{-1}z^2b_1zb_1^{-1}zu_3^{-1}a^{-1}u_2z^3,$
$zb_1z^{-3}b_1^{-1}z^{-3}b_1zu_1^{-1}za^{-1}u_2u_3z^{-3}b_1z^{-1}b_1^{-1}$
$\quad z^{-1}b_1z^3b_1z^2b_1^{-1}zb_1^{-1}z^{-1}u_1^{-1}a^{-1}u_2u_3z^{-3}b_1z^{-1}$
$\quad b_1^{-1}z^{-1}b_1z^3b_1z^2b_1^{-1}zb_1^{-1}z^{-1}u_1^{-1}a^{-1}u_2$
$\quad u_3b_1^{-1}z^2b_1zb_1^{-1}\}.$

By [1], the dimension of $H_2(G;\mathbb{F}_7)$ as a vector space over $\mathbb{F}_7$ is at most 6. We now seek generators of of this vector space. For simplicity, we denote $F(S')$ by F and $N(R')$ by N. An application of the FindBasis algorithm from the same paper gives that $\frac{N}{N^7[F,N]}$ is generated by the 12 elements

```
[ f1*f5*f1*f5^-1,
f2*f3*f2^-1*f3^-1,
f2*f5*f2*f5^-1,
f7*f5^-1*f7*f5^-1*f7*f5,
f3*f7*f1^-2*f7*f1*f7^-2*f5^-1*f3*f1^-1*f5^-1*f7^-1,
f4*f7*f1^2*f7^-2*f1^2*f7*f5^-1*f1^-2*f4*f1^-1*f5^-1*f7^-1,
f2*f7*f1*f5^-1*f1^-2*f5*f7^-2*f1^3*f7*f5^-1*f2*f1^-1*f5^-1*f7^-1,
f1^2*f7^-7*f1^-1*f3^-1*f1^-1*f4^-1*f2^-1*f1^-2*f7^-1*f1^2*f2*f3*f4,
f7*f1*f7*f1^2*f7*f1*f7*f1^2*f7*f1^3*f7*f1^3*f7*f1*f5^-1*f1^-1*f5^-1,
f7*f1^2*f7^-1*f1^-1*f4^-1*f1^-1*f5^-1*f7*f1^2*f7^-1*f4^-1*f1^-2*
        f5^-1*f7*f1^2*f7^-1*f4^-1*f1^-2*f5^-1,
f1^-1*f7^-1*f1*f7*f1*f7*f1*f7^-1*f4^-1*f5^-1*f3*f1*f7^-1*f1*f7*f1^2*
        f7^-1*f1^-1*f7*f1^-1*f4^-1*f5^-1*f3*f1^-1*f7^-1*f1*f7*f1^2*f7^-1*
        f1^-1*f7*f1^-1*f4^-1*f5^-1*f3,
f7*f1^-2*f7*f1*f7^-1*f1^2*f7*f1^2*f7^-1*f1*f7^-1*f1*f5^-1*f1^-2*f2*f3*
        f4*f7*f1^-2*f7*f1*f7^-1*f1^2*f7*f1^2*f7^-1*f1*f7^-1*f5^-1*f1^-3
        *f2*f3*f4*f7*f1^-2*f7*f1*f7^-1*f1^2*f7*f1^2*f7^-1*f1*
        f7^-1*f5^-1*f1^-3*f2*f3*f4 ]
```

By reducing these elements in $\frac{F}{[F,F]N^7}$ we obtain

```
[ <identity ...>, <identity ...>, <identity ...>,
<identity ...>, <identity ...>, <identity ...>,
f1^3, f7^-1, f2^-1, f1*f2, f1, f1^-1 ]
```

Thus the last six elements form a basis for the 6-dimensional vector space over $\mathbb{F}_7$, $F/[F,F]N^7$. This implies that the 6 vanishing elements are in the kernel of

$$\frac{N}{N^7[F,N]} \to \frac{F}{N^7[F,F]}$$

and therefore are generators of H_2. Thus the following theorem is established.

Theorem 3.3. *The Hopf second homology mod 7 of the group $SL_2(\mathbb{Z}[\zeta_7, \frac{1}{7}])$ has the following six generators*

$$\begin{aligned}
&\{zaza^{-1},\\
&u_1u_2u_1^{-1}u_2^{-1},\\
&u_1au_1a^{-1},\\
&b_1a^{-1}b_1a^{-1}b_1a,\\
&u_2b_1z^{-2}b_1zb_1^{-2}a^{-1}u_2z^{-1}a^{-1}b_1^{-1},\\
&u_3b_1z^2b_1^{-2}z^2b_1a^{-1}z^{-2}u_3z^{-1}a^{-1}b_1^{-1}\}
\end{aligned}$$

in terms of the group presentation in Proposition 3.2.

However, the possibility still exists that any, or all, of these may be trivial in H_2.

4. Homology Calculations

The following tables give the results of the algorithms in [1] applied to various linear groups. For the second table, a "less than" symbols indicates that the rewriting system involved in the calculation was not confluent, so only an upper bound was found. Otherwise, the rewriting system was confluent and the exact dimension was found; the code to implement these groups in GAP [14] is given below. We note that while none of the results in the table below are new, the previous results were found by a wide variety of methods, many of which are not computational in nature.

	$H_1(-;\mathbb{F}_2)$	$H_1(-;\mathbb{F}_3)$	$H_1(-;\mathbb{F}_5)$	$H_1(-;\mathbb{F}_7)$
$GL_2(\mathbb{Z})$	2	0	0	0
$SL_2(\mathbb{Z})$	1	1	0	0
$SL_2(\mathbb{Z}_2)$	1	0	0	0
$SL_2(\mathbb{Z}_3)$	0	1	0	0
$SL_2(\mathbb{Z}_5)$	0	0	1	0
$SL_2(\mathbb{Z}[i])$	1	0	0	0
$SL_2(\mathbb{Z}[\omega]), \omega^3 = -1$	0	1	0	0
$SL_2(\mathbb{Z}[\sqrt{-5}])$	3	2	1	1
$PSL_2(\mathbb{Z})$	1	1	0	0

Table 1. Dimensions of First Homology Groups

	$H_2(-;\mathbb{F}_2)$	$H_2(-;\mathbb{F}_3)$	$H_2(-;\mathbb{F}_5)$	$H_2(-;\mathbb{F}_7)$
$GL_2(\mathbb{Z})$	≤ 4	≤ 2	≤ 2	≤ 2
$SL_2(\mathbb{Z})$	≤ 2	≤ 2	≤ 1	≤ 1
$SL_2(\mathbb{Z}_2)$	1	0	0	0
$SL_2(\mathbb{Z}_3)$	0	1	0	0
$SL_2(\mathbb{Z}_5)$	0	0	1	0
$SL_2(\mathbb{Z}[i])$	1	0	0	0
$SL_2(\mathbb{Z}[\omega]), \omega^3 = -1$	≤ 1	≤ 2	≤ 1	≤ 1
$SL_2(\mathbb{Z}[\sqrt{-5}])$	≤ 3	≤ 3	0	0
$PSL_2(\mathbb{Z})$	≤ 1	≤ 1	0	0

Table 2. Dimensions of Second Homology Groups

5. Conclusion

The motivation for the algorithms used in this paper and in [1] grew from work on Quillen's conjecture. The utility of these algorithms is more general. In theory, they can be used to calculate or estimate the first and second homology of any finitely-presented group, provided homology coefficients are in a finite field.

Future work will involve refining and using the algorithms on a larger collection of groups with the goal of constructing the aforementioned database of calculations. In the context of the original problem, however, work to calculate the image of the transgression τ in Figure 1 is necessary to make progress on the conjecture.

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