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On the s-Integral Inequalities Concerning k-Fractional Conformable Integrals via Pre-Invexity

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Abstract In this paper, we dealt with generalized k -fractional conformable integrals. We established some integral inequalities of the s-Hermite-Hadamard type concerning k-fractional conformable integrals operators for pre-invex functions. In detail, we generalized Hermite-Hadamard type inequalities via Riemann-Liouville k-fractional integrals by the way of s-preinvex mapping and whose absolute value of 1^{st} derivative are pre-invex.

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1. Introduction

In recent years, inequalities are playing a very significant role in all fields of mathematics or precisely, we can say mathematical analysis, and present a very active and attractive field of research. We have seen many articles on the field of integration which is dominated by inequalities involving functions and their integrals. One of the famous integral inequalities is call up as:
Let $f : [a, b] \subset \mathbb{R} \to \mathbb{R}$ is said to be convex i-e, $f(tu + (1 - t)v) \leq tf(u) + (1 - t)f(v)$ for all $u, v \in [a, b]$ and $t \in [0, 1]$. The classical Hermite-Hadamard type inequality provides lower and upper estimates for the interval average of any convex function defined on a compact interval, involving the midpoint and endpoints of the domain. More precisely, if $f : I \to \mathbb{R}$ is a convex function then it is integrable in the Riemannian sense and

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{(b-a)}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2} \tag{1.1}$$

where $a, b \in I$ with $a < b$.
The classical definition of pre invex function follows

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Definition 1.1. [1] Let $I \subseteq \mathrm{R}^n$ be an invex set with respect to $\eta : \mathrm{R}^n \times \mathrm{R}^n \to \mathrm{R}^n$. The function $\phi : I \to \mathrm{R}^n$ is said to be a pre-invex on I, if and only if, $\forall u, v \in I$, $\forall \nu \in [0, 1]$.

$$\phi(u + \nu\eta(u, v)) \leq \nu\phi(u) + (1 - \nu)\phi(v). \tag{1.2}$$

Let us now consider $f : I \subseteq \mathbb{R} \to \mathbb{R}$ is said to be s-pre-invex, if the following inequality hold

$$f(a + t\eta(b, a)) \leq (1 - t)^s f(a) + t^s f(b),$$

where I is an interval in the real line $\mathbb{R}$, $t \in [0, 1]$ and for some fixed $s \in (0, 1]$.
It can be easily seen that for $s = 1$, s-pre-invexity reduces to ordinary pre-invexity of function defined on all positive real numbers. Since every convex function is pre-invex with respect to the mapping $\eta(b, a) = b - a$.
If $f : I = [a, a + \eta(b, a)] \to \mathbb{R}$ be a preinvex function on the interval of the real number I and and $a, b \in I$ with $\eta(b, a) > 0$, then the following inequality hold

$$f\left(\frac{2a + \eta(b, a)}{2}\right) \leq \frac{1}{\eta(b, a)} \int_a^{a+\eta(b,a)} f(x)dx \leq \frac{f(a) + f(b)}{2}. \tag{1.3}$$

The both inequality (1.1) and (1.3) are same. The result is analogous to the orginal Hermite-Hadamard inequality. If $\eta(b, a) = b - a$, then the inequality (1.3) reduces to the remarkable Hermite-Hadamard inequality (1.1). For detail information, please refer to [2–7] and closely related reference therein.

Definition 1.2. Let $f \in L[a, b]$.The symbol ${}_{\mathcal{R}L}J^{\alpha}_{a+} f$ and ${}_{\mathcal{R}L}J^{\alpha}_{b-} f$ denote the left and right Riemann-Liouville fractional integral of order $\alpha \in \mathbb{R}^+$ are defined by

$${}_{\mathcal{R}L}J^{\alpha}_{a+} f(b) = \frac{1}{\Gamma(a)} \int_a^{a+\eta(b,a)} (a + \eta(b, a) - t)^{\alpha-1} f(t)\, dt,\ a + \eta(b, a) > a \tag{1.4}$$

and

$${}_{\mathcal{R}L}J^{\alpha}_{b-} f(a) = \frac{1}{\Gamma(a)} \int_{b+\eta(a,b)}^{b} (t - (b + \eta(a, b)))^{\alpha-1} f(t)\, dt,\ b + \eta(a, b) < b \tag{1.5}$$

respectively. Here Γ is the classical Euler gamma function also discussed in [8].

Definition 1.3. The Riemann-Liouville k-fractional integral are respectively reproduced as

$${}_{\mathcal{R}L}J^{\alpha}_{a+} f(b) = \frac{1}{k\Gamma_k(a)} \int_a^{a+\eta(b,a)} (a + \eta(b, a) - t)^{\alpha/k-1} f(t)\, dt,\ a + \eta(b, a) > a. \tag{1.6}$$

$${}_{\mathcal{R}L}J^{\alpha}_{b-} f(a) = \frac{1}{k\Gamma_k(a)} \int_{b+\eta(a,b)}^{b} (t - (b + \eta(a, b)))^{\alpha/k-1} f(t)\, dt,\ b + \eta(a, b) < b. \tag{1.7}$$

For $\alpha > 0$.

Please refer to the papers [9–13] and close related therein for the importance of fractional integral operators reproduced in (1.4), (1.5), (1.6) and (1.7).
We now recall form [14] to inequalities of the Hermite-Hadamard type-concerning the Riemann-Liouville fractional integrals as follows.

Theorem 1.4. [14] *Let* $f : [a,b] \to \mathbb{R}$ *be a positive function with* $0 \le a < b$ *and* $f \in L_1[a,b]$ *. Then*

$$f\left(\frac{a+b}{2}\right) \le \frac{\Gamma(\beta+1)}{2(b-a)^{\alpha}} \left[{}_{RL}\mathcal{J}^{\alpha}_{a+} f(b) + {}_{\mathcal{R}L}J^{\alpha}_{b-} f(a)\right] \le \frac{f(a)+f(b)}{2}. \tag{1.8}$$

Theorem 1.5. [14] *Let* $f : [a,b] \to \mathbb{R}$ *be a differentiable mapping on* (a,b) *such that* $a < b$ *and* $f' \in L_1[a,b]$*. Then*

$$\left|\frac{f(a)+f(b)}{2} \le \frac{\Gamma(\beta+1)}{2(b-a)^{\alpha}} \left[{}_{\mathcal{R}L}J^{\alpha}_{a+} f(b) + {}_{\mathcal{R}L}J^{\alpha}_{b-} f(a)\right]\right|$$
$$\le \frac{b-a}{2(\alpha+1)}\left(1-\frac{1}{2^{\alpha}}\right)\left(|f'(a)|+|f'(b)|\right). \tag{1.9}$$

Definition 1.6. The left and right fractional conformable integral operations are defined by

$${}_{RL}{}^{\beta}_{k}\mathcal{J}^{\alpha}_{a+} f(b) = \frac{1}{\Gamma(\beta)} \int_a^{a+\eta(b,a)} \left[\frac{(a+\eta(b,a)-a)^{\alpha} - (t-a)^{\alpha}}{\alpha}\right]^{\beta-1} \frac{f(t)}{(t-a)^{1-\alpha}}\, dt$$

and

$${}_{RL}{}^{\beta}_{k}\mathcal{J}^{\alpha}_{b-} f(a) = \frac{1}{\Gamma(\beta)} \int_{b+\eta(a,b)}^{b} \left[\frac{(b-(b+\eta(a,b))^{\alpha} - (b-t)^{\alpha}}{\alpha}\right]^{\beta-1} \frac{f(t)}{(b-t)^{1-\alpha}}\, dt$$

for $\alpha > 0$ and $\beta > 0$. Obviously, if taking $\alpha = 0$ and $\alpha = 1$,then (1.6) reduce to the Riemann-liouville fractional integrals (1.2) respectively.

Definition 1.7. The generalized k-fractional conformable integrals are defined by

$${}_{RL}{}^{\beta}_{k}\mathcal{J}^{\alpha}_{a+} f(b) = \frac{1}{k\Gamma_k(\beta)} \int_a^{a+\eta(b,a)} \left[\frac{(a+\eta(b,a)-a)^{\alpha} - (t-a)^{\alpha}}{\alpha}\right]^{\beta/k-1} \frac{f(t)}{(t-a)^{1-\alpha}}\, dt$$

and

$${}_{RL}{}^{\beta}_{k}\mathcal{J}^{\alpha}_{b-} f(a) = \frac{1}{k\Gamma_k(\beta)} \int_{b+\eta(a,b)}^{b} \left[\frac{(b-(b+\eta(a,b))^{\alpha} - (b-t)^{\alpha}}{\alpha}\right]^{\beta/k-1} \frac{f(t)}{(b-t)^{1-\alpha}}\, dt$$

where $\alpha > 0$, $\beta > 0$ and $\Gamma k(x)$ is defined [15–19] by

$$\Gamma k(x) = \lim_{n\to\infty} \frac{n!k^n (nk)^x/k - 1}{(x)n, k}.$$

In term of

$$(\lambda)_{n,k} = \begin{cases} 1\,; & n = 0 \\ \lambda(\lambda+k)...(\lambda+(n-1)k)\,; & n \in \mathbb{N} \end{cases}$$

In this paper we will establish some inequality of the s-hermite-hadamard type concerning generalized k-fractional conformable integral operators and generalize several known inequalities of the hermite-hadamard type concerning k-fractional conformable integral operators.

2. Frist Main Results

For proving our main results, we need the following lemma.

Lemma 2.1. *Let $f : I \to \mathbb{R}$ be a differentiable function such that $a < b$ and $f' \in L[a, b]$*

$$\frac{2f(a) + f(\eta(b,a))}{2} - \frac{k\Gamma_k(\beta+k)\alpha^{\beta/k}}{\eta(b,a)^{\alpha\beta/k}}\left[{}_{RL}{}^{\beta}_{k}\mathcal{J}^{\alpha}_{b-}f(a) + {}_{RL}{}^{\beta}_{k}\mathcal{J}^{\alpha}_{a+}f(b)\right]$$
$$= \frac{\eta(b,a)\alpha^{\beta/k}}{2}\int_0^1\left[\left(\frac{1-t^\alpha}{\alpha}\right)^{\beta/k} - \left(\frac{1-(1-t)^\alpha}{\alpha}\right)^{\beta/k}\right] f'(a + t\eta(b,a))\,dt \tag{2.1}$$

$\alpha, \beta > 0.$

Proof. Let

$$I_1 = \int_0^1 \left(\frac{1-t^\alpha}{\alpha}\right)^{\beta/k} f'(a + t\eta(b,a))\,dt$$

$$I_2 = \int_0^1 \left(\frac{1-(1-t)^\alpha}{\alpha}\right)^{\beta/k} f'(a + t\eta(b,a))\,dt.$$

Now integrating I_1 by parts, we get

$$I_1 = \left(\frac{1-t^\alpha}{\alpha}\right)^{\beta/k} \frac{f(a+t\eta(b,a))}{\eta(b,a)}\Big|_0^1 - \int_0^1 \beta\left(\frac{(1-t)^\alpha}{\alpha}\right)^{\beta/k} f'(a+t\eta(b,a))dt$$
$$= \frac{1}{\eta(b,a)}\left[\frac{f(b)}{\alpha^{\beta/k}} - \frac{\Gamma_k(\beta+k)}{\eta(b,a)^{\alpha\beta/k}}\,{}_{RL}{}^{\beta}_{k}\mathcal{J}^{\alpha}_{b-}f(a)\right]. \tag{2.2}$$

In similar way

$$I_2 = -\frac{1}{\eta(b,a)}\left[\frac{f(a)}{\alpha^{\beta/k}} - \frac{\Gamma_k(\beta+k)}{\eta(b,a)^{\alpha\beta/k}}\,{}_{RL}{}^{\beta}_{k}\mathcal{J}^{\alpha}_{a+}f(b)\right] \tag{2.3}$$

(2.2) and (2.3) together imply (2.1). ∎

Remark 2.2. when $\alpha = 1$, the equality (2.1) in lemma 2.1 reduce to

$$\frac{2f(a) + f(\eta(b,a))}{2} - \frac{k\Gamma_k(\beta+k)}{\eta(b,a)^{\beta/k}}\left[{}_{RL}{}^{\beta}_{k}\mathcal{J}_{b-}f(a) + {}_{RL}{}^{\beta}_{k}\mathcal{J}_{a+}f(b)\right]$$
$$= \frac{\eta(b,a)}{2}\int_0^1\left[(1-t)^{\frac{\beta}{k}} - t^{\frac{\beta}{k}}\right] f'(a+t\eta(b,a))dt$$

for $\beta > 0$.
When $k = 1$ the equality (2.1) in lemma 2.1, becomes [14, lemma 3.1].
When $\alpha = 1$ and $k = 1$ the equality (2.1) in lemma 2.1 can be written as

$$\frac{\Gamma(\alpha+1)}{2(\eta(b,a))^\alpha}\left[{}_{RL}{}^{\beta}_{k}\mathcal{J}^{\alpha}_{b-}f(a) + {}_{RL}{}^{\beta}_{k}\mathcal{J}^{\alpha}_{a+}f(b)\right]$$
$$= \frac{\eta(b,a)}{2}\int_0^1 \left[(1-t)^\alpha - t^\alpha\right] f'(a+t\eta(b,a))dt$$

which can be found in [19, lemma 2].

3. Second Main Results

We now in the position to establish some inequalities of the s-Hermite-Hadamard type for pre-invex mapping generalized k-fractional conformable integral operators.

Theorem 3.1. *Let $f:[a,b]\to\mathbb{R}$ be such that $f\in L[a,b]$ and $a<b$. If f is pre-invex on $[a,b]$, then*

$$f\left(\frac{2a+\eta(b,a)}{2}\right)-\frac{k\Gamma_k(\beta+k)\alpha^{\beta/k}}{2(\eta(b,a)^{\alpha\beta/k})}\left[{}_{RL}{}^{\beta}_{k}\mathcal{J}^{\alpha}_{b-}f(a)+{}_{RL}{}^{\beta}_{k}\mathcal{J}^{\alpha}_{a+}f(b)\right]\le\frac{f(a)+f(b)}{s+1}, \tag{3.1}$$

for $\alpha,\beta>0$.

Proof. Since f is a pre-invex function on $[a,b]$ we have

$$f\left(\frac{2x+\eta(y,x)}{2}\right)\le\frac{2f(x)+f(\eta(y,x))}{2}\quad x,y\in[a,b].$$

Letting $x=a+t\eta(b,a)$ and $\eta(y,x)=b+t\eta(a,b)$ gives

$$2f\left(\frac{2a+\eta(b,a)}{2}\right)\le 2f(a+t\eta(b,a))+f(b+\eta(a,b)). \tag{3.2}$$

Multiplying on both side of (3.2) by $(\frac{1-t^{\alpha}}{\alpha})^{\beta/k-1}t^{\alpha-1}\,dt$ and integrating with respect t over $[0,1]$ leads to

$$2f\left(\frac{2a+\eta(b,a)}{2}\right)\int_0^1\left(\frac{1-t^{\alpha}}{\alpha}\right)^{\beta/k-1}t^{\alpha}-1dt$$

$$\le\int_0^1\left(\frac{1-t^{\alpha}}{\alpha}\right)^{\beta/k-1}t^{\alpha-1}2f(a+t\eta(b,a))dt+\int_0^1\left(\frac{1-t^{\alpha}}{\alpha}\right)^{\beta/k-1}t^{\alpha-1}f(b+t\eta(a,b))dt$$

$$=\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)}\left[\frac{1-(\frac{b-u}{b-a})^{\alpha}}{\alpha}\right]^{\beta/k-1}\left(\frac{b-u}{b-a}\right)^{\alpha-1}2f(u)\,du$$
$$+\frac{1}{\eta(b,a)}\int_{b+\eta(a,b)}^{b}\left[\frac{1-(\frac{v-a}{b-a})}{\alpha}\right]^{\beta/k-1}\left(\frac{v-a}{b-a}\right)^{\alpha-1}f(v)\,dv$$

$$=\frac{1}{\eta(b,a)^{\alpha\beta/k}}\int_a^{a+\eta(b,a)}\left[\frac{\eta(b,a)^{\alpha}-(b-u)^{\alpha}}{\alpha}\right]^{\beta/k-1}\frac{2f(u)}{(b-u)^{1-\alpha}}\,du$$
$$+\frac{1}{\eta(b,a)^{\alpha\beta/k}}\int_{b+\eta(a,b)}^{b}\left[\frac{\eta(b,a)^{\alpha}-(v-a)^{\alpha}}{\alpha}\right]^{\beta/k-1}\frac{f(v)}{(v-a)^{1-\alpha}}\,dv$$

$$\frac{2k\Gamma_k(\beta)}{\eta(b,a)^{\alpha\beta/k}}\left[{}_{RL}{}^{\beta}_{k}\mathcal{J}^{\alpha}_{b-}f(a)+{}_{RL}{}^{\beta}_{k}\mathcal{J}^{\alpha}_{a+}f(b)\right].$$

From

$$\int_a^{a+\eta(b,a)}\left(\frac{1-t^{\alpha}}{\alpha}\right)^{\beta/k-1}t^{\alpha-1}\,dt=\frac{k}{\beta\alpha^{\beta/k}},$$

it follow that

$$f\left(\frac{2a+\eta(b,a)}{2}\right) \leq \frac{k\Gamma_k(\beta+k)\alpha^{\beta/k}}{\eta(b,a)^{\alpha\beta/k}}\left[{}_{RL}{}_k^{\beta}\mathcal{J}_{b^-}^{\alpha} f(a) + {}_{RL}{}_k^{\beta}\mathcal{J}_{a^+}^{\alpha} f(b)\right]$$

which can be written as the left side of inequality (3.1).
Making use of the pre-invexity arrives at

$$f(a+t\eta(b,a)) \leq (1-t)^s f(a) + t^s f(b)$$

and

$$f(b+t\eta(a,b)) \leq (1-t)^s f(b) + t^s f(a)$$

adding the above two inequality yields

$$f(a+t\eta(b,a)) + f(b+t\eta(a,b)) \leq ((1-t)^s + t^s)(f(a)+f(b)).$$

Now, integrating above inequality, we get

$$\frac{k\Gamma_k(\beta+k)\alpha^{\beta/k}}{\eta(b,a)^{\alpha\beta/k}}\left[{}_{RL}{}_k^{\beta}\mathcal{J}_{b^-}^{\alpha} f(a) + {}_{RL}{}_k^{\beta}\mathcal{J}_{a^+}^{\alpha} f(b)\right] \leq 2\left(\frac{f(a)+f(b)}{s+1}\right)$$

which can be rewritten as the right hand side of inequality (3.1). The proof of theorem (3.1) is complete. ∎

Remark 3.2. If $\alpha = 1$ then the inequality (3.1) reduce to

$$f\left(\frac{2a+\eta(b,a)}{2}\right) - \frac{k\Gamma_k(\beta+k)}{2\eta(b,a)^{\beta/k}}\left[{}_{RL}{}_k^{\beta}\mathcal{J}_{b^-} f(a) + {}_{RL}{}_k^{\beta}\mathcal{J}_{a^+} f(b)\right] \leq \frac{f(a)+f(b)}{s+1}$$

for $\beta > 0$.
If $k = 1$ then then the inequality (3.1) in theorem 3.1, becomes [14, Theorem 2.1].
If $\alpha = 1$,$s = 1$ and $k = 1$ then the inequality (3.1) in theorem 3.1 can be rearranged as (1.8).

Theorem 3.3. *Let $f : [a,b] \to \mathbb{R}$ be a differentiable function such that $a < b$ and $f' \in L[a,b]$. If $|f'|$ is preinvex function on $[a,b]$, then*

$$\left|\frac{2f(a)+f(\eta(b,a))}{2} - \frac{k\Gamma_k(\beta+k)\alpha^{\beta/k}}{\eta(b,a)^{\alpha\beta/k}}\left[{}_{RL}{}_k^{\beta}\mathcal{J}_{b^-}^{\alpha} f(a) + {}_{RL}{}_k^{\beta}\mathcal{J}_{a^+}^{\alpha} f(b)\right]\right|$$
$$\leq \frac{\eta(b,a)}{2\alpha}\left[2B_{\frac{1}{2^\alpha}}\left(\frac{1}{\alpha},\frac{\beta}{k}+1\right) - B\left(\frac{1}{\alpha},\frac{\beta}{k}+1\right)\right](|f'(a)|+|f'(b)|) \tag{3.3}$$

for $\alpha, \beta > 0$.

Proof. By lemma 2.1 and the pre-invexity of $|f'|$, we have

$$\left|\frac{2f(a)+f(\eta(b,a))}{2} - \frac{k\Gamma_k(\beta+k)\alpha^{\beta/k}}{\eta(b,a)^{\alpha\beta/k}}\left[{}_{RL}{}_k^{\beta}\mathcal{J}_{b^-}^{\alpha} f(a) + {}_{RL}{}_k^{\beta}\mathcal{J}_{a^+}^{\alpha} f(b)\right]\right|$$
$$= \frac{\eta(b,a)\alpha^{\beta/k}}{2}\left|\int_0^1 \left(\frac{1-t^\alpha}{\alpha}\right)^{\beta/k} - \left(\frac{1-(1-t)^\alpha}{\alpha}\right)^{\beta/k} f'(a+t\eta(b,a))dt\right|$$

$$\leq \frac{\eta(b,a)\alpha^{\beta/k}}{2}\left|\int_0^1 \left(\frac{1-t^\alpha}{\alpha}\right)^{\beta/k} - \left(\frac{1-(1-t)^\alpha}{\alpha}\right)^{\beta/k}\right|((1-t)^s|f'(a)|+t^s|f'(b)|)$$

$$\leq \frac{\eta(b,a)\alpha^{\beta/k}}{2}\int_0^{1/2}\left(\frac{1-t^\alpha}{\alpha}\right)^{\beta/k}-\left(\frac{1-(1-t)^\alpha}{\alpha}\right)^{\beta/k}((1-t)^s|f'(a)|+t^s|f'(b)|)$$
$$+\frac{\eta(b,a)\alpha^{\beta/k}}{2}\int_{1/2}^{1}\left(\frac{1-(1-t)^\alpha}{\alpha}\right)^{\beta/k}-\left(\frac{1-t^\alpha}{\alpha}\right)^{\beta/k}((1-t)^s|f'(a)|+t^s|f'(b)|)$$

$$\begin{aligned}=\frac{\eta(b,a)\alpha^{\beta/k}}{2}\Bigg\{&|f'(a)|\int_0^{\frac{1}{2}}\left[(1-t)^s\left(\frac{1-t^\alpha}{\alpha}\right)^{\frac{\beta}{k}}-(1-t)^s\left(\frac{1-(1-t)^\alpha}{\alpha}\right)^{\frac{\beta}{k}}\right]dt\\
&+|f'(b)|\int_0^{1/2}\left[t^s\left(\frac{1-t^\alpha}{\alpha}\right)^{\beta/k}-t^s\left(\frac{1-(1-t)^\alpha}{\alpha}\right)^{\beta/k}\right]dt\\
&+|f'(a)|\int_{\frac{1}{2}}^{1}\left[(1-t)^s\left(\frac{1-(1-t)^\alpha}{\alpha}\right)^{\frac{\beta}{k}}-(1-t)^s\left(\frac{1-t^\alpha}{\alpha}\right)^{\frac{\beta}{k}}\right]dt\\
&+|f'(b)|\int_{\frac{1}{2}}^{1}\left[t^s\left(\frac{1-(1-t)^\alpha}{\alpha}\right)^{\frac{\beta}{k}}-t^s\left(\frac{1-t^\alpha}{\alpha}\right)^{\frac{\beta}{k}}\right]dt\Bigg\},\end{aligned} \tag{3.4}$$

changing variable by $x=t^\alpha$ and $y=(1-t)^\alpha$ results in

$$\int_0^{\frac{1}{2}}(1-t)^s\left(\frac{1-t^\alpha}{\alpha}\right)^{\frac{\beta}{k}}dt=\frac{1}{\alpha^{\frac{\beta}{k}+1}}\left[B\left(\frac{1}{\alpha},\frac{\beta}{k}-1\right)-B_{\frac{1}{2^\alpha}}\left(\frac{s+1}{\alpha},\frac{\beta}{k}+1\right)\right] \tag{3.5}$$

$$\begin{aligned}\int_0^{\frac{1}{2}}t^s\left(\frac{1-(1-t)^\alpha}{\alpha}\right)^{\frac{\beta}{k}}dt=\frac{1}{\alpha^{\frac{\beta}{k}+1}}&\left[B\left(\frac{1}{\alpha},\frac{\beta}{k}+1\right)-B\left(\frac{s+1}{\alpha},\frac{\beta}{k}+1\right)\right.\\
&\left.+B_{\frac{1}{2^\alpha}}\left(\frac{s+1}{\alpha},\frac{\beta}{k}+1\right)-B_{\frac{1}{2^\alpha}}\left(\frac{1}{\alpha},\frac{\beta}{k}+1\right)\right]\end{aligned} \tag{3.6}$$

$$\int_0^{\frac{1}{2}}(1-t)^s\left(\frac{1-(1-t)^\alpha}{\alpha}\right)^{\beta/k}dt=\frac{1}{\alpha^{\beta/k+1}}\left[B\left(\frac{s+1}{\alpha},\frac{\beta}{k}+1\right)-B_{\frac{1}{2^\alpha}}\left(\frac{s+1}{\alpha},\frac{\beta}{k}+1\right)\right] \tag{3.7}$$

substituting the equalities (3.5), (3.6) and (3.7) into the equality (3.4) leads to (3.3). The proof of theorem 3.3 is completed. ∎

Remark 3.4. If $\alpha=1$ then the inequality (3.3) in theorem 3.3 reduce to

$$\begin{aligned}&\left|\frac{f(a)+f(\eta(b,a))}{2}-\frac{k\Gamma_k(\beta+k)}{\eta(b,a)^{\beta/k}}\left[{}_{RL}{}^{\beta}_{k}\mathcal{J}_{b^-}f(a)+{}_{RL}{}^{\beta}_{k}\mathcal{J}_{a^+}f(b)\right]\right|\\
&\leq\frac{\eta(b,a)}{2(\beta+1)}\left(1-\frac{1}{2^\beta}\right)(|f'(a)|+|f'(b)|)\end{aligned}$$

for $\beta>0$.

When $k=1$ then the inequality (3.3) in theorem 3.3, becomes [14, Theorem 3.1].

If $\alpha=1$, and $k=1$, then the inequality (3.3) in theorem 3.3 can be reformulated as (1.9).

4. Concluding Remarks

In this article, we obtained some s-Hermite-Hadamard type concerning k-fractional conformable integrals for pre-invex functions via Riemann-liouville fractional integrals operators. The analogous results for convex functions also established. The results obtained in this monograph generalizing the existing results in literature cited herein.

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