Thai Journal of Mathematics
Volume 20 Number 3 (2022)
Pages 1441–1450

http://thaijmath.in.cmu.ac.th

ISSN 1686-0209

A New Type of Statistically Convergent Complex Uncertain Triple Sequence

Birojit Das[1,*], Binod Chandra Tripathy[2] and Piyali Debnath[1]

[1] *Department of Mathematics, NIT Agartala, India, 799046*
e-mail : dasbirojit@gmail.com (B. Das); piyalimaths@gmail.com (P. Debnath)

[2] *Department of Mathematics, Tripura University, India, 799022*
e-mail : tripathybc@rediffmail.com (B. C. Tripathy)

Abstract Statistical convergence of complex uncertain triple sequence is already introduced in four aspects of uncertainty. In this article, we initiate the notion of statistical convergence of complex uncertain triple sequence with respect to uniformly almost surely. We prove that statistical convergence with respect to uniformly almost surely of a complex uncertain triple sequence implies its statistical convergence in almost surely. Moreover, we characterized a statistically convergent complex uncertain triple sequence via natural density operator.

MSC: 60B10; 60B1; 60E05; 60F25; 40A05; 40A30; 40D25; 40F05

Keywords: Complex uncertain sequences; uncertainty space; triple sequence; statistical convergence

Submission date: 18.03.2021 / Acceptance date: 31.08.2021

1. Introduction

The theory of uncertainty came into existence to deal with certain types of indeterminacy, where the distribution function is not closer to the frequencies/ samples or when no samples are available to estimate a probability distribution. In the year 2007, Liu [1] introduced uncertainty theory to design belief degree of events which occurs in the state of indeterminacy. Through out the last decade several researchers have shown their great interest to develop the theory and now-a-days it became a separate branch of mathematics. Uncertainty theory have been applied in different field of mathematics, like uncertain logic [2], uncertain process and uncertain renewal process [3], uncertain differential equation [3], uncertain inference [4], uncertain calculus [5], uncertain risk and reliability analysis [6], uncertain programming [7], uncertain graph [8], uncertain finance [5, 9], uncertain entailment [10], uncertain set theory [4], uncertain statistics [11], uncertain random variable [12] and many more. Liu [1] introduced the notion of uncertain measure in which he adopted normality, duality, subadditivity, product uncertainty and conditional uncertainty axiom of uncertain measure. Also, he initiated the notions of expected value

*Corresponding author.

Published by The Mathematical Association of Thailand.

operator and distribution function of uncertain variable in uncertain environment and proved the linearity property of expected value operator. Using the notions of uncertain measure, expected value operator and uncertain distribution, Liu [1] introduced four types of convergence concepts of uncertain sequences which are termed as convergence in measure, convergence in mean, convergence in distribution and convergence in almost surely and established interrelationship among these notions. In the year 2009, You [13] came up with a new type of convergence concept of uncertain sequence namely convergence with respect to uniformly almost surely. All these five kinds of convergence concepts mentioned as above were in case of real uncertain sequences. Peng [14] in his doctoral thesis introduced complex uncertain variable and by considering sequences of complex uncertain variable, Chen et. al [15] studied the notions of different convergences and established interrelationships among those ideas. In recent days, researchers are trying to explore different types of convergence concepts, for example almost convergence, statistical convergence, by considering sequences / double sequences / triple sequences of real or complex uncertain variables. Almost convergence of complex uncertain sequences was studied by Saha et al. [16] and the same has been extended to complex uncertain double and triple sequences by Das et al. [17]. On the other hand, statistical convergence of complex uncertain sequence is introduced by Tripathy et al. [18]. Characterization of statistical convergence by considering complex uncertain double and triple are made by Das et al. [19] and Das et al. [20] respectively. Moreover, matrix transformation of complex uncertain sequences is studied by introducing the notion of convergence of complex uncertain series by Das et al. [21]. Few recent studies on statistical convergence of double and triple sequences in different aspects can be seen in [22–26]. In this article, we extend the study of statistically convergent complex uncertain triple sequence via uniformly almost surely and figure out the nature of relation of such kind of statistically convergent triple sequences with the triple sequences mentioned in [20].
Before going to the main section, we need some basic and preliminary ideas about the existing definitions and results which will play a major role in this study.

2. Preliminaries

Definition 2.1 ([1])**.** Let $\mathscr{L}$ be a σ-algebra on a non-empty set Γ. A set function $\mathscr{M}$ on Γ is called an uncertain measure if it satisfies the following axioms:
Axiom 1 (Normality Axiom). $\mathscr{M}\{\Gamma\}$=1;
Axiom 2 (Duality Axiom). $\mathscr{M}\{\Lambda\} + \mathscr{M}\{\Lambda^c\}$=1, for any $\Lambda \in \mathscr{L}$;
Axiom 3 (Subadditivity Axiom). For every countable sequence of events $\{\Lambda_j\} \in \mathscr{L}$, we have

$$\mathscr{M}\left\{\bigcup_{j=1}^{\infty} \Lambda_j\right\} \leq \sum_{j=1}^{\infty} \mathscr{M}(\Lambda_j)$$

The triplet $(\Gamma, \mathscr{L}, \mathscr{M})$ is called an uncertainty space and each element Λ in $\mathscr{L}$ is called an event. In order to obtain an uncertain measure of compound events, a product uncertain measure is defined by as follows:
Axiom 4 (Product Axiom). Let $(\Gamma_k, \mathscr{L}_k, \mathscr{M}_k)$ be uncertainty spaces, for k = 1,2,3..... The product uncertain measure $\mathscr{M}$ is an uncertain measure satisfying

$$\mathscr{M}\left\{\prod_{j=1}^{\infty} \Lambda_j\right\} = \bigwedge_{j=1}^{\infty} \mathscr{M}(\Lambda_j),$$

where Λ_k are arbitrarily chosen events from Γ_k, for k=1,2,3,...... respectively.

Definition 2.2 ([15]). A complex uncertain variable is a measurable function ζ from an uncertainty space $(\Gamma, \mathscr{L}, \mathscr{M})$ to the set of complex numbers, i.e., for any Borel set B of complex numbers, the set $\{\zeta \in B\}=\{\gamma \in \Gamma : \zeta(\gamma) \in B\}$ is an event.

Definition 2.3 ([20]). A complex uncertain triple sequence $\{\zeta_{nml}\}$ is said to be statistically convergent in mean to ζ if for any given $\varepsilon > 0$

$$\lim_{p,q,r\to\infty} \frac{1}{pqr}|\{(n,m,l) : n \leq p, m \leq q, l \leq r, E[||\zeta_{nml} - \zeta||] \geq \varepsilon\}| = 0.$$

Definition 2.4 ([20]). A complex uncertain triple sequence $\{\zeta_{nml}\}$ is said to be statistically convergent in measure to ζ if for any given $\varepsilon, \delta > 0$

$$\lim_{p,q,r\to\infty} \frac{1}{pqr}|\{(n,m,l) : n \leq p, m \leq q, l \leq r, \mathscr{M}(||\zeta_{nml} - \zeta|| \geq \varepsilon) \geq \delta\}| = 0.$$

Definition 2.5 ([20]). Let Φ, Φ_{nml} be the distribution function for the complex uncertain variables ζ, ζ_{nml} respectively, where $n, m, l \in \mathbb{N}$. Then the complex uncertain triple sequence $\{\zeta_{nml}\}$ is said to be statistically convergent in distribution to ζ if for any preassigned positive number ε,

$$\lim_{p,q,r\to\infty} \frac{1}{pqr}|\{(n,m,l) : n \leq p, m \leq q, l \leq r, ||\Phi_{nml}(z) - \Phi(z)|| \geq \varepsilon\}| = 0,$$

for all points z on complex plane, where Φ is continuous.

Definition 2.6 ([20]). A triple sequence $\{\zeta_{nml}\}$ of complex uncertain variable is said to be statistically converges to ζ with respect to almost surely if for every $\varepsilon > 0$ there exists some event Λ whose uncertain measure is 1 such that

$$\lim_{p,q,r\to\infty} \frac{1}{pqr}|\{(n,m,l) : n \leq p, m \leq q, l \leq r, ||\zeta_{nml}(\gamma) - \zeta(\gamma)|| \geq \varepsilon\}| = 0,$$

for all $\gamma \in \Lambda$.

3. Main Results

In this section, at first we introduce the notion of statistical convergence with respect to uniformly almost surely by considering triple sequences of complex uncertain variables. Then, we show the existence of such type of triple sequence by providing a suitable example. We establish few properties of such triple sequence and characterize statistical convergence of complex uncertain triple sequences via natural density operator.

Definition 3.1. A complex uncertain triple sequence $\{\zeta_{nml}\}$ is called statistically convergent to the limit ζ with respect to uniformly almost surely if for any preassigned $\varepsilon > 0$, there exists a sequence of events $\{E_x\}$, each of whose uncertain measure tends to zero,

$$\lim_{p,q,r\to\infty} \frac{1}{pqr}|\{(n,m,l) : n \leq p, m \leq q, l \leq r, ||\zeta_{nml}(\gamma) - \zeta(\gamma)|| \geq \varepsilon\}| = 0,$$

for all $\gamma \in \Gamma - E_x$.
The collection of all statistically convergent complex uncertain sequence with respect to uniformly almost surely is denoted by $st_3(\Gamma_{u.a.s})$

Example 3.2. We consider an uncertainty space $(\Gamma, \mathscr{L}, \mathscr{M})$ with $\Gamma = \{\gamma_1, \gamma_2, \gamma_3\}$ and $\mathscr{L} = P(\Gamma)$.
Also, we define the uncertain measurable function $\mathscr{M}$ as follows:

$$\mathscr{M}\{\Lambda\} = \begin{cases} 0, & \text{if } \Lambda = \emptyset \text{ or } \gamma_1, \gamma_2 \in \Lambda. \\ 1, & \text{otherwise.} \end{cases}$$

We also define the complex uncertain variables ζ_{nml} and ζ by

$$\zeta_{nml}(\gamma) = \begin{cases} i, & \text{if } \gamma = \gamma_1; \\ 2i, & \text{if } \gamma = \gamma_2; \\ 0, & \text{otherwise}, \end{cases}$$

for $n, m, l \in \mathbb{N}$ and $\zeta(\gamma) = 0, \ \forall \ \gamma \in \Gamma$.
Here, the complex uncertain triple sequence $\{\zeta_{nml}\}$ is statistically convergent to ζ with respect to uniformly almost surely.

Theorem 3.3. *If a complex uncertain triple sequence $\{\zeta_{nml}\}$ is statistically convergent in almost surely to ζ, then for any $\varepsilon > 0$ and $\delta > 0$ there exists $N \in \mathbb{N}$ such that*

$$\mathscr{M}\left\{\bigcap_{N=1}^{\infty}\bigcup_{n=N}^{\infty}\bigcup_{m=N}^{\infty}\bigcup_{l=N}^{\infty}\left\{\gamma : \tfrac{1}{nml}|\{(x,y,z) : x \le n, y \le m, z \le l, ||\zeta_{xyz}(\gamma) - \zeta(\gamma)|| \ge \delta\}| \ge \varepsilon\right\}\right\}$$
$$= 0.$$

Proof. Let $\{\zeta_{nml}\}$ be a complex uncertain triple sequence which is statistically convergent in almost surely to ζ.
From the definition of statistical convergence in almost surely, for a given $\delta > 0$ there exists some event Λ with unit uncertain measure such that

$$\lim_{n,m,l\to\infty} \tfrac{1}{nml} |\, \{(x,y,z) : x \le n, y \le m, z \le l, ||\zeta_{xyz}(\gamma) - \zeta(\gamma)|| \ge \delta\}\,| = 0,$$

for all $\gamma \in \Lambda$.
Then, for a given $\varepsilon > 0$ there exists a positive integer N such that for all $n, m, l \ge N$

$$\tfrac{1}{nml} |\, \{(x,y,z) : x \le n, y \le m, z \le l, ||\zeta_{xyz}(\gamma) - \zeta(\gamma)|| \ge \delta\}\,| < \varepsilon,$$

which is equivalent to

$$\mathscr{M}\left\{\bigcap_{N=1}^{\infty}\bigcup_{n=N}^{\infty}\bigcup_{m=N}^{\infty}\bigcup_{l=N}^{\infty}\left\{\gamma : \tfrac{1}{nml}|\,\{(x,y,z) : x \le n, y \le m, z \le l, ||\zeta_{xyz}(\gamma) - \zeta(\gamma)|| \ge \delta\}\,| < \varepsilon\right\}\right\}$$
$$= 1.$$

Then applying the duality axiom of uncertainty measure, we get

$$\mathscr{M}\left\{\bigcap_{N=1}^{\infty}\bigcup_{n=N}^{\infty}\bigcup_{m=N}^{\infty}\bigcup_{l=N}^{\infty}\left\{\gamma : \tfrac{1}{nml}|\,\{(x,y,z) : x \le n, y \le m, z \le l, ||\zeta_{xyz}(\gamma) - \zeta(\gamma)|| \ge \delta\}\,| \ge \varepsilon\right\}\right\}$$
$$= 0.$$ ∎

Theorem 3.4. *The triple sequence $\{\zeta_{nml}\}$ of complex uncertain variable converges statistically with respect to uniformly almost surely to ζ if and only if for any given $\varepsilon > 0$ and $\delta > 0$ there exists $N \in \mathbb{N}$ such that*

$$\lim_{n,m,l\to\infty}\mathscr{M}\left\{\bigcup_{n=N}^{\infty}\bigcup_{m=N}^{\infty}\bigcup_{l=N}^{\infty}\left\{\gamma : \tfrac{1}{nml}|\{(x,y,z) : x \le n, y \le m, z \le l, ||\zeta_{xyz}(\gamma) - \zeta(\gamma)|| \ge \delta\}\,| \ge \varepsilon\right\}\right\}$$
$$= 0.$$

Proof. Let $\{\zeta_{nml}\}$ be a uniformly almost surely statistically convergent complex uncertain triple sequence, which converges statistically to ζ.
Then, for given numbers $\varepsilon > 0$ and $\delta > 0$ there exists an event $\mathcal{B}$ with uncertain measure

less than ν ($\nu \to 0^+$) such that $\{\zeta_{nml}\}$ statistically converges uniformly almost surely to ζ on $\Gamma - \mathcal{B}$.
In other words , there exists a positive integer N such that

$$\bigcup_{n=N}^{\infty}\bigcup_{m=N}^{\infty}\bigcup_{l=N}^{\infty}\left\{\gamma : \tfrac{1}{nml}|\{(x,y,z) : x \le n, y \le m, z \le l, ||\zeta_{xyz}(\gamma) - \zeta(\gamma)|| \ge \delta\}| \ge \varepsilon\right\} \subseteq \mathcal{B}.$$

Taking uncertain measure of both the terms in the above, we get

$$\mathscr{M}\left\{\bigcup_{n=N}^{\infty}\bigcup_{m=N}^{\infty}\bigcup_{l=N}^{\infty}\left\{\gamma : \tfrac{1}{nml}|\{(x,y,z) : x \le n, y \le m, z \le l, ||\zeta_{xyz}(\gamma) - \zeta(\gamma)|| \ge \delta\}| \ge \varepsilon\right\}\right\}$$
$$\le \mathscr{M}\{\mathcal{B}\} < \nu.$$

Taking limits of n, m and l to infinity, we have

$$\lim_{n,m,l\to\infty}\mathscr{M}\left\{\bigcup_{n=N}^{\infty}\bigcup_{m=N}^{\infty}\bigcup_{l=N}^{\infty}\left\{\gamma : \tfrac{1}{nml}|\{(x,y,z) : x \le n, y \le m, z \le l, ||\zeta_{xyz}(\gamma) - \zeta(\gamma)|| \ge \delta\}| \ge \varepsilon\right\}\right\}$$
$$= 0.$$

For the converse part, let us assume that the given conditions hold. Then, for any given $\varepsilon > 0$ and $\delta > 0$ with $a \ge 1$ there exists $N_0 \in \mathbb{N}$ such that

$$\mathscr{M}\left\{\bigcup_{n=N_0}^{\infty}\bigcup_{m=N_0}^{\infty}\bigcup_{l=N_0}^{\infty}\left\{\gamma : \tfrac{1}{nml}|\{(x,y,z) : x \le n, y \le m, z \le l, ||\zeta_{xyz}(\gamma) - \zeta(\gamma)|| \ge \delta\}| \ge \tfrac{1}{a}\right\}\right\}$$
$$\le \tfrac{\nu}{2^a}$$

Let $\mathcal{B}$ be an event such that

$$\mathcal{B} = \bigcup_{a=1}^{\infty}\bigcup_{n=N_0}^{\infty}\bigcup_{m=N_0}^{\infty}\bigcup_{l=N_0}^{\infty}\left\{\gamma : \tfrac{1}{nml}|\{(x,y,z) : x \le n, y \le m, z \le l, ||\zeta_{xyz}(\gamma) - \zeta(\gamma)|| \ge \delta\}| \ge \tfrac{1}{a}\right\}.$$

Then, $\mathscr{M}\{\mathcal{B}\} \le \sum_{a=1}^{\infty} \frac{\nu}{2^a} = \nu$, which tends to 0.
Moreover,

$$\sup_{\gamma\in\Gamma-\mathcal{B}} \tfrac{1}{nml}|\{(x,y,z) : x \le n, y \le m, z \le l, ||\zeta_{xyz}(\gamma) - \zeta(\gamma)|| \ge \delta\}| < \tfrac{1}{a},$$

for all $n, m, l \ge N_0$ and uniformly for all $m = 1, 2, \ldots..$
Hence, the complex uncertain triple sequence $\{\zeta_{nml}\}$ is statistically convergent with respect to uniformly almost surely. ∎

Theorem 3.5. *Statistical convergence of complex uncertain triple sequence with respect to uniformly almost surely imply statistical convergence in almost surely.*

Proof. Statistical convergence of a complex uncertain triple sequence $\{\zeta_{nml}\}$ with respect to uniformly almost surely to ζ gives the following from the Theorem 3.4.

$$\lim_{n,m,l\to\infty}\mathscr{M}\left\{\bigcup_{n=N}^{\infty}\bigcup_{m=N}^{\infty}\bigcup_{l=N}^{\infty}\left\{\gamma : \tfrac{1}{nml}|\{(x,y,z) : x \le n, y \le m, z \le l, ||\zeta_{xyz}(\gamma) - \zeta(\gamma)|| \ge \delta\}| \ge \varepsilon\right\}\right\}$$
$$= 0.$$

Since, $\mathscr{M}\left\{\bigcap_{N=1}^{\infty}\bigcup_{n=N}^{\infty}\bigcup_{m=N}^{\infty}\bigcup_{l=N}^{\infty}\left\{\gamma : \tfrac{1}{nml}|\{(x,y,z) : x \le n, y \le m, z \le l, ||\zeta_{xyz}(\gamma) - \zeta(\gamma)|| \ge \delta\}| \ge \varepsilon\right\}\right\}$

$$\le \mathscr{M}\left\{\bigcup_{n=N}^{\infty}\bigcup_{m=N}^{\infty}\bigcup_{l=N}^{\infty}\left\{\gamma : \tfrac{1}{nml}|\{(x,y,z) : x \le n, y \le m, z \le l, ||\zeta_{xyz}(\gamma) - \zeta(\gamma)|| \ge \delta\}| \ge \varepsilon\right\}\right\}.$$

Taking the limits $n \to \infty, m \to \infty$ and $l \to \infty$, we get

$$\lim_{n,m,l\to\infty} \mathscr{M}\left\{\bigcap_{N=1}^{\infty}\bigcup_{n=N}^{\infty}\bigcup_{m=N}^{\infty}\bigcup_{l=N}^{\infty}\left\{\gamma : \tfrac{1}{nml}\left|\left\{(x,y,z) : x\leq n, y\leq m, z\leq l, ||\zeta_{xyz}(\gamma)-\zeta(\gamma)|| \geq \delta\right\}\right| \geq \varepsilon\right\}\right\}$$
$$\leq \lim_{n,m,l\to\infty} \mathscr{M}\left\{\bigcup_{n=N}^{\infty}\bigcup_{m=N}^{\infty}\bigcup_{l=N}^{\infty}\left\{\gamma : \tfrac{1}{nml}\left|\left\{(x,y,z) : x\leq n, y\leq m, z\leq l, ||\zeta_{xyz}(\gamma)-\zeta(\gamma)|| \geq \delta\right\}\right| \geq \varepsilon\right\}\right\}$$
$= 0.$

Hence, the triple sequence $\{\zeta_{nml}\}$ converges statistically to ζ in almost surely. ■

Theorem 3.6. *A complex uncertain triple sequence which converges with respect to uniformly almost surely, is also statistically convergent to the same limit in the same aspect.*

Proof. Let the complex uncertain triple sequence $\{\zeta_{nml}\}$ converges to ζ in almost surely. Then there exists a sequence of uncertain events $\{E_x\}$ with uncertain measure of each of the events approaching to zero, so that for all $\varepsilon > 0$, there must have an $n_0 \in \mathbb{N}$ such that

$$||\zeta_{nml}(\gamma) - \zeta(\gamma)|| < \varepsilon,\ \forall\ \gamma \in \Gamma - E_x \text{ and } n, m, l \geq n_0,\ \text{ for each } x.$$

Then we have,

$$\max_{n,m,l} |\{(n,m,l) : ||\zeta_{nml}(\gamma) - \zeta(\gamma)|| \geq \varepsilon\}| = n_0^3$$

Consequently, $\lim\limits_{p,q,r\to\infty} \frac{1}{pqr}|\{(n,m,l) : n \leq p, m \leq q, l \leq r, ||\zeta_{nml}(\gamma) - \zeta(\gamma)|| \geq \varepsilon\}|$

$\leq \lim\limits_{p,q,r\to\infty} \frac{n_0^3}{pqr} = 0,\ \forall\ \gamma \in \Gamma - E_x.$

Hence, the complex uncertain triple sequence $\{\zeta_{nml}\}$ statistically converges to ζ with respect to uniformly almost surely. ■

Remark 3.7. The converse of the above theorem is not true, that is, a statistically convergent complex uncertain triple sequence in uniformly almost surely may not be convergent therein. This claim is justified in the following example.

Example 3.8. Let $(\Gamma, \mathscr{L}, \mathscr{M})$ be an uncertainty space with $\Gamma = \{\gamma_1, \gamma_2, \gamma_3, ...\}, \mathscr{L} = P(\Gamma)$. Let the uncertainty measure of any uncertain event Λ be defined as follows:

$$\mathscr{M}\{\Lambda\} = \sum_{\gamma_i \in \Lambda} \frac{1}{2^i},\ \text{ where } \Lambda \subseteq \Gamma.$$

Let us now consider the complex uncertain triple sequence $\{\zeta_{nml}\}$, where the complex uncertain variable ζ_{nml} is defined by

$$\zeta_{nml}(\gamma) = \begin{cases} nml, & \text{if } l, m, n \text{ are all squares;} \\ \\ 0, & \text{otherwise,} \end{cases}$$

for all $\gamma \in \Gamma$.

Here, the triple sequence $\{\zeta_{nml}\}$ statistically converges to ζ, which is defined by $\zeta(\gamma) = 1,\ \forall\ \gamma \in \Gamma$.

The reason is that for any given $\varepsilon > 0$ and $\gamma \in \Gamma$, the set $\{(n,m,l) : ||\zeta_{nml}(\gamma) - \zeta(\gamma)|| \geq \varepsilon\}$, that is, $\{(n,m,l) : ||\zeta_{nml}(\gamma) - 1|| \geq \varepsilon\}$ has cardinality less than or equal to $\sqrt{nml}$.

Consequently, $\lim\limits_{p,q,r\to\infty} \frac{1}{pqr}|\{(n,m,l) : n \leq p, m \leq q, l \leq r, ||\zeta_{nml}(\gamma) - 1|| \geq \varepsilon\}|$

$\leq \lim\limits_{p,q,r\to\infty} \frac{1}{pqr}.\sqrt{pqr} = \lim\limits_{p,q,r\to\infty} \frac{1}{\sqrt{pqr}} = 0.$

But it is obvious that the complex uncertain triple sequence $\{\zeta_{nml}\}$ does not converge to any finite limit in respect of uniformly almost surely.

Remark 3.9. From the above example, we can easily verify that the complex uncertain triple sequence $\{\zeta_{nml}\}$ is not bounded in uniformly almost surely. Thus we can make a conclusion that a uniformly almost surely statistically convergent complex uncertain triple sequence is not bounded with respect to uniformly almost surely.

Theorem 3.10. *A convergent complex uncertain triple sequence converges statistically with preservation of limit in view of the concept of uncertain measure.*

Proof. Let $\zeta = \{\zeta_{nml}\}$ be a convergent complex uncertain triple sequence in measure to the limit ζ.
Then for any given $\varepsilon > 0$, there exists a number $n_0 \in \mathbb{N}$ such that

$$\lim_{n,m,l\to\infty} \mathscr{M}\{||\zeta_{nml} - \zeta|| \geq \varepsilon\} = 0, \ \forall\ n,m,l > n_0,$$

which means that for any $\varepsilon > 0$ and $\delta > 0$ there exists $n_0 \in \mathbb{N}$ such that

$$\mathscr{M}\{||\zeta_{nml} - \zeta|| \geq \varepsilon\} > \delta, \text{ for some } \ n,m,l \leq n_0.$$

This implies that, the cardinality of the set $\{(n,m,l) : \mathscr{M}\{||\zeta_{nml} - \zeta|| \geq \varepsilon\} \geq \delta\}$ is at most n_0^3.
Hence, $\lim\limits_{p,q,r\to\infty} \frac{1}{pqr}|\{(n,m,l) : n \leq p, m \leq q, l \leq r, \mathscr{M}\{||\zeta_{nml} - \zeta|| \geq \varepsilon\} \geq \delta\}|$
$\leq \lim\limits_{p,q,r\to\infty} \frac{n_0^3}{pqr} = 0.$
As a consequence, the complex uncertain triple sequence is statistically convergent to ζ in measure. ∎

Remark 3.11. Theorem 3.6 is true for the cases of convergence in mean, in distribution and with respect to almost surely.
By considering the expected value operator and complex uncertainty distribution function respectively in the Theorem 3.10, we can easily prove that every convergent complex uncertain triple sequence is statistically convergent in mean and distribution respectively.
Also, considering the uncertain set events Λ i.e., $\Lambda \subseteq \Gamma$, with unit uncertain measure and then by taking the events $\gamma \in \Lambda$ in the Theorem 3.6, it can be proved that every convergent complex uncertain triple sequence with respect to almost surely is statistically convergent therein.

Theorem 3.12. *The triple sequence $\{\zeta_{nml}\}$ of complex uncertain variable converges statistically to ζ with respect to uniformly almost surely if there exists a subset K of $\mathbb{N}\times\mathbb{N}\times\mathbb{N}$ with $\delta_3(K) = 1$ and a sequence of events E_t having uncertain measure of each events approaching zero such that*

$$\lim_{x,y,z\to\infty} \zeta_{xyz}(\gamma) = \zeta(\gamma), \ \forall\ (x,y,z) \in K, \ \gamma \in \Gamma - \{E_t\}.$$

Proof. Let $\{\zeta_{nml}\}$ be a statistically convergent complex uncertain triple sequence in uniformly almost surely to ζ.
We construct the following two sets S_r and L_r as

$$S_r = \{(x,y,z) \in \mathbb{N}\times\mathbb{N}\times\mathbb{N} : ||\zeta_{xyz}(\gamma) - \zeta(\gamma)|| \geq \tfrac{1}{r}\}, \ \forall\ \gamma \in \Gamma - E_t$$

and

$$L_r = \{(x, y, z) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N} : ||\zeta_{xyz}(\gamma) - \zeta(\gamma)|| < \tfrac{1}{r}\}, \ \forall \ \gamma \in \Gamma - E_t,$$

where $r \in \mathbb{N}$.
Then

$$\delta_3(S_r) = 0, \ L_1 \supset L_2 \supset .... \supset L_i \supset L_{i+1} \supset .... \tag{3.1}$$

and

$$\delta_3(L_r) = 1, \ r = 1, 2, ... \tag{3.2}$$

At this stage, we aim to show that for any $(x, y, z) \in L_r$, the complex uncertain triple sequence $\{\zeta_{xyz}\}$ is convergent in uniformly almost surely to ζ.
Let $\{\zeta_{xyz}\}$ is not convergent in uniformly almost surely to ζ. Then, there exists $\varepsilon > 0$ such that

$$||\zeta_{xyz}(\gamma) - \zeta(\gamma)|| \geq \varepsilon,$$

for infinitely many $(x, y, z) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N}$ and $\gamma \in \Gamma - E_t$.
Let us suppose that $L_\varepsilon = \{(x, y, z) : ||\zeta_{xyz}(\gamma) - \zeta(\gamma)|| < \varepsilon\}$ and $\varepsilon > \frac{1}{r}$ $(r = 1, 2, 3, ...)$.
Then,

$$\delta_3(L_\varepsilon) = 0 \tag{3.3}$$

and by the Equation 3.1, $L_r \subset L_\varepsilon$.
Hence, $\delta_3(L_\varepsilon) = 0$, which is a contradiction to the Equation 3.2.
Thus $\{\zeta_{xyz}\}$ is convergent in uniformly almost surely to ζ.
Conversely, let there exists subset $K \subseteq \mathbb{N} \times \mathbb{N} \times \mathbb{N}$ such that $\delta_3(K) = 1$ and

$$\lim_{x,y,z \to \infty} ||\zeta_{xyz}(\gamma) - \zeta(\gamma)|| = 0, \ (x, y, z) \in K,$$

which implies there exists $n_0 \in \mathbb{N}$ such that for every $\varepsilon > 0$

$$||\zeta_{xyz}(\gamma) - \zeta(\gamma)|| < \varepsilon, \forall \ x, y, z \geq n_0.$$

Now, $S_\varepsilon = \{(x, y, z) : ||\zeta_{xyz}(\gamma) - \zeta(\gamma)|| \geq \varepsilon\}$
$\subseteq \mathbb{N} \times \mathbb{N} \times \mathbb{N} - \{(x_{n_0+1}, y_{n_0+1}, z_{n_0+1}), (x_{n_0+2}, y_{n_0+2}, z_{n_0+2}), ...\}$,
that is, $\delta_3(S_\varepsilon) \leq 1 - 1 = 0$.
Hence, $\{\zeta_{xyz}\}$ is statistically convergent in uniformly almost surely to ζ. ■

References

[1] B. Liu, Uncertainty theory, 2nd edn, Springer-Verlag, Berlin, 2007.

[2] B. Liu, Uncertain logic for modeling human language, J. Uncer. Syst. 5 (1) (2011) 3–20.

[3] B. Liu, Fuzzy process, hybrid process and uncertain process, J. Uncer. Syst. 2 (1) (2008) 3–16.

[4] B. Liu, Uncertain set theory and uncertain inference rule with application to uncertain control, J. Uncer. Syst. 4 (2) (2010) 83–98.

[5] B. Liu, Some research problems in uncertain theory, J. Uncer. Syst. 3 (1) (2009) 3–10.

[6] B. Liu, Uncertain risk analysis and uncertain stability analysis, J. Uncer. Syst. 4 (3) (2010) 163–170.

[7] B. Liu, Theory and practice of uncertain programming, 2nd edn, Springer-Verlag, Berlin, 2009.

[8] X. Gao, Y. Gao, Connectedness index of uncertain graphs, Int. J. Uncer. Fuzzy. Knowl. Based Syst. 21 (1) (2013) 127–137.

[9] X. Chen, American option pricing formula for uncertain finance market, Int. J. Oper. Res. 8 (2) (2011) 32–37.

[10] B. Liu, Uncertain entailment and modus penes in the framework of uncertain logic, J. Uncer. Syst. 3 (4) (2009) 243–251.

[11] B. Liu, Uncertainty theory: a branch of mathematics for modeling human uncertainty, Springer, Berlin, 2010.

[12] Y.H. Liu, Uncertain random variables: a mixture of uncertainty and randomness, Soft Comp. 17 (4) (2013) 625–634.

[13] C. You, On the convergence of uncertain sequences, Math. Comp. Model. 49 (2009) 482–487.

[14] Z. Peng, Complex uncertain variable, Doctoral Dissertation, Tsinghua University, 2012.

[15] X. Chen, Y. Ning, X. Wang, Convergence of complex uncertain sequences. J. Int. Fuzzy Syst. 30 (2016) 3357–3366.

[16] S. Saha, B.C. Tripathy, S. Roy, On almost convergent of complex uncertain sequences, New Math. Nat. Comp. 16 (3) (2020) 573–580.

[17] B. Das, B.C. Tripathy, P. Debnath, J. Nath, B. Bhattacharya, Almost convergence of complex uncertain triple sequences, Proc. Nat. Aca. Sci. Ser. A. (2020) doi: 10.1007/s40010-020-00721-w.

[18] B.C. Tripathy, P.K. Nath, Statistical convergence of complex uncertain sequences, New Math. Nat. Comp. 13 (3) (2017) 359–374.

[19] B. Das, B.C. Tripathy, P. Debnath, B. Bhattacharya, Characterization of statistical convergence of complex uncertain double sequences, Anal. Math. Phy. 10 (4) (2020) 1–20.

[20] B. Das, B.C. Tripathy, P. Debnath, B. Bhattacharya, Statistical convergence of complex uncertain triple sequences, Comm. Stat. Theo. Meth. (2021) doi: 10.1080/03610926.2020.1871016.

[21] B. Das, B.C. Tripathy, P. Debnath, B. Bhattacharya, Study of matrix transformation of uniformly almost surely convergent complex uncertain sequences, Filomat 34 (14) (2020) 4907–4922.

[22] B. Hazarika, N. Subramanian, M. Mursaleen, Korovkin-type approximation theorem for Bernstein operator of rough statistical convergence of triple sequences, Adv. Oper. Theo. 5:2 (2020) 1–12.

[23] B. Hazarika, A. Alotaibi, S.A. Mohiuddine, Statistical convergence in measure for double sequences of fuzzy-valued functions, Soft Comp. 24 (2020) 6613–6622.

[24] A. Indumathi, S. Nithyanandam, B. Hazarika, On the Borel summability method for convergence of triple sequences of Bernstein–Stancu operators of fuzzy numbers, Soft Comp. 25:1 (2021) 683–697.

[25] J. Nath, B.C. Tripathy, B. Das, B. Bhattacharya, On strongly almost λ-convergence and statistically almost λ-convergence in the environment of uncertainty, Int. J. Gen. Sys. 51 (3) (2022) 262–276.

[26] B. Das, P. Debnath, B.C. Tripathy, On statistically convergent complex uncertain sequences, Car. Math. Pub. 14 (1) (2022) 135–146.