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On the Difference Equations

$$x_{n+1} = \frac{x_{n-5}}{\pm 1 - x_{n-2}x_{n-5}}$$

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Abstract : We study the solutions of the following difference equations

$$\begin{aligned} x_{n+1} &= \frac{x_{n-5}}{1 - x_{n-2}x_{n-5}}, \\ x_{n+1} &= \frac{x_{n-5}}{-1 - x_{n-2}x_{n-5}}, \quad n = 0, 1, ... \end{aligned}$$

where initial values are non zero real numbers.

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1 Introduction

Recently there has been a lot of interest in studying the global attractivity, boundedness character and the periodic nature of nonlinear difference equations. For some results in this area, see for example [1-10]. Cinar [2-4] investigated the solutions of the following difference equations

$$x_{n+1} = \frac{x_{n-1}}{1 + x_n x_{n-1}}, \quad x_{n+1} = \frac{x_{n-1}}{-1 + x_n x_{n-1}}, \quad x_{n+1} = \frac{a x_{n-1}}{1 + b x_n x_{n-1}}.$$

Aloqeili [1] has obtained the solutions of the difference equation

$$x_{n+1} = \frac{x_{n-1}}{a - x_n x_{n-1}},$$

Elabbasy et al. [5] investigated the global stability, periodicity character and gave the solution of special case of the following recursive sequence

$$x_{n+1} = a x_n - \frac{b x_n}{c x_n - d x_{n-1}}.$$

Elabbasy et al. [7] investigated the global stability, periodicity character and gave the solution of some special cases of the difference equation

$$x_{n+1} = \frac{d x_{n-l} x_{n-k}}{c x_{n-s} - b} + a.$$

Elsayed [8] has obtained the solutions of the difference equations

$$x_{n+1} = \frac{x_n}{x_{n-1}(x_n \pm 1)}.$$

Karatas et al.[9] gave that the solution of the difference equation

$$x_{n+1} = \frac{x_{n-5}}{1 + x_{n-2}x_{n-5}}.$$

In this paper a solution of the following two difference equations

$$x_{n+1} = \frac{x_{n-5}}{1 - x_{n-2}x_{n-5}}, \quad n = 0, 1, ... \tag{1.1}$$

$$x_{n+1} = \frac{x_{n-5}}{-1 - x_{n-2}x_{n-5}}, \quad n = 0, 1, ... \tag{1.2}$$

where the initial conditions $x_{-5},\ x_{-4},\ x_{-3},\ x_{-2},\ x_{-1},\ x_0$ are arbitrary non zero real numbers are investigated.

Definition 1. (Periodicity)

A sequence $\{x_n\}_{n=-k}^{\infty}$ is said to be periodic with period p if $x_{n+p} = x_n$ for all $n \geq -k$.

2 MAIN RESULTS

2.1 Equation (1)

In this section we give a specific form of the solutions of Eq.(1)

Theorem 2.1. *Let $\{x_n\}_{n=-5}^{\infty}$ be a solution of Eq.(1). Then for $n = 0, 1, ...$*

$$\begin{aligned} x_{6n-5} &= f_{i=0}^{n-1}\left(\frac{1-2ifc}{1-(2i+1)fc}\right), \qquad x_{6n-4} = e_{i=0}^{n-1}\left(\frac{1-2ieb}{1-(2i+1)eb}\right), \\ x_{6n-3} &= d_{i=0}^{n-1}\left(\frac{1-2ida}{1-(2i+1)da}\right), \qquad x_{6n-2} = c_{i=0}^{n-1}\left(\frac{1-(2i+1)fc}{1-(2i+2)fc}\right), \\ x_{6n-1} &= b_{i=0}^{n-1}\left(\frac{1-(2i+1)eb}{1-(2i+2)eb}\right), \qquad x_{6n} = a_{i=0}^{n-1}\left(\frac{1-(2i+1)da}{1-(2i+2)da}\right), \end{aligned} \tag{2.1}$$

where $x_{-5} = f,\ x_{-4} = e,\ x_{-3} = d,\ x_{-2} = c,\ x_{-1} = b,\ x_{-0} = a.$

Proof: For $n = 0$ the result holds. Now suppose that $n > 0$ and that our assumption holds for $n - 1$. That is;

$$\begin{aligned} x_{6n-11} &= f_{i=0}^{n-2}\left(\frac{1-2ifc}{1-(2i+1)fc}\right), \qquad x_{6n-10} = e_{i=0}^{n-2}\left(\frac{1-2ieb}{1-(2i+1)eb}\right), \\ x_{6n-9} &= d_{i=0}^{n-2}\left(\frac{1-2ida}{1-(2i+1)da}\right), \qquad x_{6n-8} = c_{i=0}^{n-2}\left(\frac{1-(2i+1)fc}{1-(2i+2)fc}\right), \\ x_{6n-7} &= b_{i=0}^{n-2}\left(\frac{1-(2i+1)eb}{1-(2i+2)eb}\right), \qquad x_{6n-6} = a_{i=0}^{n-2}\left(\frac{1-(2i+1)da}{1-(2i+2)da}\right). \end{aligned}$$

Now, it follows from Eq.(1) that

$$\begin{aligned} x_{6n-5} &= \frac{x_{6n-11}}{1-x_{6n-8}x_{6n-11}} = \frac{f_{i=0}^{n-2}\left(\dfrac{1-2ifc}{1-(2i+1)\,fc}\right)}{1-c_{i=0}^{n-2}\left(\dfrac{1-(2i+1)fc}{1-(2i+2)\,fc}\right)f_{i=0}^{n-2}\left(\dfrac{1-2ifc}{1-(2i+1)\,fc}\right)} \\ &= \frac{f_{i=0}^{n-2}\left(\dfrac{1-2ifc}{1-(2i+1)\,fc}\right)}{1-fc\left(\dfrac{1}{1-(2n-2)\,fc}\right)} = \frac{f_{i=0}^{n-2}\left(\dfrac{1-2ifc}{1-(2i+1)\,fc}\right)(1-(2n-2)\,fc)}{(1-(2n-2)\,fc)-fc} \\ &= f_{i=0}^{n-2}\left(\frac{1-2ifc}{1-(2i+1)\,fc}\right)\left(\frac{1-2\,(n-1)\,fc}{1-(2n-1)\,fc}\right). \end{aligned}$$

Hence, we have

$$x_{6n-5} = f_{i=0}^{n-1}\left(\frac{1-2ifc}{1-(2i+1)\,fc}\right).$$

Also, we see from Eq.(1) that

$$x_{6n-3} = \frac{x_{6n-9}}{1-x_{6n-6}x_{6n-9}} = \frac{d_{i=0}^{n-2}\left(\dfrac{1-2ida}{1-(2i+1)\,da}\right)}{1-a_{i=0}^{n-2}\left(\dfrac{1-(2i+1)da}{1-(2i+2)\,da}\right)d_{i=0}^{n-2}\left(\dfrac{1-2ida}{1-(2i+1)\,da}\right)}$$

$$= \frac{d_{i=0}^{n-2}\left(\dfrac{1-2ida}{1-(2i+1)\,da}\right)}{1-\left(\dfrac{da}{1-(2n-2)\,da}\right)}\left(\frac{1-(2n-2)\,da}{1-(2n-2)\,da}\right) = d_{i=0}^{n-2}\left(\frac{1-2ida}{1-(2i+1)\,da}\right)\left(\frac{1-2(n-1)da}{1-(2n-1)\,da}\right).$$

Hence, we have

$$x_{6n-2} = d_{i=0}^{n-1}\left(\frac{1-2ida}{1-(2i+1)\,da}\right).$$

Similarly, one can easily obtain the other relations. Thus, the proof is completed.

Theorem 2.2. *Eq.(1) has a unique equilibrium point which is the number zero.*

Proof: For the equilibrium points of Eq.(1), we can write

$$\overline{x} = \frac{\overline{x}}{1-\overline{x}^2}.$$

Then we have

$$\overline{x} - \overline{x}^3 = \overline{x},$$

or,

$$-\overline{x}^3 = 0.$$

Thus the equilibrium point of Eq.(1) is $\overline{x} = 0$.

Numerical examples

For confirming the results of this section, we consider numerical examples which represent different types of solutions to Eq. (1).

Example 1. We consider $x_{-5} = 8,\ x_{-4} = 3,\ x_{-3} = 5,\ x_{-2} = 11,\ x_{-1} = 1,\ x_0 = 2$ See Fig. 1.

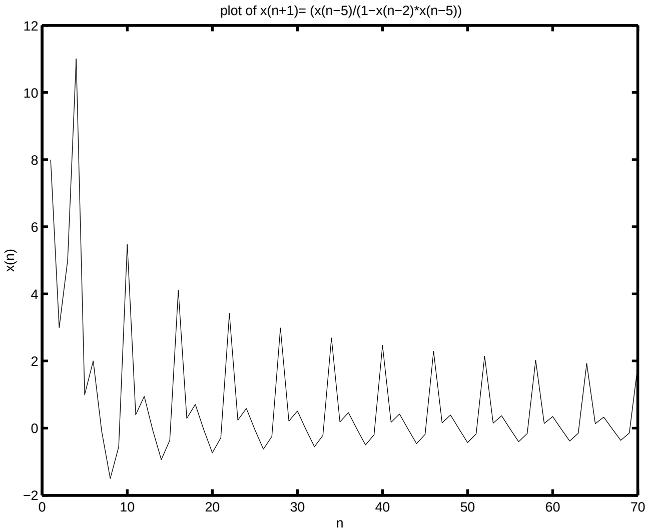


Fig.1

Example 2. See Fig. 2, since $x_{-5} = -2,\ x_{-4} = 3,\ x_{-3} = 5,\ x_{-2} = -3,\ x_{-1} = 11,\ x_0 = -9.$

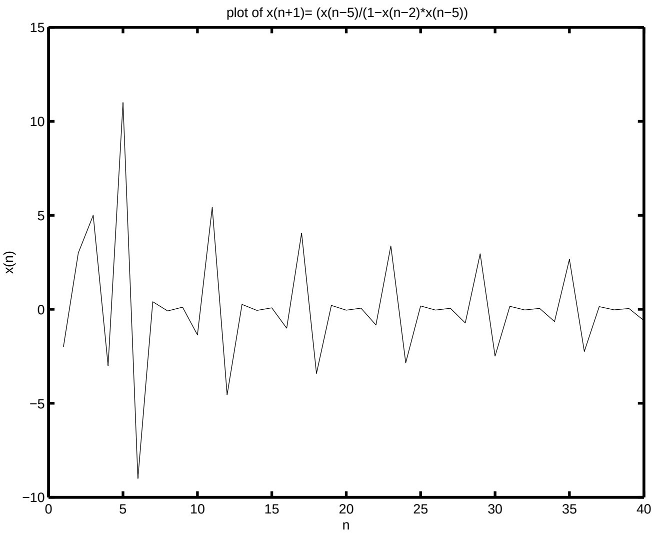


Fig.1

2.2 Equation (2)

Here we obtain a form of the solutions of Eq.(2)

Theorem 2.3. *Let* $\{x_n\}_{n=-5}^{\infty}$ *be a solution of Eq.(2). Then every solution of Eq.(2) is unbounded and for* $n = 0, 1, ...$

$$\begin{aligned} x_{6n-5} &= \frac{(-1)^n f}{(1+cf)^n}, \qquad x_{6n-4} = \frac{e(-1)^n}{(1+be)^n}, \\ x_{6n-3} &= \frac{(-1)^n d}{(1+ad)^n}, \qquad x_{6n-2} = c(-1)^n (1+cf)^n, \\ x_{6n-1} &= (-1)^n b(1+be)^n, \qquad x_{6n} = a(-1)^n (1+ad)^n, \end{aligned} \tag{2.2}$$

where $x_{-5}=f,\ x_{-4}=e,\ x_{-3}=d,\ x_{-2}=c,\ x_{-1}=b,\ x_{-0}=a,\ x_{-5}x_{-2}\neq -1,\ x_{-4}x_{-1}\neq -1,\ x_{-3}x_{0}\neq -1.$

Proof: For $n=0$ the result holds. Now suppose that $n>0$ and that our assumption holds for $n-1$. That is;

$$\begin{aligned} x_{6n-11} &= \frac{(-1)^{n-1}f}{(1+cf)^{n-1}}, \qquad x_{6n-10}=\frac{e(-1)^{n-1}}{(1+be)^{n-1}},\\ x_{6n-9} &= \frac{(-1)^{n-1}d}{(1+ad)^{n-1}}, \qquad x_{6n-8}=c(-1)^{n-1}(1+cf)^{n-1},\\ x_{6n-7} &= (-1)^{n-1}b(1+be)^{n-1}, \qquad x_{6n-6}=a(-1)^{n-1}(1+ad)^{n-1}. \end{aligned}$$

Now, it follows from Eq.(2) that

$$\begin{aligned} x_{6n-5} &= \frac{x_{6n-11}}{-1-x_{6n-8}x_{6n-11}}=\frac{\frac{(-1)^{n-1}f}{(1+cf)^{n-1}}}{-1-c(-1)^{n-1}(1+cf)^{n-1}\frac{(-1)^{n-1}f}{(1+cf)^{n-1}}}\\ &= \frac{\frac{(-1)^{n-1}f}{(1+cf)^{n-1}}}{-1-cf}. \end{aligned}$$

Hence, we have

$$x_{6n-5}=\frac{(-1)^{n}f}{(1+cf)^{n}}.$$

Similarly

$$\begin{aligned} x_{6n-3} &= \frac{x_{6n-9}}{-1-x_{6n-6}x_{6n-9}}=\frac{\frac{(-1)^{n-1}d}{(1+ad)^{n-1}}}{-1+a(-1)^{n-1}(1+ad)^{n-1}\frac{(-1)^{n-1}d}{(1+ad)^{n-1}}}\\ &= \frac{\frac{(-1)^{n-1}d}{(1+ad)^{n-1}}}{-1-ad}. \end{aligned}$$

Hence, we have

$$x_{6n-2}=\frac{(-1)^{n}d}{(1+ad)^{n}}.$$

Similarly, one can easily obtain the other relations. Thus, the proof is completed.

Theorem 2.4. *Eq.(2) has a unique equilibrium point which is the number zero.*

Proof: For the equilibrium points of Eq.(1), we can write

$$\overline{x}=\frac{\overline{x}}{-1-\overline{x}^{2}}.$$

Then we have

$$-\overline{x}-\overline{x}^{3}=\overline{x},$$

or,

$$\overline{x}(\overline{x}^2+2)=0.$$

Thus the equilibrium point of Eq.(2) is $\overline{x}=0$.

Theorem 2.5. *Eq.(2) has a periodic solutions of period six iff $cf=be=ad=-2$ and will be take the form $\{f,e,d,c,b,a,f,e,d,c,b,a,...\}$.*

Proof: First suppose that there exists a prime period six solution

$$f,e,d,c,b,a,f,e,d,c,b,a,...$$

of Eq.(2), we see from Eq.(4) that

$$\begin{aligned} f &= \frac{(-1)^n f}{(1+cf)^n}, \qquad e=\frac{e\,(-1)^n}{(1+be)^n}, \\ d &= \frac{(-1)^n d}{(1+ad)^n}, \qquad c=c\,(-1)^n\,(1+cf)^n\,, \\ b &= (-1)^n\, b\,(1+be)^n\,, \qquad a=a\,(-1)^n\,(1+ad)^n\,, \end{aligned}$$

or,

$$(1+cf)^n=(-1)^n\,,\ (1+be)^n=(-1)^n\,,\ (1+ad)^n=(-1)^n\,.$$

Then

$$cf=-2,\quad be=-2,\quad ad=-2.$$

Second suppose $cf=-2,\ be=-2,\ ad=-2$. Then we see from Eq.(4) that

$$\begin{aligned} x_{6n-5} &= f, \qquad x_{6n-4}=e, \\ x_{6n-3} &= d, \qquad x_{6n-2}=c, \\ x_{6n-1} &= b, \qquad x_{6n}=a. \end{aligned}$$

Thus we have a period six solution and the proof is complete.

Numerical examples

Example 3. Consider $x_{-5}=8,\ x_{-4}=7,\ x_{-3}=12,\ x_{-2}=4,\ x_{-1}=3,\ x_0=5$ See Fig. 3.

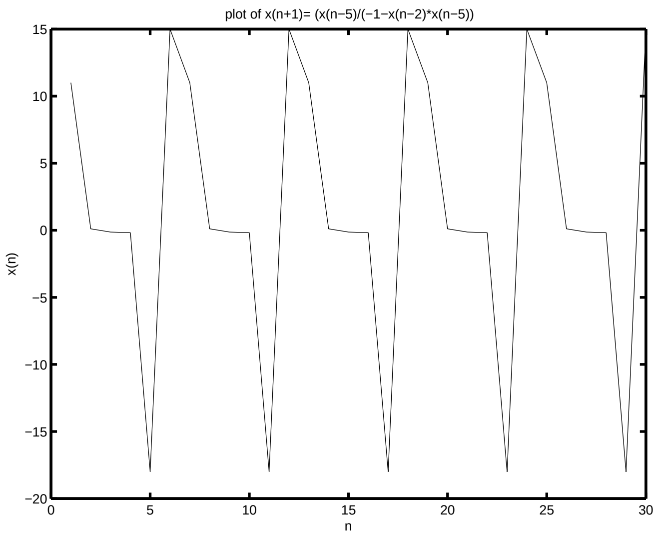
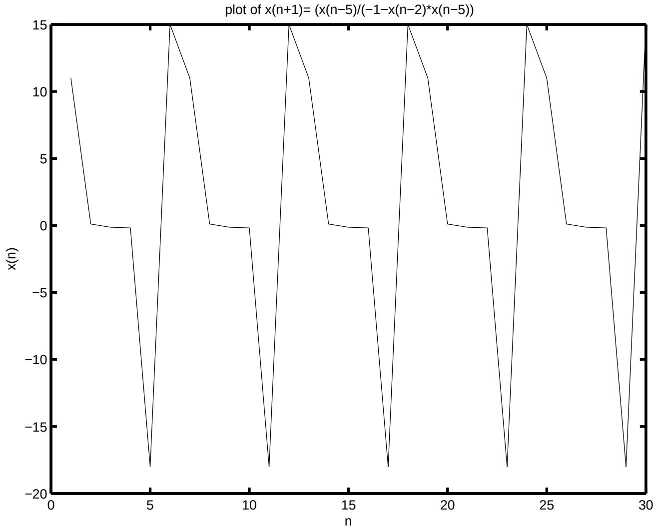


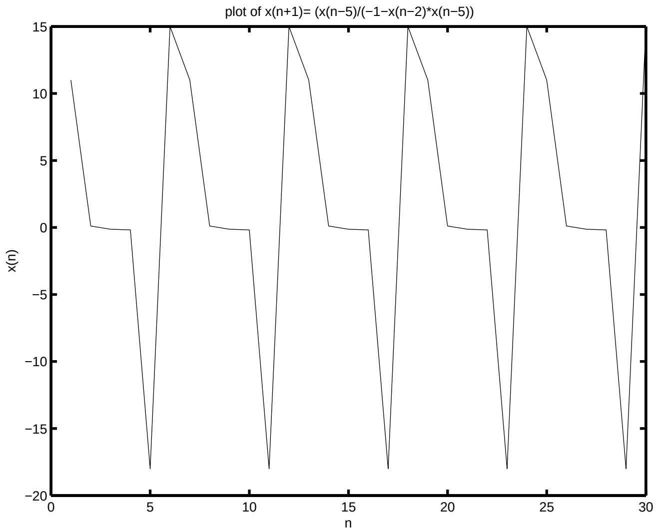
*Fig.*3

Example 4. See Fig. 4, since $x_{-5} = 0.2,\ x_{-4} = 0.7,\ x_{-3} = 1.2,\ x_{-2} = 0.4,\ x_{-1} = 0.3,\ x_0 = 0.2$.



$Fig.4$

Example 5. In Fig. 5, we assume $x_{-5} = 11,\ x_{-4} = 1/9,\ x_{-3} = -2/15,\ x_{-2} = -2/11,\ x_{-1} = -18,\ x_0 = 15$.



$Fig.5$

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