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Rough Statistical Cluster Points In 2-Normed Spaces

Mukaddes Arslan[1] **and Erdinç Dündar**[1,*]

[1] *Department of Mathematics, Afyon Kocatepe University, 03200, Afyonkarahisar, Turkey*
e-mail : mukad.deu@gmail.com (Mukaddes Arslan); edundar@aku.edu.tr (Erdinç Dündar)

Abstract In this study, we introduce the concepts of rough statistical cluster point and rough statistical limit point of a sequence in 2-normed space and investigate some properties of these concepts. Also, we obtain an ordinary statistical convergence criteria associated with rough statistical cluster point of a sequence in 2-normed space.

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1. Introduction

Throughout the paper, $\mathbb{N}$ denotes the set of all positive integers and $\mathbb{R}$ the set of all real numbers. The concept of convergence of a sequence of real numbers has been extended to statistical convergence independently by Fast [1] and Schoenberg [2].

The concept of 2-normed spaces was initially introduced by Gähler [3, 4] in the 1960's. Since then, this concept has been studied by many authors. Gürdal and Pehli̇van [5] studied statistical convergence, statistical Cauchy sequence and investigated some properties of statistical convergence in 2-normed spaces. Gürdal and Açık [6] investigated $\mathcal{I}$-Cauchy and $\mathcal{I}^*$-Cauchy sequences in 2-normed spaces. Sarabadan and Talebi [7] studied statistical convergence and ideal convergence of sequences of functions in 2-normed spaces. Arslan and Dündar [8, 9] investigated the concepts of $\mathcal{I}$-convergence, $\mathcal{I}^*$-convergence, $\mathcal{I}$-Cauchy and $\mathcal{I}^*$-Cauchy sequences of functions in 2-normed spaces. Futhermore, a lot of development have been made in this area (see [10–18]).

The idea of rough convergence was first introduced by Phu [19] in finite-dimensional normed spaces. In [19], he showed that the set $\mathtt{LIM}^r x_i$ is bounded, closed, and convex; and he introduced the notion of rough Cauchy sequence. He also investigated the relations between rough convergence and other convergence types and the dependence of $\mathtt{LIM}^r x_i$ on the roughness degree r. In another paper [20] related to this subject, he defined the rough continuity of linear operators and showed that every linear operator $f : X \to Y$ is r-continuous at every point $x \in X$ under the assumption $dimY < \infty$ and $r > 0$ where X and

*Corresponding author.

Y are normed spaces. In [21], he extended the results given in [19] to infinite-dimensional normed spaces. Aytar [22] studied of rough statistical convergence and defined the set of rough statistical limit points of a sequence and obtained two statistical convergence criteria associated with this set and prove that this set is closed and convex. Also, Aytar [23] studied that the r-limit set of the sequence is equal to the intersection of these sets and that r-core of the sequence is equal to the union of these sets. Recently, Dündar and Çakan [24–26] introduced the notion of rough $\mathcal{I}$-convergence and the set of rough $\mathcal{I}$-limit points of a sequence and studied the notions of rough convergence, $\mathcal{I}_2$-convergence and the sets of rough limit points and rough $\mathcal{I}_2$-limit points of a double sequence. Arslan and Dündar [27, 28] introduced some concepts of rough convergence in 2-normed spaces. Also, Arslan and Dündar [29] studied rough statistical convergence in 2-normed spaces.

In this paper, we study the concepts of rough statistical cluster point and rough statistical limit point of a sequence in 2-normed space and investigate some properties of these concepts. Also, we obtain an ordinary statistical convergence criteria associated with rough statistical cluster point of a sequence in 2-normed space. We note that our results and proof techniques presented in this paper are analogues of those in Aytar's [30] paper. Namely, the actual origin of most of these results and proof techniques is them papers. The following our theorems and results are the extension of theorems and results in [30].

Now, we recall the concept of 2-normed space, ideal convergence and some fundamental definitions and notations (See [5–9, 11, 13, 15–17, 19–23, 27–43]).

Let r be a nonnegative real number and $\mathbb{R}^n$ denotes the real n-dimensional space with the norm $\|.\|$. Consider a sequence $x = (x_n) \subset \mathbb{R}^n$.

The sequence $x = (x_n)$ is said to be r-convergent to L, denoted by $x_n \xrightarrow{r} L$ provided that

$$\forall \varepsilon > 0,\ \exists n_\varepsilon \in \mathbb{N} :\ n \geq n_\varepsilon \Rightarrow \|x_n - L\| < r + \varepsilon.$$

The set $\mathrm{LIM}^r x := \{L \in \mathbb{R}^n : x_n \xrightarrow{r} L\}$ is called the r-limit set of the sequence $x = (x_n)$. A sequence $x = (x_n)$ is said to be r-convergent if $\mathrm{LIM}^r x \neq \emptyset$. In this case, r is called the convergence degree of the sequence $x = (x_n)$. For $r = 0$, we get the ordinary convergence.

Let K be a subset of the set of positive integers $\mathbb{N}$, and let us denote the set $\{k \in K : k \leq n\}$ by K_n. Then the natural density of K is given by

$$\delta(K) := \lim_{n\to\infty} \frac{|K_n|}{n},$$

where $|K_n|$ denotes the number of elements in K_n. Clearly, a finite subset has natural density zero and we have $\delta(K^c) = 1 - \delta(K)$, where $K^c := \mathbb{N} \setminus K$ is the complement of K. If $K_1 \subseteq K_2$, then $\delta(K_1) \leq \delta(K_2)$.

A sequence $x = (x_n)$ is said to be r-statistically convergent to L, denoted by $x_n \xrightarrow{r-st} L$, provided that the set $\{n \in \mathbb{N} : \|x_n - L\| \geq r + \varepsilon\}$ has natural density zero for $\varepsilon > 0$; or equivalently, if the condition $st - \limsup \|x_n - L\| \leq r$ is satisfied. In addition, we can write $x_n \xrightarrow{r-st} L$ if and only if, the inequality $\|x_n - L\| < r + \varepsilon$ holds for every $\varepsilon > 0$ and almost all n.

Here r is called the statistical convergence degree. If we take $r = 0$, then we obtain the ordinary statistical convergence.

In general, the rough statistical limit of a sequence $x = (x_n)$ may not be unique for roughness degree $r > 0$. So we have to consider the so-called r-statistical limit set of the sequence x, which is defined by $st - \mathrm{LIM}^r x := \{L \in X : x_n \overset{r-st}{\longrightarrow} L\}$.

The sequence x is said to be r-statistically convergent provided that $st - \mathrm{LIM}^r x \neq \emptyset$.

Let $r \geq 0$. The vector $\lambda \in X$ is called the r-statistical cluster point of the sequence $x = (x_n)$ provided that $\delta(\{n \in \mathbb{N} : \|x_n - \lambda\| < r + \varepsilon\}) \neq 0$, for every $\varepsilon > 0$. We denote the set of all r-statistically cluster points the sequence x by Γ_x^r.

Let $r \geq 0$. The vector $\nu \in X$ is called the r-statistical limit point of the sequence $x = (x_n)$, provided that there is a nonthin subsequence (x_{n_k}) of (x_n) such that for every $\varepsilon > 0$ there exists a number $k_0 = k_{0(\varepsilon)} \in \mathbb{N}$ with $\|x_{n_k} - \nu\| < r + \varepsilon$ for all $k \geq k_0$. We denote the set of all r-statistical limit points the sequence x by Λ_x^r.

Let X be a real vector space of dimension d, where $2 \leq d < \infty$. A 2-norm on X is a function $\|\cdot,\cdot\| : X \times X \to \mathbb{R}$ which satisfies the following statements:

(i) $\|x, y\| = 0$ if and only if x and y are linearly dependent.
(ii) $\|x, y\| = \|y, x\|$.
(iii) $\|\alpha x, y\| = |\alpha| \|x, y\|$, $\alpha \in \mathbb{R}$.
(iv) $\|x, y + z\| \leq \|x, y\| + \|x, z\|$.

As an example of a 2-normed space we may take $X = \mathbb{R}^2$ being equipped with the 2-norm $\|x, y\| :=$ the area of the parallelogram based on the vectors x and y which may be given explicitly by the formula

$$\|x, y\| = |x_1 y_2 - x_2 y_1|; \quad x = (x_1, x_2), y = (y_1, y_2) \in \mathbb{R}^2.$$

In this study, we suppose X to be a 2-normed space having dimension d; where $2 \leq d < \infty$. The pair $(X, \|\cdot,\cdot\|)$ is then called a 2-normed space.

A sequence $x = (x_n)$ in 2-normed space $(X, \|\cdot,\cdot\|)$ is said to be convergent to L in X if $\lim_{n\to\infty} \|x_n - L, z\| = 0$, for every $z \in X$. In such a case, we write $\lim_{n\to\infty} x_n = L$ and call L the limit of (x_n).

$c \in X$ is called a statistical cluster point of a sequence $x = (x_n)$ provided that the natural density of the set $\{n \in \mathbb{N} : \|x_n - c, z\| < \varepsilon\}$ is different from zero for every $\varepsilon > 0$ and each nonzero $z \in X$. We deneto the set of all statistical cluster points of the sequence x by Γ_x^2.

Let (x_n) be a sequence in $(X, \|.,.\|)$ 2-normed linear space and r be a non-negative real number. $x = (x_n)$ is said to be rough convergent (r-convergent) to L denoted by $x_n \overset{\|.,.\|}{\longrightarrow}_r L$ if

$$\forall \varepsilon > 0, \exists n_\varepsilon \in \mathbb{N} : n \geq n_\varepsilon \Rightarrow \|x_n - L, z\| < r + \varepsilon \tag{1.1}$$

or equivalently, if $\limsup \|x_n - L, z\| \leq r$, for every $z \in X$.

If (1.1) holds L is an r-limit point of $x = (x_n)$, which is usually no more unique (for $r > 0$). So, we have to consider the so-called r-limit set (or shortly r-limit) of (x_n) defined by $\mathrm{LIM}_2^r x := \{L \in X : x_n \overset{\|.,.\|}{\longrightarrow}_r L\}$.

The sequence $x = (x_n)$ is said to be rough convergent if $\mathrm{LIM}_2^r x \neq \emptyset$. In this case, r is called a convergence degree of (x_n). For $r = 0$ we have the classical convergence in 2-normed space again. But our proper interest is case $r > 0$. There are several reasons for this interest. For instance, since an orginally convergent sequence (y_n) (with $y_n \to L$) in 2-normed space often cannot be determined (i.e., measured or calculated) exactly, one has to do with an approximated sequence (x_n) satisfying $\|x_n - y_n, z\| \leq r$ for all n and for every $z \in X$, where $r > 0$ is an upper bound of approximation error. Then, (x_n) is no more convergent in the classical sense, but for every $z \in X$,

$$\|x_n - L, z\| \leq \|x_n - y_n, z\| + \|y_n - L, z\| \leq r + \|y_n - L, z\|$$

implies that is r-convergent in the sense of (1.1).

Let $(X, \|.,.\|)$ be a 2-normed space. A sequence $x = (x_n)$ in X said to be rough statistically convergent (r_2st-convergent) to L, denoted by $x_n \xrightarrow{\|.,.\|}_{r_2st} L$, provided that the set $\{n \in \mathbb{N} : \|x_n - L, z\| \geq r + \varepsilon\}$ has natural density zero, for every $\varepsilon > 0$ and each nonzero $z \in X$; or equivalently, if the condition $st - \limsup \|x_n - L, z\| \leq r$ is satisfied. In addition, we can write $x_n \xrightarrow{\|.,.\|}_{r_2st} L$, if and only if, the inequality $\|x_n - L, z\| < r + \varepsilon$ holds almost all n.

In this convergence, r is called the statistical convergence degree. For $r = 0$, rough statistically convergent coincides with ordinary statistical convergence.

In general, the rough statistical limit of a sequence $x = (x_n)$ may not be unique for the roughness degree $r > 0$. So, we have to consider the so-called r-statistically limit set of the sequence x in X, which is defined by

$$st - \text{LIM}_2^r x := \{L \in X : x_n \xrightarrow{\|.,.\|}_{r_2st} L\}. \tag{1.2}$$

The sequence x is said to be r-statistically convergent provided that $st - \text{LIM}_2^r x \neq \emptyset$.

Lemma 1.1 ([27], Theorem 2.2). *Let $(X, \|.,.\|)$ be a 2-normed space and consider a sequence $x = (x_n) \in X$. The sequence (x_n) is bounded if and only if there exist an $r \geq 0$ such that $\text{LIM}_2^r x \neq \emptyset$. For all $r > 0$, a bounded sequence (x_n) always contains a subsequence x_{n_k} with $\text{LIM}_2^{(x_{n_k}),r} x_{n_k} \neq \emptyset$.*

Lemma 1.2 ([27], Theorem 2.3). *Let $(X, \|.,.\|)$ be a 2-normed space and consider a sequence $x = (x_n) \in X$. For all $r \geq 0$, the r-limit set $\text{LIM}_2^r x$ of an arbitrary sequence (x_n) is closed.*

Lemma 1.3 ([27], Theorem 2.4). *Let $(X, \|.,.\|)$ be a 2-normed space and consider a sequence $x = (x_n) \in X$. If $y_0 \in \text{LIM}_2^{r_0} x$ and $y_1 \in \text{LIM}_2^{r_1} x$, then*

$$y_\alpha := (1 - \alpha) y_0 + \alpha y_1 \in \text{LIM}_2^{(1-\alpha)r_0 + \alpha r_1} x, \quad \textit{for} \quad \alpha \in [0, 1].$$

2. Main Results

In this section, we introduce the concept of rough statistical cluster pointand rough-statistical limit pointof a sequence in 2-normed spaces.

Definition 2.1. Let $r \geq 0$. The vector $\lambda \in X$ is called the rough statistical cluster point of the sequence $x = (x_n)$ if for every $\varepsilon > 0$ and $z \in X$

$$\delta(\{n \in \mathbb{N} : \|x_n - \lambda, z\| < r + \varepsilon\}) \neq 0.$$

We denote the set of all rough statistical cluster points the sequence x in 2-normed space X by r-Γ_x^2.

Here, if we take $r = 0$, then we obtain the notion of ordinary statistical cluster point. It is clear that

$$r_1 - \Gamma_x^2 \subseteq r_2 - \Gamma_x^2,$$

for $r_1 \leq r_2$. In [30] Aytar proved that the set Γ_x^r is closed. We will show that the set r-Γ_x^2 is closed, for each $r > 0$.

Theorem 2.2. *Let $x = (x_n)$ be a sequence in 2-normed space X. Then, for every $r \geq 0$, the set r-Γ_x^2 is closed.*

Proof. Let r-$\Gamma_x^2 \neq \emptyset$ and consider a sequence $y = (y_n) \subseteq r$-Γ_x^2 such that $\lim_{n\to\infty} y_n = L$. Let us show that

$$\delta(\{n \in \mathbb{N} : \|x_n - L, z\| < r + \varepsilon\}) \neq 0$$

for every $\varepsilon > 0$ and $z \in X$. Fix $\varepsilon > 0$. Since $\lim_{n\to\infty} y_n = L$, there exists an $n_0 = n_0(\varepsilon) \in \mathbb{N}$ such that

$$\|y_n - L, z\| < \frac{\varepsilon}{2},$$

for all $n > n_0$ and every $z \in X$. Fix m_0 such that $m_0 > n_0$. Then, we have

$$\|y_{m_0} - L, z\| < \frac{\varepsilon}{2},$$

for every $z \in X$. Let m be any point of the set

$$\left\{n \in \mathbb{N} : \|x_n - y_{m_0}, z\| < r + \frac{\varepsilon}{2}\right\}.$$

Since $\|x_m - y_{m_0}, z\| < r + \frac{\varepsilon}{2}$, we have

$$\begin{aligned}
\|x_m - L, z\| &\leq \|x_m - y_{m_0}, z\| + \|y_{m_0} - L, z\| \\
&< r + \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\
&= r + \varepsilon
\end{aligned}$$

and so,

$$m \in \{n \in \mathbb{N} : \|x_n - L, z\| < r + \varepsilon\},$$

for every $z \in X$. Hence, we have

$$\left\{n \in \mathbb{N} : \|x_n - y_{m_0}, z\| < r + \frac{\varepsilon}{2}\right\} \subseteq \{n \in \mathbb{N} : \|x_n - L, z\| < r + \varepsilon\}. \tag{2.1}$$

Since

$$\delta\left(\left\{n \in \mathbb{N} : \|x_n - y_{m_0}, z\| < r + \frac{\varepsilon}{2}\right\}\right) \neq 0$$

by (2.1), we get

$$\delta(\{n \in \mathbb{N} : \|x_n - L, z\| < r + \varepsilon\}) \neq 0,$$

for every $z \in X$. Therefore, we have $L \in r$-Γ_x^2. ∎

If we let $\lambda \in r$-Γ_x^2, then for every $z \in X$,

$$\delta(\{n : \|x_n - \lambda, z\| < r + \varepsilon\}) \neq 0.$$

By the statistical analogue of Bolzano-Weierstrass Theorem (see [43], Theorem 2), the subsequence $(x_n)_{n\in A}$ has a statistical cluster point, where

$$A = \{n : \|x_n - \lambda, z\| \leq r\},$$

for every $z \in X$. If we denote this statistical cluster point by γ, then we have $\|\lambda - \gamma, z\| \leq r$. Therefore, we have that if $\lambda \in r$-Γ_x^2, then there exists a vector $\gamma \in \Gamma_x^2$ such that $\|\lambda - \gamma, z\| \leq r$.

Theorem 2.3. *Let $r > 0$. For a sequence $x = (x_n)$ in 2-normed space X, we have $L \in r$-Γ_x^2 if and only if there exists a sequence $y = (y_n)$ such that $L \in \Gamma_y^2$ and $\|x_n - y_n, z\| \leq r$, for every $z \in X$ and almost all n.*

Proof. Necessity: Fix r and ε and suppose that $L \in r\text{-}\Gamma_x^2$. Thus, we have $\delta(A) \neq 0$, where

$$A := \{n \in \mathbb{N} : \|x_n - L, z\| < r + \varepsilon\},$$

for every $z \in X$. Define

$$y_n := \begin{cases} L, & \|x_n - L, z\| \leq r \text{ and } n \in A \\ x_n + r\frac{L-x_n}{\|x_n-L,z\|}, & \|x_n - L, z\| > r \text{ and } n \in A \\ t_n, & n \notin A \end{cases} \tag{2.2}$$

where the sequence $t = (t_n)$ is arbitrary. It is clear that

$$\|y_n - L, z\| = \begin{cases} 0, & if \|x_n - L, z\| \leq r \\ \|x_n - L, z\| - r, & otherwise \end{cases} \tag{2.3}$$

and $\|x_n - y_n, z\| \leq r$, for every $n \in A$ and $z \in X$. Now let us show that the inclusion

$$A \subseteq \{n \in \mathbb{N} : \|y_n - L, z\| < \varepsilon\} \tag{2.4}$$

holds, for every $z \in X$. If $n_0 \in A$, then we have

$$\|x_{n_0} - L, z\| < r + \varepsilon,$$

for every $z \in X$. Hence the following two cases are possible:

(i) If $\|x_{n_0} - L, z\| \leq r$, then from (2.3), we have

$$\|y_{n_0} - L, z\| = 0,$$

that is,

$$n_0 \in \{n \in \mathbb{N} : \|y_n - L, z\| < \varepsilon\},$$

for every $z \in X$.

(ii) If $\|x_{n_0} - L, z\| > r$, then from (2.3), we have

$$\|y_{n_0} - L, z\| = \|x_{n_0} - L, z\| - r < r + \varepsilon - r = \varepsilon,$$

that is,

$$n_0 \in \{n \in \mathbb{N} : \|y_n - L, z\| < \varepsilon\},$$

for every $z \in X$.

Since $\delta(A) \neq 0$, by the inclusion (2.4), we have

$$\delta\{n \in \mathbb{N} : \|y_n - L, z\| < \varepsilon\} \neq 0,$$

for every $z \in X$.

Sufficiency: Assume that $L \in \Gamma_y^2$ and fix $\varepsilon > 0$. Then, we have

$$\delta\{n \in \mathbb{N} : \|y_n - L, z\| < \varepsilon\} \neq 0,$$

for every $z \in X$. Now, we let

$$m \in \{n \in \mathbb{N} : \|y_n - L, z\| < \varepsilon\}$$

and so, we can write

$$\|x_m - L, z\| \leq \|x_m - y_m, z\| + \|y_m - L, z\| < r + \varepsilon,$$

for every $z \in X$. Therefore, we have

$$m \in \{n \in \mathbb{N} : \|x_n - L, z\| < r + \varepsilon\}$$

and so, for every $z \in X$

$$\{n \in \mathbb{N} : \|y_n - L, z\| < \varepsilon\} \subseteq \{n \in \mathbb{N} : \|x_n - L, z\| < r + \varepsilon\}$$

holds. From this inclusion, we have

$$\delta(\{n \in \mathbb{N} : \|x_n - L, z\| < r + \varepsilon\}) \neq 0.$$

■

The following theorem presents a simple way to find the set r-Γ_x^2

Theorem 2.4.

$$r - \Gamma_x^2 = \bigcup_{c \in \Gamma_x^2} \overline{B}_r(c), \tag{2.5}$$

where $\overline{B}_r(c) := \{y \in X : \|y - c, z\| \leq r\}$, *for every* $z \in X$.

Proof. Let $\lambda \in \bigcup_{c \in \Gamma_x^2} \overline{B}_r(c)$. Then, there exists a vector $c \in \Gamma_x^2$ such that $\lambda \in \overline{B}_r(c)$, that is, $\|c - \lambda, z\| \leq r$, for every $z \in X$. Fix $\varepsilon > 0$. Since $c \in \Gamma_x^2$, there exists a set

$$A(\varepsilon) := \{n \in \mathbb{N} : \|x_n - c, z\| < \varepsilon\}$$

with $\delta(A(\varepsilon)) \neq 0$. Hence, we have

$$\|x_n - \lambda, z\| \leq \|x_n - c, z\| + \|c - \lambda, z\| < \varepsilon + r,$$

and so,

$$\delta(\{n \in \mathbb{N} : \|x_n - \lambda, z\| < \varepsilon + r\}) \neq 0,$$

for every $n \in A(\varepsilon)$ and every $z \in X$. Therefore, $\lambda \in r$-Γ_x^2 and so,

$$\bigcup_{c \in \Gamma_x^2} \overline{B}_r(c) \subset r - \Gamma_x^2.$$

For the converse inclusion, take $\lambda \in r$-Γ_x^2. Then, we have

$$\delta(\{n \in \mathbb{N} : \|x_n - \lambda, z\| < \varepsilon + r\}) \neq 0, \tag{2.6}$$

for every $\varepsilon > 0$ and every $z \in X$. We must show that $\lambda \in \bigcup_{c \in \Gamma_x^2} \overline{B}_r(c)$. Suppose that this is not satisfied. Then, it is clear that $\lambda \notin \overline{B}_r(c)$, that is, $\|\lambda - c, z\| > r$, for every $c \in \Gamma_x^2$ and every $z \in X$. Since the set Γ_x^2 is closed, there exists a vector $\widetilde{c} \in \Gamma_x^2$ such that

$$\|\lambda - \widetilde{c}, z\| = \min\{\|\lambda - c, z\| : c \in \Gamma_x^2\}.$$

We can write $k := \|\lambda - \widetilde{c}, z\| > r$, because $\|\lambda - c, z\| > r$, for all $c \in \Gamma_x^2$ and every $z \in X$. Define $\widetilde{\varepsilon} := \frac{k-r}{3}$. Then, we get

$$X \backslash B_{\widetilde{\varepsilon}}(\Gamma_x^2) \supseteq \{y \in X : \|\lambda - y, z\| < \widetilde{\varepsilon} + r\}, \tag{2.7}$$

for every $z \in X$, where

$$B_{\widetilde{\varepsilon}}(\Gamma_x^2) = \{y \in X : \min\{\|y - c, z\| : c \in \Gamma_x^2\} < \widetilde{\varepsilon}\}.$$

By definition of Γ_x^2 we can say that the set

$$\{n : x_n \notin B_{\widetilde{\varepsilon}}(\Gamma_x^2)\}$$

has density zero. Then, by the inclusion (2.7), we have

$$\{n : x_n \notin B_{\widetilde{\varepsilon}}(\Gamma_x^2)\} \supseteq \{n : \|x_n - \lambda, z\| < \widetilde{\varepsilon} + r, \} \tag{2.8}$$

for every $z \in X$. Thus, from the inclusion (2.8), for every $z \in X$ we have that the set

$$\{n : \|x_n - \lambda, z\| < \widetilde{\varepsilon} + r\}$$

has natural density zero, which contradicts to (2.6) and so,

$$r-\Gamma_x^2 \subset \bigcup_{c\in\Gamma_x^2} \overline{B}_r(c).$$

Hence, we have

$$r-\Gamma_x^2 = \bigcup_{c\in\Gamma_x^2} \overline{B}_r(c).$$

∎

Now, we present an ordinary statistical convergence criterion associated with the set r-Γ_x^2.

Theorem 2.5. *The sequence $x=(x_n)$ in X is statistically convergent if and only if*

$$r-\Gamma_x^2 = st-\mathrm{LIM}_2^r x.$$

Proof. Necessity. Assume that the sequence $x=(x_n)$ statistically convergent to L. Then, we have $\Gamma_x^2=\{L\}$. By Theorem 2.4, we can write r-$\Gamma_x^2=\overline{B}_r(L)$. Therefore, from [[29], Theorem 2.7], we get

$$r-\Gamma_x^2 = \overline{B}_r(L) = st-\mathrm{LIM}_2^r x.$$

Sufficiency. By Theorem 2.4 and [[29], Theorem 2.9(ii)], we have

$$\bigcup_{c\in\Gamma_x^2} \overline{B}_r(c) = \bigcap_{c\in\Gamma_x^2} \overline{B}_r(c). \tag{2.9}$$

The equality (2.9) is valid if and only if, either the set Γ_x^2 is empty or it is a singleton. Since

$$st-\mathrm{LIM}_2^r x = \bigcap_{c\in\Gamma_x^2} \overline{B}_r(c) = \overline{B}_r(L)$$

(see [[29], Theorem 2.7]), we have $st-\lim_{n\to\infty} x_n = L$. ∎

We note that in Theorem 2.5, the sequence $x=(x_n)$ need not be statistically convergent in order that the inclusion

$$r-\Gamma_x^2 \subseteq st-\mathrm{LIM}_2^r x$$

holds, but this sequence must be statistically convergent in order that the converse inclusion holds.

Definition 2.6. Ler $r\geq 0$. The vector γ in 2-normed space X is called the rough statistical limit point of the sequence $x=(x_n)$ in X, provided that there is a nonthin subsequence (x_{n_k}) of (x_n) such that for every $\varepsilon>0$ there exists a number $k_0=k_0(\varepsilon)\in\mathbb{N}$ with

$$\|x_{n_k}-\gamma, z\| < r+\varepsilon,$$

for every $z\in X$ and all $k\geq k_0$. We denote the set of all rough statistical limit points the sequence $x=(x_n)$ by r-Λ_x^2.

Now we present a result which characterizes the set r-Λ_x^2. The proof is immediate by definitions.

Proposition 2.7. *We have* $\gamma \in r\text{-}\Lambda_x^2$ *if and only if there exists a nonthin subsequence* (x_{n_k}) *of* (x_n) *such that*

$$\limsup_{k\to\infty} \|x_{n_k} - \gamma, z\| \leq r,$$

for every $z \in X$.

Theorem 2.8. *Let* $x = (x_n)$ *be a sequence in 2-normed space* X. *Then we have*

$$r - \Lambda_x^2 \subseteq r - \Gamma_x^2.$$

Proof. The proof of the theorem above is similar to that of [[33], Proposition 1]. ■

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