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Some Results on Generalized Frames

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Abstract The concept of a generalized frame or simply a g-frame in a Hilbert space H was introduced by Wenchang Sun in [4]. Given a g-frame $\{\Lambda_i\}_{i\in I}$ in a Hilbert space H and a bounded operator T on H, we show that the sequence $\{\Lambda_i T\}_{i\in I}$ is a g-frame for H if and only if T is invertiable on H. Moreover, we prove that add a g-frame to its canonical dual g-frame and the canonical Parseval g-frame are also g-frames. At the end, we provide sufficient conditions under which a subsequence of a g-frame in a Hilbert space H is itself a g-frame for H.

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1. Introduction

Frames were first introduced in 1952 by Duffin and Schaeffer [1, 2]. Today, frames play important roles in several applications in mathematics, science and engineering. Frames are an extension of bases in Hilbert spaces. In fact, a frame is a sequence $\{f_k\}_{k=1}^{\infty}$ in a Hilbert space H which allows every element $f \in H$ can be written as: $f = \sum_{k=1}^{\infty} c_k(f) f_k$, whereas the coefficients $c_k(f)$ are not unique [3]. The notion of a generalized frame or simply a g-frame for a Hilbert space H was first defined by Wenchang Sun in his work [4]. Afterwards, some generalizations of frames have been attracted much more attentions. In what follows: In section 1, it is reviewed preliminaries from operator theory and frame theory which needed in the sequel. In section 2, for a given g-frame $\{\Lambda_i\}_{i\in I}$ in a Hilbert space H and a bounded operator T on H, we show that the sequence $\{\Lambda_i T\}_{i\in I}$ is a g-frame for H if and only if T is invertiable on H. Also, we conclude that add a g-frame to its canonical dual g-frame and the canonical Parseval g-frame are g-frames. Moreover, some results on g-frames are derived. Throughout the paper, H and K denote two separable Hilbert spaces over $\mathbb{F}(\mathbb{R}$ *or* $\mathbb{C})$ and $\{H_i\}_{i\in I}$ denotes a sequence of closed subspaces of K, where I is a subset of positive integers. As usual, $B(H, K)$ consists of all bounded operators from H to K and $B(H)$ is abbreviated for $H = K$. The next definition can be seen in any context on operators (for example, see [5]).

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Definition 1.1. An operator $T \in B(H)$ is said to be an isometry if for all $h \in H$; $||Th|| = ||h||$. It is a partial isometry if it is an isometry on the orthogonal complement of its kernel.

Recall that an operator in $B(H)$ is called a co-isometry whenever its adjoint is an isometry. A unitary operator defines as a linear transformation which is a surjective isometry.

Lemma 1.2 ([6]). *Let $U \in B(K, H)$ be a bounded operator. Then the following holds:*

(i) $||U|| = ||U^*||$, *and* $||UU^*|| = ||U||^2$.

(ii) R_U *is closed in* H *if and only if* R_{U^*} *is closed in* K.

(iii) U *is surjective if and only if there exists a constant* $C > 0$ *such that*

$$||U^*h|| \geq C||h||, \ \ \forall h \in H, \text{ i.e., } U^* \text{ is bounded below.}$$

Definition 1.3. A frame for a Hilbert space H is a family of vectors $F = \{f_i\}_{i\in I}$ in H such that there exist constants A and $B > 0$ satisfying:

$$A||f||^2 \leq \sum_{i\in I} |< f,\ f_i >|^2 \leq B||f||^2, \quad \forall\, f \in H.$$

The constants A and B are called lower and upper frame bounds, respectively. Those are not unique. If only the right-hand inequality is assumed, then it is called a B-Bessel sequence. If $A = B$, it is said to be an A-tight frame for H.
For a Bessel sequence $F = \{f_i\}_{i\in I}$ in H the synthesis (pre-frame) operator is defined by

$$T : l^2(I) \longrightarrow H, \qquad T(\{c_i\}) = \sum_{i\in I} c_i f_i.$$

The analysis operator T^* for F is given by $T^*f = \{< f,\ f_i >\}_{i\in I}$, for all $f \in H$. The frame operator is $S = TT^*$ and it satisfies: $S_F f = \sum_{i\in I} < f,\ f_i > f_i, \ \forall f \in H$.

It is a known fact that if $F = \{f_i\}_{i\in I}$ is an A-tight frame for H with the frame operator S, then $S = AI$ and so we obtain

$$f = \frac{1}{A}\sum_{i\in I} < f,\ f_i > f_i, \ \ \forall f \in H.$$

Definition 1.4. A family $\{\Lambda_i\}_{i\in I} \subset B(H, H_i)$ is called a g-frame for H with respect to $\{H_i\}_{i\in I}$ if there exist positive constants A and B such that

$$A||f||^2 \leq \sum_{i\in I} ||\Lambda_i f||^2 \leq B||f||^2, \ \ \forall f \in H. \tag{1.1}$$

The constants A and B are called lower and upper g-frame bounds, respectively. If $A = B$, it is said to be an A-tight g-frame. A Parseval g-frame is a A-tight g-frame whenever $A = 1$. The family $\{\Lambda_i\}_{i\in I}$ is called a g-Bessel sequence with g-Bessel bound B if only the right-hand inequality (1.1) is satisfied. We define the following set for the family $\{\Lambda_i\}_{\in I} \subset B(H, H_i)$ as follows:

$$(\sum_{i\in I} \oplus H_i)_{l^2} = \{\{g_i\}_{i\in I} : g_i \in H_i,\ \ \sum_{i\in I} ||g_i||^2 < \infty\},$$

with the inner product given by $\langle \{f_i\}, \{g_i\} \rangle = \sum_{i\in I} < f_i, g_i >$. It is well-known that $(\sum_{i\in I} \oplus H_i)_{l^2}$ is a Hilbert space with respect to the pointwise operations. It is shown in [7] that if $\{\Lambda_i\}_{i\in I} \subset B(H, H_i)$ is a g-Bessel sequence for H, then the linear operator $T : (\sum_{i\in I} \oplus H_i)_{l^2} \longrightarrow H$ defined by

$$T(\{g_i\}) = \sum_{i\in I} \Lambda_i^* g_i,\ \forall g_i \in H_i,$$

is well-defined and bounded. Also, its adjoint operator is $T^*(f) = \{\Lambda_i f\}_{i\in I},\ \forall f \in H$. The operators T and T^* are called the synthesis and the analysis operators of $\{\Lambda_i\}_{i\in I}$, respectively.

Lemma 1.5 ([4]). *The family $\{\Lambda_i \in B(H, H_i) : i \in I\}$ is a g-frame for H if and only if T is a well-defined bounded and surjective operator.*

Also, the next proposition can be seen in [4].

Proposition 1.6. *Let $\{\Lambda_i\}_{i\in I} \subset B(H, H_i)$ be a g-Bessel sequence for H. Then, the operator S defined by*

$$S : H \longrightarrow H,\ \ Sf = \sum_{i\in I} \Lambda_i^* \Lambda_i f,\ \forall\ f \in H,$$

is a positive and bounded operator.

Clearly, we obtain $\langle Sf,\ f\rangle = \sum_{i\in I} ||\Lambda_i f||^2,\ \forall f \in H$. From this, we can conclude that a g-Bessel sequence $\{\Lambda_i\}_{i\in I}$ is a g-frame for H if and only if S is invertible. This proposition implies that every $f \in H$ can be written as:

$$f = SS^{-1}f = \sum_{i\in I} \Lambda_i^* \Lambda_i S^{-1} f,$$

and

$$f = S^{-1}Sf = \sum_{i\in I} S^{-1}\Lambda_i^* \Lambda_i f.$$

The operator S is called the g-frame operator of $\{\Lambda_i\}_{i\in I}$. It is easy to see that if $\{\Lambda_i\}_{i\in I}$ is a g-Bessel sequence, then S is a well-defined bounded operator. Furthermore $S = TT^*$.

Definition 1.7. Let $\Lambda = \{\Lambda_i\}_{i\in I}$ be a g-frame for H with respect to $\{H_i\}_{i\in I}$. A g-Bessel sequence $\Gamma = \{\Gamma_i\}_{i\in I}$ is called a dual g-frame of Λ if $f = \sum_{i\in I} {\Lambda_i}^* \Gamma_i f, \quad \forall\ f \in H$.

A simple calculation shows that if $\Lambda = \{\Lambda_i\}_{i\in I} \subset B(H, H_i)$ is a g-frame for H with the g-frame operator S, then the sequence $\{\Lambda_i S^{-1}\}_{\in I} \subset B(H, H_i)$ is a dual g-frame of Λ with g-frame operator S^{-1}. The family $\{\Lambda_i S^{-1}\}_{i\in I}$ is called canonical dual g-frame of Λ.

Definition 1.8. A family $\{\Lambda_i\}_{i\in I} \subset B(H, H_i)$ is called a g-orthonormal basis for H with respect to $\{H_i\}_{i\in I}$ if the following properties hold:

(i) $\langle \Lambda_i^* g_i,\ \Lambda_j^* g_j\rangle = \delta_{ij}\langle g_i,\ g_j\rangle,\ \forall i, j \in I,\ \forall g_i \in H_i, g_j \in H_j;$

(ii) $\sum_{i\in I} ||\Lambda_i f||^2 = ||f||^2,\ \ \forall f \in H.$

It is clear that every g-orthonormal basis is a Parseval g-frame.

Lemma 1.9 ([8]). *Let $\{\Theta_i\}_{i\in I} \subset B(H, H_i)$ be a g-orthonormal basis for H with respect to $\{H_i\}_{i\in I}$. Then, $\{\Lambda_i\}_{i\in I} \subset B(H, H_i)$ is a g-Bessel sequence for H with respect to $\{H_i\}_{i\in I}$ if and only if there exists a unique bounded operator $V : H \longrightarrow H$ such that $\Lambda_i = \Theta_i V^*$, for all $i \in I$.*

Remark 1.10 ([8])**.** Given a g-orthonormal basis $\{\Theta_i\}_{i\in I} \subset B(H, H_i)$, the unique operator V in Lemma 1.9, associated to g-Bessel sequence $\{\Lambda_i\}_{i\in I}$ with the g-frame operator S satisfies: $S = VV^*$. As a result, the synthesis operator T of a g-Bessel sequence $\{\Lambda_i\}_{i\in I}$ and the operator V have similar effects although they are defined in different ways.

2. Main Results

For an operator $T \in B(H)$, in general, add a g-frame $\{\Lambda_i\}_{i\in I}$ to $\{\Lambda_i T\}_{i\in I}$ can go wrong. For example, $\{\Lambda_i T\}_{i\in I} = \{-\Lambda_i\}_{i\in I}$. In this section, for a given g-frame $\{\Lambda_i\}_{i\in I}$ in a Hilbert space H and a bounded operator T on H, we first show that the sequence $\{\Lambda_i T\}_{i\in I}$ is a g-frame for H if and only if T is invertiable on H. Then, it is concluded that add a g-frame to its canonical dual g-frame and also canonical Parseval g-frame are g-frames. At the end, we provide sufficient conditions under what a subsequence of a g-frame for a Hilbert space H is itself a g-frame for H.

Proposition 2.1. *Let $\{\Lambda_i\}_{i\in I} \subset B(H, H_i)$ be a g-Bessel sequence for H with the synthesis operator T and $\{\Theta_i\}_{i\in I}$ be a g-orthonormal basis for H with respect to $\{H_i\}_{i\in I}$. Then, the sequence $\{\Lambda_i\}_{i\in I}$ is a Parseval g-frame for H if and only if T is a co-isometry.*

Proof. Given g-orthonormal basis $\{\Theta_i\}_{i\in I}$ for H, by Lemma 1.9, $\Lambda_i = \Theta_i T^*, \forall\, i \in I$. Thus, we have

$$\sum_{i\in I} ||\Lambda_i f||^2 = \sum_{i\in I} ||\Theta_i T^* f||^2 = ||T^* f||^2, \quad \forall f \in H. \tag{2.1}$$

This equality shows $\{\Lambda_i\}_{i\in I}$ is a Parseval g-frame for H if and only if $||T^* f||^2 = ||f||^2, \forall\, f \in H$. That is, if and only if T^* is an isometry. ∎

Proposition 2.2. *If $\{\Lambda_i\}_{i\in I} \subset B(H, H_i)$ is a g-frame for H with the g-frame operator S, then the sequence $\{\Lambda_i S^{-\frac{1}{2}}\}_{i\in I}$ is a Parseval g-frame for H with respect to $\{H_i\}_{i\in I}$.*

Proof. Because S is a positive and invertible bounded operator, S^{-1} has a unique positive square root $S^{-\frac{1}{2}}$ which is limit of a sequence of polynomials in S^{-1}. Thus it commutes with S^{-1} and S. By definition, for all $f \in H$, we have $Sf = \sum_{i\in I} \Lambda_i^* \Lambda_i f$.
Therefore, each $f \in H$ can be written as:

$$\begin{aligned} f &= S^{-\frac{1}{2}} S S^{-\frac{1}{2}} f = S^{-\frac{1}{2}} (\sum_{i\in I} \Lambda_i^* \Lambda_i S^{-\frac{1}{2}} f) \\ &= \sum_{i\in I} S^{-\frac{1}{2}} \Lambda_i^* \Lambda_i S^{-\frac{1}{2}} f. \end{aligned}$$

Now, by taking the inner product this with f, we obtain

$$\begin{aligned} ||f||^2 &= < f, f > = < \sum_{i\in I} S^{-\frac{1}{2}} \Lambda_i^* \Lambda_i S^{-\frac{1}{2}} f, f > \\ &= \sum_{i\in I} < S^{-\frac{1}{2}} \Lambda_i f, S^{-\frac{1}{2}} \Lambda_i f > = \sum_{i\in I} ||\Lambda_i S^{-\frac{1}{2}} f||^2. \end{aligned}$$

∎

Parseval g-frame $\{\Lambda_i S^{-\frac{1}{2}}\}_{i\in I}$ is called the canonical Parseval g-frame of $\{\Lambda_i\}_{i\in I}$.
The next result generalizes a known result on frames in [6] to the situation of g-frames.

Proposition 2.3. *If $\{\Lambda_i\}_{i\in I} \subset B(H, H_i)$ is a g-frame for a Hilbert space H with the frame operator S, then for any $f \in H$, we have*

(i) $||\sum_{i\in I} \Lambda_i^*\Lambda_i f||^2 \leq ||S|| \sum_{i\in I} ||\Lambda_i f||^2$.

(ii) $\sum_{i\in I} ||\Lambda_i f||^2 \leq ||S^{-1}|| || \sum_{i\in I} \Lambda_i^*\Lambda_i f||^2$.

In particular, both inequalities for Parseval g-frames are equal.

Proof. On the one hand, $||S|| = ||TT^*|| = ||T||^2$, and for every $f \in H$, we have

$$||T^*f||^2 = \sum_{i\in I} ||\Lambda_i f||^2 \ \ \textit{and} \ \ ||Sf||^2 = ||\sum_{i\in I} \Lambda_i^*\Lambda_i f||^2. \tag{2.2}$$

We now substitute (2.2) in the following inequality:

$$||Sf||^2 = ||TT^*f||^2 \leq ||T||^2||T^*f||^2 = ||S||||T^*f||^2.$$

This concludes (i). To do (ii), for any $f \in H$, we can write

$$||T^*f||^2 =< T^*f,\ T^*f >=< Sf,\ f >\leq ||Sf||||f||.$$

On the other hand, we obtain

$$\begin{aligned} ||Sf||||f|| &= ||Sf||||S^{-1}Sf|| \\ &\leq ||Sf||||S^{-1}||||Sf|| = ||S^{-1}||||Sf||^2. \end{aligned}$$

Therefore, it yields that

$$||T^*f||^2 \leq ||S^{-1}||||Sf||^2, \ \ \forall\ f \in H.$$

Again, similar to (i) if we substitute the relations (2.2) in this inequality, it gets the conclusion. ■

Let us prove a simple lemma on operators. It uses in the sequel. (for example, see [9]).

Lemma 2.4. *If $T \in B(H)$ is a positive, self-adjoint operator, then $I + T$ is an invertible bounded operator on H.*

Proof. Because T is self-adjoint, for every $h \in H$, we obtain

$$||(I + T)h||^2 = ||h||^2 + 2 < Th,\ h > +||Th||^2.$$

But all the terms in the middle of this relation are nonnegative, hence for all $h \in H$, we get $||(I + T)h|| \geq ||h||$. That is, $I + T$ is bounded below, so $(I + T)$ is injective and $(I+T)^* = I+T$ is surjective. Furthermore, $||(I+T)h|| \geq ||h||$ implies that for all $h \in H$,

$$||(I + T)^{-1}h|| \leq ||(I + T)(I + T)^{-1}h|| = ||h||.$$

Therefore, $I + T$ is invertible in $B(H)$. ■

It is well-known that if $\{f_i\}_{i\in I}$ is a frame for a Hilbert space H and $T \in B(H)$, then the sequence $\{Tf_i\}_{i\in I}$ is a frame for H if and only if T is invertible on H (see [10]). The next theorem generalizes this fact to the situation of g-frames.

Theorem 2.5. *Let $\{\Lambda_i\}_{i\in I} \subset B(H, H_i)$ be a g-frame for H with bounds A, B and $T \in B(H)$. Then, $\{\Lambda_i T\}_{i\in I}$ is a g-frame for H if and only if T is invertible on H. In this case, the g-frame operator is T^*ST, where S is the g-frame operator of $\{\Lambda_i\}_{i\in I}$.*

Proof. We first prove necessary condition. Suppose T is invertible on H. On the one hand, since $\{\Lambda_i\}_{i\in I}$ is a g-frame for H, hence we have

$$\sum_{i\in I} ||\Lambda_i Th||^2 \leq B||Th||^2 \leq B||T||^2||h||^2, \ \forall h \in H.$$

On the other hand, since T^* is bounded below, by Lemma 1.2, we get

$$\sum_{i\in I} ||\Lambda_i Th||^2 \geq A||T^*h||^2 \geq A||T^{-1}||^2||h||^2, \forall h \in H.$$

Therefore, these relations imply that $\{\Lambda_i T\}_{i\in I}$ is a g-frame for H with respect to $\{H_i\}_{i\in I}$. In contrast, if $\{\Lambda_i T\}_{i\in I}$ is a g-frame for H, then by Proposition 1.6, its g-frame operator, saying U is invertible and we obtain

$$\begin{aligned} Uh &= \sum_{i\in I}(\Lambda_i T)^*(\Lambda_i T)h = \sum_{i\in I} T^*\Lambda_i^*\Lambda_i Th \\ &= T^*(\sum_{i\in I}\Lambda_i^*\Lambda_i Th) = T^*STh, \ \forall h \in H. \end{aligned}$$

That is, $U = T^*ST$. Since U and S are invertible, it yields that T is invertible on H. ∎

Corollary 2.6. *Let $\{\Lambda_i\}_{i\in I} \subset B(H, H_i)$ be a g-frame for H and $T \in B(H)$. Then, the sequence $\{\Lambda_i + \Lambda_i T\}_{i\in I}$ is a g-frame for H if and only if $I + T$ is invertible on H.*

In this case, the g-frame operator is $(I+T)S(I+T)^*$, where S is the g-frame operator of $\{\Lambda_i\}_{i\in I}$. In particular, if T is positive, then $\{\Lambda_i+\Lambda_i T\}_{i\in I}$ is a g-frame with the g-frame operator $S + ST + TS + TST$.

Corollary 2.7. *If $\{\Lambda_i\}_{i\in I} \subset B(H, H_i)$ is a g-frame for H with the g-frame operator S, then for all $\alpha \in \mathbb{R}$, the sequence $\{\Lambda_i + \Lambda_i S^\alpha\}_{i\in I}$ is a g-frame for H with g-frame operator $(I + S^\alpha)^2 S$. Specially, $\{\Lambda_i + \Lambda_i S\}_{i\in I}$, $\{\Lambda_i + \Lambda_i S^{-1}\}_{i\in I}$, and $\{\Lambda_i + \Lambda_i S^{-\frac{1}{2}}\}_{i\in I}$ (i.e., the sum of a g-frame with its canonical dual g-frame and canonical Parseval g-frame)are g-frames.*

Corollary 2.8. *If $\{\Lambda_i\}_{i\in I} \subset B(H, H_i)$ is a g-frame for H and $P \in B(H)$ is an idempotent, then $\{\Lambda_i + \Lambda_i P\}_{i\in I}$ is a g-frame for H with respect to $\{H_i\}_{i\in I}$.*

Corollary 2.9. *Suppose that $\{\Lambda_i\}_{i\in I} \subset B(H, H_i)$ is a g-frame and $T \in B(H)$ is an isometry. Then, $\{\Lambda_i T\}_{i\in I}$ is a g-frame for H with respect to $\{H_i\}_{i\in I}$.*

Example 2.10. Let $\{e_i\}_{i\in I}$ be an orthonormal basis for a separable Hilbert space H. For any $f \in H$, we define operators $\Lambda_i : H \longrightarrow H$ as follows:

$$\Lambda_i(f) = \begin{cases} 0, & i = 2k \\ \langle f, e_i\rangle e_i, & i = 2k+1 \end{cases}$$

We have $\sum_{i\in I} ||\Lambda_i f||^2 = ||f||^2$, $\forall f \in H$. Hence, the sequence $\{\Lambda_i\}_{i\in I}$ is a g-frame for H, but the subsequence $\{\Lambda_{2k}\}_{k\in I}$ is not a g-frame for H.

The next result provide sufficient conditions under which a subsequence of a g-frame is itself a g-frame.

Theorem 2.11. *Assume that $\{\Lambda_i\}_{i\in I}$ is a g-frame for H with respect to $\{H_i\}_{i\in I}$ with g-frame bounds A, B. If $J \subset I$ and for some $0 < C_J < A$, we have*

$$\sum_{i\in J^c} ||\Lambda_i f||^2 \leq C_J||f||^2, \ \forall f \in H.$$

Then, the sequence $\{\Lambda_i\}_{i\in J}$ is a g-frame for H as well.

Proof. It is enough to satisfy the lower condition g-frame. In other words, for any $f \in H$, we get

$$\begin{aligned}\sum_{i\in J} ||\Lambda_i f||^2 &= \sum_{i\in I} ||\Lambda_i f||^2 - \sum_{i\in J^c} ||\Lambda_i f||^2 \\ &\geq A||f||^2 - C_J||f||^2 = (A - C_J)||f||^2.\end{aligned}$$

Therefore, $(A - C_J)$ is a lower g-frame bound and the proof is complete. ■

Corollary 2.12. *Suppose that $\{\Lambda_i\}_{i\in I}$ is a Parseval g-frame for H and $J \subset I$. Then, $\{\Lambda_i\}_{i\in J}$ is a g-frame for H if and only if $C_J < 1$.*

Proof. Since $\{\Lambda_i\}_{i\in I}$ is a Parseval g-frame, for any $f \in H$, we obtain

$$\begin{aligned}\sum_{i\in J} ||\Lambda_i f||^2 &= \sum_{i\in I} ||\Lambda_i f||^2 - \sum_{i\in J^c} ||\Lambda_i f||^2 \\ &\geq ||f||^2 - C_J||f||^2 = (1 - C_J)||f||^2.\end{aligned}$$

This shows that $\{\Lambda_i\}_{i\in J}$ is a g-frame for H if and only if $1 - C_J > 0$. ■

Recall that a maximal partial isometry, either itself or its adjoint is isometry.

Proposition 2.13. *If $T \in B(H)$ is a normal maximal partial isometry and $\{\Theta_i\}_{i=1}^{\infty}$ is a g-orthonormal basis for H, then $\{\Theta_i T\}_{i=1}^{\infty}$ is a Parseval g-frame for H.*

Proof. Because T is a normal operator, for all $h \in H$, we have $||Th|| = ||T^*h||$.
If T^* is isometry, then we get

$$||h||^2 = ||T^*h||^2 = ||Th||^2 = \sum_{i=1}^{\infty} ||\Theta_i Th||^2, \ \forall\, h \in H.$$

If T is an isometry, then for all $h \in H$, we have

$$||h||^2 = ||Th||^2 = ||T^*h||^2 = \sum_{i=1}^{\infty} ||\Theta_i Th||^2.$$

Therefore, in each case, it concludes that $\sum_{i=1}^{\infty} ||\Theta_i Th||^2 = ||h||^2, \quad \forall\, h \in H$.
That is, $\{\Theta_i T\}_{i=1}^{\infty}$ is a Parseval g-frame for H. ■

Corollary 2.14. *If $T \in B(H)$ is a unitary and $\{\Theta_i\}_{i=1}^{\infty}$ is a g-orthonormal basis for H, then $\{\Theta_i T\}_{i=1}^{\infty}$ is a Parseval g-frame for H.*

Proposition 2.15. *Let $T \in B(H)$ be an isometry and $\{\Lambda_i\}_{i\in I} \subset B(H, H_i)$ be a g-frame for H with lower and upper bounds A and B, respectively. Then $\{\Lambda_i T\}_{i\in I}$ is a g-frame for H with lower and upper bounds A and $B||T||^2$, respectively.*

Proof. For all $h \in H$, we have

$$A||h||^2 = A||Th||^2 \leq \sum_{i\in I} ||\Lambda_i Th||^2,$$

and

$$\sum_{i\in I} ||\Lambda_i Th||^2 \leq B||Th||^2 \leq B||T||^2||h||^2.$$

These relations show that $\{\Lambda_i T\}_{i\in I}$ is a g-frame for H with respect to $\{H_i\}_{i\in I}$. ■

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