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End Point of Multivalued Cyclic Admissible Mappings

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Abstract In this paper we introduce an admissibility condition for multivalued mappings. Also we introduce two types of multivalued almost contractions with the help of δ- distance. We show that these contractions, under the assumption of the admissibility condition defined here, have end point property. The results are obtained without any reference of continuity. There is also an illustrative example.

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1. Introduction and Mathematical Background

It is a widely recognized fact that the metric fixed point theory has its origination in the celebrated work of S. Banach [1] in which the contraction mapping principle was established. Afterwards a very large number of works have been published where fixed points of operators are shown to be existent in metric spaces under the assumptions of various types by contractive conditions on these operators. This line of research constitutes what is called the fixed point theory of contractive mappings. It has a vast literature and even today, after about a century of its initiation, remains an active branch of research. A comprehensive account of its development is given in [2, 3].

A close examination of the proofs of various types of contractive fixed point theorems reveal that for the purpose of the proof it is not a necessity to assume the contractive inequality for every pairs of points from the underlying metric space. Instead, the proofs are valid if the contraction inequality is suitably restricted to hold for appropriate pairs of points.

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One way of achieving this goal of suitably restricting the contraction condition is to introduce a partial order in the metric space and to restrict the contraction for pairs of points which are related by the partial ordering. This restriction, along with some other suitably chosen conditions could successfully reproduce several contractive fixed point theorems and thus fixed point theory in partially ordered metric spaces developed. The development has been rapid during last one decade. There are results of both singlevalued as well as multivalued mappings in metric spaces with a partial ordering. It may be mentioned that Nadler [4] in 1969 proved a multivalued extension of the Banach's contraction mapping principal [1] after which metric fixed point theory had substantially developed in the domain of setvalued analysis. This development was also extended through a good number of works to the context of partially ordered metric spaces. Some works from this domain of study are [5–9].

An alternative to this approach is the use of admissibility conditions which achieves the same goal of restricting the contraction to suitable pairs of points. Its difference with the previously mentioned approach is that while the partial order is defined on the space, the admissibility conditions are assumed on the respective mappings. The approach was initiated by Samet et al [10] and was developed through the works like [11–15].

In the present paper we adopt the later approach. We prove here two end point results of multivalued mappings in a complete metric space. Precisely, we introduce the concept of multivalued cyclic $(\alpha,\ \beta)$ - admissible mappings and $(\alpha-\beta)$ - almost contractions to establish end point results for multivalued mappings.

We review below some essential concepts for our discussions in this paper. Let $(X,\ d)$ be a metric space. We denote the class of nonempty subsets of X by $N(X)$ and the class of nonempty bounded subsets of X by $B(X)$. For $A,\ B\in B(X)$, the functions D and δ are defined as

$$D(A,\ B)=\inf\ \{d(a,\ b):a\in A,\ b\in B\}$$

and

$$\delta(A,\ B)=\sup\ \{d(a,\ b):a\in A,\ b\in B\}.$$

If $A=\{a\}$, then we write $D(A,\ B)=D(a,\ B)$ and $\delta(A,\ B)=\delta(a,\ B)$. Also in addition, if $B=\{b\}$, then $D(A,\ B)=d(a,\ b)$ and $\delta(A,\ B)=d(a,\ b)$. For all $A,\ B,\ C\in B(X)$, the definition of $\delta(A,\ B)$ yields that $\delta(A,\ B)=\delta(B,\ A),\ \ \delta(A,\ B)\le\delta(A,\ C)+\delta(C,\ B)$, $\delta(A,\ B)=0$ iff $A=B=\{a\},\ \ \delta(A,\ A)=\text{diam } A$ [16].

Definition 1.1 ([10])**.** Let X be a nonempty set, $T:X\to X$ and $\alpha:X\times X\to[0,\infty)$. The mapping T is *α-admissible* if for $x,y\in X$, $\alpha(x,y)\ge 1\Longrightarrow\alpha(Tx,Ty)\ge 1$.

In a separate vein the following definition was introduced in [11].

Definition 1.2 ([11])**.** Let X be a nonempty set, $T:X\to X$ and $\alpha,\ \beta:X\to[0,\ \infty)$. We say that T is a *cyclic $(\alpha-\beta)$-admissible mapping* if for $x\in X$,
(i) $\alpha(x)\ge 1\implies\beta(Tx)\ge 1$, (ii) $\beta(x)\ge 1\implies\alpha(Tx)\ge 1$.

In the following we define multivalued cyclic $(\alpha-\beta)$ admissible mappings.

Definition 1.3. Let X be a nonempty set, $T:X\to N(X)$ be a multivalued mapping and $\alpha,\ \beta:X\to[0,\ \infty)$. We say that T is a *cyclic $(\alpha,\ \beta)$- admissible mapping* if for $x\in X$,
(i) $\alpha(x)\ge 1\Longrightarrow\beta(u)\ge 1$ for all $u\in Tx$,
(ii)$\beta(x)\ge 1\Longrightarrow\alpha(u)\ge 1$ for all $u\in Tx$.

Definition 1.4. Let $(X,\ d)$ be a metric space and $\gamma : X \to [0,\ \infty)$. Then X is said to have γ- *regular property* if $\{x_n\}$ is a sequence in X with $\gamma(x_n) \geq 1$ for all n and $x_n \to x$ as $n \to \infty$, then $\gamma(x) \geq 1$.

Khan et al. [17] initiated the use of a control function in metric fixed point theory which they called Altering distance function. Fixed point studies using various types of control functions have been considered in a large number of works like in [18–22]. In this paper we use the following classes of functions.

Let Ψ denote the family of all functions $\psi : [0,\ \infty)^4 \to [0,\ \infty)$ such that (i_ψ) ψ is nondecreasing in each coordinate and continuous; (ii_ψ) $\psi(t,\ t,\ t,\ t) < t$ for $t > 0$. By Θ we denote the family of all functions $\varphi : [0,\ \infty)^3 \to [0,\ \infty)$ such that (i_φ) φ is nondecreasing in each coordinate and continuous; (ii_φ) $\varphi(t,\ t,\ t) < t$ for $t > 0$.

The following are the well known definitions of fixed point and end point of a multivalued operator.

Definition 1.5. Let X be a nonempty set and $T : X \to N(X)$ be a multivalued mapping. An element $x \in X$ is called a *fixed point of* T if $x \in Tx$ and an element $x \in X$ is called an *end point of* T if $\{x\} = Tx$.

The set of all fixed points and the set of all end points of T are respectively denoted as Fix (T) and End (T). Every end point of T is a fixed point of it but the converse is not true. So End $(T) \subseteq$ Fix (T).

2. Main Results

Definition 2.1. Let $(X,\ d)$ be a metric space, $T : X \to N(X)$ be a multivalued mapping and $\alpha,\ \beta : X \to [0,\ \infty)$. We say that T is an $(\alpha,\ \beta)$- *almost contraction of type I* if there exists $\psi \in \Psi$ such that for $x,\ y \in X$,

$\alpha(x)\ \beta(y) \geq 1 \Longrightarrow$

$$\begin{aligned}\delta(Tx,\ Ty) \ &\leq\ \psi\Big(d(x,\ y),\ D(x,\ Tx),\ D(y,\ Ty),\ \frac{D(x,\ Ty) + D(y,\ Tx)}{2}\Big)\\ &\quad +L\ \min\Big\{D(x,\ Tx),\ D(y,\ Ty),\ D(x,\ Ty),\ D(y,\ Tx)\Big\}.\end{aligned}$$

Definition 2.2. Let $(X,\ d)$ be a metric space, $T : X \to N(X)$ be a multivalued mapping and $\alpha,\ \beta : X \to [0,\ \infty)$. We say that T is an $(\alpha,\ \beta)$- *almost contraction of type II* if there exists $\varphi \in \Theta$ such that for $x,\ y \in X$,

$\alpha(x)\ \beta(y) \geq 1 \Longrightarrow$

$$\begin{aligned}\delta(Tx,Ty) \ &\leq\ \varphi\Big(d(x,y),\ \frac{D(y,Ty)\ [1 + D(x,Tx)]}{1 + d(x,y)},\ \frac{D(y,Tx)\ [1 + D(x,Ty)]}{1 + d(x,y)}\Big)\\ &\quad +L\ \min\Big\{D(x,Tx),\ D(y,Ty),\ D(x,Ty),\ D(y,Tx)\Big\}.\end{aligned}$$

Theorem 2.3. *Let $(X,\ d)$ be a complete metric space, $T : X \to B(X)$ be a multivalued mapping and $\alpha,\ \beta : X \to [0,\ \infty)$. Suppose that (i) there exists $x_0 \in X$ such that $\alpha(x_0) \geq 1$ and $\beta(x_0) \geq 1$; (ii) X is regular with respect to α and β; (iii) T is cyclic $(\alpha,\ \beta)$- admissible; (iv) T is an $(\alpha,\ \beta)$- almost contraction of type I. Then End (T) is nonempty.*

Proof. By the condition (i), there exists $x_0 \in X$ such that $\alpha(x_0) \geq 1$ and $\beta(x_0) \geq 1$. Let $x_1 \in Tx_0$. By the condition (iii), $\beta(x_1) \geq 1$. Let $x_2 \in Tx_1$. By the condition (iii),

$\alpha(x_2) \geq 1$. Let $x_3 \in Tx_2$. By the condition (iii), we have $\beta(x_3) \geq 1$. Continuing this process we obtain a sequence $\{x_n\}$ in X satisfying

$$x_{n+1} \in Tx_n \text{ with } \alpha(x_{2n}) \geq 1 \text{ and } \beta(x_{2n+1}) \geq 1. \tag{2.1}$$

Since $\beta(x_0) \geq 1$, using the condition (iii) for the sequence $\{x_n\}$, we also have

$$\alpha(x_{2n+1}) \geq 1 \text{ and } \beta(x_{2n}) \geq 1. \tag{2.2}$$

From (2.1) and (2.2), we have

$$x_{n+1} \in Tx_n \text{ with } \alpha(x_n) \geq 1 \text{ and } \beta(x_n) \geq 1 \text{ for all } n \geq 0. \tag{2.3}$$

As $\alpha(x_n)\ \beta(x_{n+1}) \geq 1$, applying the condition (iv), we have

$$\begin{aligned} d(x_{n+1},\ x_{n+2}) &\leq \delta(Tx_n,\ Tx_{n+1}) \\ &\leq \psi\Big(d(x_n,\ x_{n+1}),\ D(x_n,\ Tx_n),\ D(x_{n+1},\ Tx_{n+1}), \\ &\qquad \frac{D(x_n,\ Tx_{n+1}) + D(x_{n+1},\ Tx_n)}{2}\Big) \\ &\quad +L \min\Big\{D(x_n,\ Tx_n),\ D(x_{n+1},\ Tx_{n+1}), \\ &\qquad D(x_n,\ Tx_{n+1}),\ D(x_{n+1},\ Tx_n)\Big\} \end{aligned}$$

$$\begin{aligned} &\leq \psi\Big(d(x_n,\ x_{n+1}),\ d(x_n,\ x_{n+1}),\ d(x_{n+1},\ x_{n+2}), \\ &\qquad \frac{d(x_n,\ x_{n+2}) + d(x_{n+1},\ x_{n+1})}{2}\Big) \\ &\quad +L \min\Big\{d(x_n,\ x_{n+1}),\ d(x_{n+1},\ x_{n+2}), \\ &\qquad d(x_n,\ x_{n+2}),\ d(x_{n+1},\ x_{n+1})\Big\} \\ &\leq \psi\Big(d(x_n,\ x_{n+1}),\ d(x_n,\ x_{n+1}),\ d(x_{n+1},\ x_{n+2}), \\ &\qquad \frac{d(x_n,\ x_{n+2})}{2}\Big) \\ &\leq \psi\Big(d(x_n,\ x_{n+1}),\ d(x_n,\ x_{n+1}),\ d(x_{n+1},\ x_{n+2}), \\ &\qquad \frac{d(x_n,\ x_{n+1}) + d(x_{n+1},\ x_{n+2})}{2}\Big) \\ &\leq \psi\Big(d(x_n,\ x_{n+1}),\ d(x_n,\ x_{n+1}),\ d(x_{n+1},\ x_{n+2}), \\ &\qquad \max\ [d(x_n,\ x_{n+1}), d(x_{n+1},\ x_{n+2})]\Big). \end{aligned}$$

Therefore, we have

$$\begin{aligned} d(x_{n+1},\ x_{n+2}) &\leq \psi\Big(d(x_n,\ x_{n+1}),\ d(x_n,\ x_{n+1}),\ d(x_{n+1},\ x_{n+2}), \\ &\qquad \max\ [d(x_n,\ x_{n+1}), d(x_{n+1},\ x_{n+2})]\Big). \end{aligned} \tag{2.4}$$

Suppose that $d(x_{n+1},\ x_{n+2}) > d(x_n,\ x_{n+1})$. Then $d(x_{n+1},\ x_{n+2}) > 0$. From the above inequality and using a property of ψ, we have

$$\begin{aligned} d(x_{n+1}, x_{n+2}) &\leq \psi\Big(d(x_n, x_{n+1}),\ d(x_n, x_{n+1}),\ d(x_{n+1}, x_{n+2}),\ d(x_{n+1}, x_{n+2})\Big) \\ &\leq \psi\Big(d(x_{n+1}, x_{n+2}), d(x_{n+1}, x_{n+2}), d(x_{n+1}, x_{n+2}), d(x_{n+1}, x_{n+2})\Big) \\ &< d(x_{n+1}, x_{n+2}), \end{aligned}$$

which is a contradiction. Therefore, $d(x_{n+1},\ x_{n+2}) \leq d(x_n,\ x_{n+1})$ for all $n \geq 0$. So $\{d(x_n,\ x_{n+1}\}$ is a decreasing sequence of nonnegative real numbers. Hence there exists a real number $r \geq 0$ such that

$$d(x_n,\ x_{n+1}) \to r \ \text{ as } \ n \to \infty. \tag{2.5}$$

Taking $n \to \infty$ in (2.4) and using a property of ψ, we have

$$r \leq \psi(r,\ r,\ r,\ r).$$

Suppose that $r > 0$. Then using a property of a ψ we have from the above inequality that

$$r \leq \psi(r,\ r,\ r,\ r) < r,$$

which is a contradiction. Hence $r = 0$. So, we have

$$\lim_{n\to\infty} d(x_n,\ x_{n+1}) = 0. \tag{2.6}$$

Next we show that $\{x_n\}$ is a Cauchy sequence. If otherwise, there exists an $\epsilon > 0$ for which we can find two sequences of positive integers $\{m(k)\}$ and $\{n(k)\}$ such that for all positive integers k, $n(k) > m(k) > k$ and $d(x_{m(k)},\ x_{n(k)}) \geq \epsilon$. Assuming that $n(k)$ is the smallest such positive integer, we get

$$n(k) > m(k) > k, \ \ d(x_{m(k)},\ x_{n(k)}) \geq \epsilon \ \text{ and } \ d(x_{m(k)},\ x_{n(k)-1}) < \epsilon.$$

Now,

$$\epsilon \leq d(x_{m(k)},\ x_{n(k)}) \leq d(x_{m(k)},\ x_{n(k)-1}) + d(x_{n(k)-1},\ x_{n(k)}),$$

that is,

$$\epsilon \leq d(x_{m(k)},\ x_{n(k)}) < \epsilon + d(x_{n(k)-1},\ x_{n(k)}).$$

Taking limit as $k \to \infty$ in the above inequality and using (2.6), we have

$$\lim_{k\to\infty} d(x_{m(k)},\ x_{n(k)}) = \epsilon. \tag{2.7}$$

Again,

$$d(x_{m(k)}, x_{n(k)}) \leq d(x_{m(k)}, x_{m(k)+1}) + d(x_{m(k)+1}, x_{n(k)+1}) + d(x_{n(k)+1}, x_{n(k)})$$

and

$$d(x_{m(k)+1}, x_{n(k)+1}) \leq d(x_{m(k)+1}, x_{m(k)}) + d(x_{m(k)}, x_{n(k)}) + d(x_{n(k)}, x_{n(k)+1}).$$

Taking limit as $k \to \infty$ in the above inequalities and using (2.6) and (2.7), we have

$$\lim_{k\to\infty} d(x_{m(k)+1},\ x_{n(k)+1}) = \epsilon. \tag{2.8}$$

Again,

$$d(x_{m(k)},\ x_{n(k)}) \leq d(x_{m(k)},\ x_{n(k)+1}) + d(x_{n(k)+1},\ x_{n(k)})$$

and

$$d(x_{m(k)},\ x_{n(k)+1}) \leq d(x_{m(k)},\ x_{n(k)}) + d(x_{n(k)},\ x_{n(k)+1}).$$

Letting $k \to \infty$ in the above inequalities and using (2.6) and (2.7), we have

$$\lim_{k\to\infty} d(x_{m(k)},\ x_{n(k)+1}) = \epsilon. \tag{2.9}$$

Similarly, we have that

$$\lim_{k\to\infty} d(x_{n(k)},\ x_{m(k)+1}) = \epsilon. \tag{2.10}$$

As $\alpha(x_{n(k)})\beta(x_{m(k)}) \geq 1$, applying the condition (iv), we have

$$\begin{aligned}
d(x_{n(k)+1},\ x_{m(k)+1}) &\leq \delta(Tx_{n(k)},\ Tx_{m(k)}) \\
&\leq \psi\Big(d(x_{n(k)},\ x_{m(k)}),\ D(x_{n(k)},\ Tx_{n(k)}),\ D(x_{m(k)},\ Tx_{m(k)}), \\
&\qquad\qquad \frac{D(x_{n(k)},\ Tx_{m(k)}) + D(x_{m(k)},\ Tx_{n(k)})}{2}\Big) \\
&\quad + L \min\Big\{D(x_{n(k)},\ Tx_{n(k)}),\ D(x_{m(k)},\ Tx_{m(k)}), \\
&\qquad\qquad D(x_{n(k)},\ Tx_{m(k)}),\ D(x_{m(k)},\ Tx_{n(k)})\Big\} \\
&\leq \psi\Big(d(x_{n(k)},\ x_{m(k)}),\ d(x_{n(k)},\ x_{n(k)+1}),\ d(x_{m(k)},\ x_{m(k)+1}), \\
&\qquad\qquad \frac{d(x_{n(k)},\ x_{m(k)+1}) + d(x_{m(k)},\ x_{n(k)+1})}{2}\Big) \\
&\quad + L \min\Big\{d(x_{n(k)},\ x_{n(k)+1}),\ d(x_{m(k)},\ x_{m(k)+1}), \\
&\qquad\qquad d(x_{n(k)},\ x_{m(k)+1}),\ d(x_{m(k)},\ x_{n(k)+1})\Big\}.
\end{aligned}$$

Taking limit as $k \to \infty$ in the above inequality and using (2.6 - 2.10) and the properties of ψ, we have

$$\epsilon \leq \psi(\epsilon,\ \epsilon,\ \epsilon,\ \epsilon) < \epsilon,$$

which is a contradiction. Therefore, $\{x_n\}$ is a Cauchy sequence. As X is complete, there exists $y \in X$ such that

$$x_n \to y \quad \text{as} \quad n \to \infty. \tag{2.11}$$

Using (2.3), (2.11) and the condition (ii), we have $\beta(y) \geq 1$. Hence by (2.3), we get $\alpha(x_n)\ \beta(y) \geq 1$. Then applying the assumption (iv), we have

$$\begin{aligned}
\delta(x_{n+1},\ Ty) &\leq \delta(Tx_n,\ Ty) \\
&\leq \psi\Big(d(x_n, y),\ D(x_n, Tx_n),\ D(y, Ty),\ \frac{D(x_n, Ty) + D(y, Tx_n)}{2}\Big) \\
&\quad + L \min\Big\{D(x_n, Tx_n),\ D(y, Ty),\ D(x_n, Ty),\ D(y, Tx_n)\Big\} \\
&\leq \psi\Big(d(x_n, y),\ d(x_n, x_{n+1}),\ D(y, Ty),\ \frac{d(x_n, Ty) + d(y, x_{n+1})}{2}\Big) \\
&\quad + L \min\Big\{d(x_n, x_{n+1}),\ D(y, Ty),\ D(x_n, Ty),\ d(y, x_{n+1})\Big\}.
\end{aligned}$$

Taking limit as $n \to \infty$ in the above inequality and using (2.11) and the properties of ψ, we have

$$\begin{aligned}\delta(y,\ Ty) &\le \psi\Big(0,\ 0,\ D(y,\ Ty),\ \frac{D(y,\ Ty)}{2}\Big)\\ &\le \psi\Big(D(y,\ Ty),\ D(y,\ Ty),\ D(y,\ Ty),\ D(y,\ Ty)\Big)\\ &\le \psi\Big(\delta(y,\ Ty),\ \delta(y,\ Ty),\ \delta(y,\ Ty),\ \delta(y,\ Ty)\Big).\end{aligned}$$

Suppose that $\delta(y,\ Ty) > 0$. Then from the above inequality, we have

$$\delta(y,\ Ty) \le \psi\Big(\delta(y,\ Ty),\ \delta(y,\ Ty),\ \delta(y,\ Ty),\ \delta(y,\ Ty)\Big) < \delta(y,\ Ty),$$

which is a contradiction. Therefore, $\delta(y,\ Ty) = 0$, that is, $\{y\} = Ty$. So, y is an end point of T and hence End (T) is nonempty. ∎

In Theorem 2.3, considering $\psi(x_1, x_2, x_3, x_4) = k \max\{x_1, x_2, x_3, x_4\}$, where $0 < k < 1$, we have the following corollary.

Corollary 2.4. *Let $(X,\ d)$ be a complete metric space, $T : X \to B(X)$ be a multivalued mapping and $\alpha,\ \beta : X \to [0,\ \infty)$. Suppose that the assumptions (i), (ii) and (iii) of Theorem 2.3 are satisfied. Also, suppose that there exist $0 < k < 1$ and $L \ge 0$ such that for $x,\ y \in X$,*

$\alpha(x)\ \beta(y) \ge 1 \Longrightarrow$

$$\begin{aligned}\delta(Tx, Ty) \le\ & k\ max \Big\{d(x, y),\ D(x, Tx),\ D(y, Ty),\ \frac{D(x, Ty) + D(y, Tx)}{2}\Big\}\\ &+L\ min \Big\{D(x,\ Tx),\ D(y,\ Ty),\ D(x,\ Ty),\ D(y,\ Tx)\Big\}.\end{aligned}$$

Then End (T) is nonempty.

In Theorem 2.3, considering $\psi(x_1, x_2, x_3, x_4) = \frac{k}{2}\ (x_2 + x_3)$, where $0 < k < 1$, we have the following corollary.

Corollary 2.5. *Let $(X,\ d)$ be a complete metric space, $T : X \to B(X)$ be a multivalued mapping and $\alpha,\ \beta : X \to [0,\ \infty)$. Suppose that the assumptions (i), (ii) and (iii) of Theorem 2.3 are satisfied. Also, suppose that there exist $0 < k < 1$ and $L \ge 0$ such that for $x,\ y \in X$,*

$\alpha(x)\ \beta(y) \ge 1 \Longrightarrow$

$$\begin{aligned}\delta(Tx,\ Ty) \le\ & \frac{k}{2}\ \Big[D(x,\ Tx) + D(y,\ Ty)\Big]\\ &+L\ min \Big\{D(x,\ Tx),\ D(y,\ Ty),\ D(x,\ Ty),\ D(y,\ Tx)\Big\}.\end{aligned}$$

Then End (T) is nonempty.

In Theorem 2.3, considering $\psi(x_1, x_2, x_3, x_4) = k\ x_4$, where $0 < k < 1$, we have the following corollary.

Corollary 2.6. *Let $(X,\ d)$ be a complete metric space, $T : X \to B(X)$ be a multivalued mapping and $\alpha,\ \beta : X \to [0,\ \infty)$. Suppose that the assumptions (i), (ii) and (iii) of Theorem 2.3 are satisfied. Also, suppose that there exist $0 < k < 1$ and $L \geq 0$ such that for $x,\ y \in X$,*

$$\alpha(x)\ \beta(y) \geq 1 \Longrightarrow$$

$$\begin{aligned}\delta(Tx,\ Ty) \ &\leq\ \frac{k}{2}\Big[D(x,\ Ty) + D(y,\ Tx)\Big]\\ &\quad +L\ min\Big\{D(x,\ Tx),\ D(y,\ Ty),\ D(x,\ Ty),\ D(y,\ Tx)\Big\}.\end{aligned}$$

Then End (T) is nonempty.

In Theorem 2.3, considering $\psi(x_1, x_2, x_3, x_4) = k\ x_1$, where $0 < k < 1$, we have the following corollary.

Corollary 2.7. *Let $(X,\ d)$ be a complete metric space, $T : X \to B(X)$ be a multivalued mapping and $\alpha,\ \beta : X \to [0,\ \infty)$. Suppose that the assumptions (i), (ii) and (iii) of Theorem 2.3 are satisfied. Also, suppose that there exist $0 < k < 1$ and $L \geq 0$ such that for $x,\ y \in X$,*

$$\alpha(x)\ \beta(y) \geq 1 \Longrightarrow$$

$$\delta(Tx, Ty) \leq k\ d(x, y) + L\ min\Big\{D(x, Tx),\ D(y, Ty),\ D(x, Ty),\ D(y, Tx)\Big\}.$$

Then End (T) is nonempty.

Theorem 2.8. *Let $(X,\ d)$ be a complete metric space, $T : X \to B(X)$ be a multivalued mapping and $\alpha,\ \beta : X \to [0,\ \infty)$. Suppose that the assumptions (i), (ii) and (iii) of Theorem 2.3 are satisfied. Also, suppose that T is an $(\alpha,\ \beta)$ - almost contraction of type II. Then End (T) is nonempty.*

Proof. Arguing similarly as in the proof of Theorem 2.3, we construct a sequence $\{x_n\}$ which satisfies (2.3). As $\alpha(x_n)\ \beta(x_{n+1}) \geq 1$ and T is a $(\alpha,\ \beta)$- almost contraction of type II, we have

$$\begin{aligned}d(x_{n+1},\ x_{n+2}) &\leq \delta(Tx_n,\ Tx_{n+1})\\ &\leq \varphi\Big(d(x_n,\ x_{n+1}),\ \frac{D(x_{n+1},\ Tx_{n+1})\ [1 + D(x_n,\ Tx_n)]}{1 + d(x_n,\ x_{n+1})},\\ &\qquad \frac{D(x_{n+1},\ Tx_n)\ [1 + D(x_n,\ Tx_{n+1})]}{1 + d(x_n,\ x_{n+1})}\Big)\\ &\quad + L\min\Big\{d(x_n, Tx_n), D(x_{n+1}, Tx_{n+1}), D(x_n, Tx_{n+1}), D(x_{n+1}, Tx_n)\Big\}\\ &\leq \varphi\Big(d(x_n,\ x_{n+1}),\ \frac{d(x_{n+1},\ x_{n+2})\ [1 + d(x_n,\ x_{n+1})]}{1 + d(x_n,\ x_{n+1})},\\ &\qquad \frac{d(x_{n+1},\ x_{n+1})\ [1 + d(x_n,\ x_{n+2})]}{1 + d(x_n,\ x_{n+1})}\Big)\\ &\quad + L\min\Big\{d(x_n, x_{n+1}), (x_{n+1}, x_{n+2}),\ d(x_n, x_{n+2}),\ d(x_{n+1}, x_{n+1})\Big\}\\ &\leq \varphi\Big(d(x_n,\ x_{n+1}),\ d(x_{n+1},\ x_{n+2}),\ 0\Big).\end{aligned}$$

Therefore, we have

$$d(x_{n+1},\ x_{n+2}) \leq \varphi\Big(d(x_n,\ x_{n+1}),\ d(x_{n+1},\ x_{n+2}),\ 0\Big). \tag{2.12}$$

Suppose that $d(x_{n+1},\ x_{n+2}) > d(x_n,\ x_{n+1})$. Then $d(x_{n+1},\ x_{n+2}) > 0$. From the above inequality and using the properties of φ, we have

$$\begin{aligned} d(x_{n+1}, x_{n+2}) &\leq \varphi\Big(d(x_n,\ x_{n+1}),\ d(x_{n+1},\ x_{n+2}),\ 0\Big) \\ &\leq \varphi\Big(d(x_{n+1},\ x_{n+2}),\ d(x_{n+1},\ x_{n+2}),\ d(x_{n+1},\ x_{n+2})\Big) \\ &< d(x_{n+1}, x_{n+2}), \end{aligned}$$

which is a contradiction. Therefore, $d(x_{n+1},\ x_{n+2}) \leq d(x_n,\ x_{n+1})$ for all $n \geq 0$. So $\{d(x_n,\ x_{n+1}\}$ is a decreasing sequence nonnegative real numbers. Hence there exists a real number $l \geq 0$ such that $d(x_n,\ x_{n+1}) \to l$. Suppose that $l \neq 0$. Taking $n \to \infty$ in (2.12) and using the properties of φ, we have

$$l \leq \varphi(l,\ l,\ 0) \leq \varphi(l,\ l,\ l) < l$$

which is a contradiction. Hence $l = 0$. Therefore, the sequence $\{x_n\}$ satisfies (2.6), that is, $\lim_{n\to\infty} d(x_n,\ x_{n+1}) = 0$.

Next we show that $\{x_n\}$ is a Cauchy sequence. If $\{x_n\}$ is not a Cauchy sequence, then using an argument similar to that given in Theorem 2.3, we can find two sequences of positive integers $\{m(k)\}$ and $\{n(k)\}$ for which we have (2.7), (2.8), (2.9) and (2.10).

As $\alpha(x_{n(k)})\beta(x_{m(k)}) \geq 1$ and T is a $(\alpha,\ \beta)$- almost contraction of type II, we have $d(x_{n(k)+1}, x_{m(k)+1}) \leq \delta(Tx_{n(k)}, Tx_{m(k)})$

$$\begin{aligned} &\leq \varphi\Big(d(x_{n(k)}, x_{m(k)}),\ \frac{D(x_{m(k)}, Tx_{m(k)})\ [1 + D(x_{n(k)}, Tx_{n(k)})]}{1 + d(x_{n(k)}, x_{m(k)})}, \\ &\qquad \frac{D(x_{m(k)}, Tx_{n(k)})\ [1 + D(x_{n(k)}, Tx_{m(k)})]}{1 + d(x_{n(k)}, x_{m(k)})}\Big) \\ &\quad + L \min\Big\{D(x_{n(k)}, Tx_{n(k)}),\ D(x_{m(k)}, Tx_{m(k)}), \\ &\qquad D(x_{n(k)}, Tx_{m(k)}),\ D(x_{m(k)}, Tx_{n(k)})\Big\} \\ &\leq \varphi\Big(d(x_{n(k)}, x_{m(k)}),\ \frac{d(x_{m(k)}, x_{m(k)+1})\ [1 + d(x_{n(k)}, x_{n(k)+1})]}{1 + d(x_{n(k)}, x_{m(k)})}, \\ &\qquad \frac{d(x_{m(k)}, x_{n(k)+1})\ [1 + d(x_{n(k)}, x_{m(k)+1})]}{1 + d(x_{n(k)}, x_{m(k)})}\Big) \\ &\quad + L \min\Big\{d(x_{n(k)}, x_{n(k)+1}),\ d(x_{m(k)}, x_{m(k)+1}), \\ &\qquad d(x_{n(k)}, x_{m(k)+1}),\ d(x_{m(k)}, x_{n(k)+1})\Big\}. \end{aligned}$$

Taking limit as $k \to \infty$ in the above inequality and using (2.6 - 2.10) and the properties of φ, we have

$$\epsilon \leq \varphi(\epsilon,\ 0,\ \epsilon) \leq \varphi(\epsilon,\ \epsilon,\ \epsilon) < \epsilon,$$

which is a contradiction. Therefore, $\{x_n\}$ is a Cauchy sequence. As X is a complete, there exists $x \in X$ such that

$$x_n \to x \quad \text{as} \quad n \to \infty. \tag{2.13}$$

Then using (2.3) and the condition (ii), we have $\beta(x) \geq 1$. By (2.3), $\alpha(x_n)\ \beta(x) \geq 1$. As T is a $(\alpha,\ \beta)$- almost contraction of type II, we have

$$\begin{aligned}\delta(x_{n+1},\ Tx) \leq \delta(Tx_n,\ Tx) &\leq \varphi\Big(d(x_n,x),\ \frac{D(x,Tx)\ [1+D(x_n,Tx_n)]}{1+d(x_n,x)}, \\ &\qquad \frac{D(x,Tx_n)\ [1+D(x_n,Tx)]}{1+d(x_n,x)}\Big) \\ &\quad + L\min\Big\{D(x_n,Tx_n),\ D(x,Tx),\ D(x_n,Tx),\ D(x,Tx_n)\Big\} \\ &\leq \varphi\Big(d(x_n,\ x),\ \frac{D(x,\ Tx)\ [1+d(x_n,\ x_{n+1})]}{1+d(x_n,\ x)}, \\ &\qquad \frac{d(x,\ x_{n+1})\ [1+D(x_n,\ Tx)]}{1+d(x_n,\ x)}\Big) \\ &\quad + L\min\Big\{d(x_n,x_{n+1}),\ D(x,Tx),\ D(x_n,Tx),\ d(x,x_{n+1})\Big\}.\end{aligned}$$

Now taking limit as $n \to \infty$ in the above inequality and using (2.13) and the properties of φ, we have

$$\begin{aligned}\delta(x,\ Tx) &\leq \varphi\Big(0,\ D(x,\ Tx),\ 0\Big) \leq \varphi\Big(D(x,\ Tx),\ D(x,\ Tx),\ D(x,\ Tx)\Big) \\ &\leq \varphi\Big(\delta(x,\ Tx),\ \delta(x,\ Tx),\ \delta(x,\ Tx)\Big).\end{aligned}$$

Suppose that $\delta(x,\ Tx) > 0$. Then from the above inequality, we have

$$\delta(x,\ Tx) \leq \varphi\Big(\delta(x,\ Tx),\ \delta(x,\ Tx),\ \delta(x,\ Tx)\Big) < \delta(x,\ Tx),$$

which is a contradiction. Therefore, $\delta(x,\ Tx) = 0$, that is, $\{x\} = Tx$. So, x is an end point of T and hence End (T) is nonempty. ∎

In Theorem 2.8, considering $\varphi(x_1, x_2, x_3) = k\ \max\ \{x_1, x_2, x_3\}$ with $0 < k < 1$, we have the following corollary.

Corollary 2.9. *Let $(X,\ d)$ be a complete metric space, $T : X \to B(X)$ be a multivalued mapping and $\alpha,\ \beta : X \to [0,\ \infty)$. Suppose that the assumptions (i), (ii) and (iii) of Theorem 2.3 are satisfied. Also, suppose that there exist $0 < k < 1$ and $L \geq 0$ such that for $x,\ y \in X$,*

$\alpha(x)\ \beta(y) \geq 1 \Longrightarrow$

$$\begin{aligned}\delta(Tx,\ Ty) \leq\ & k\ max\Big\{d(x,\ y),\ \frac{D(y,\ Ty)\ [1+D(x,\ Tx)]}{1+d(x,\ y)}, \\ &\qquad \frac{D(y,\ Tx)\ [1+D(x,\ Ty)]}{1+d(x,\ y)}\Big\} \\ &+L\ min\Big\{D(x,\ Tx),\ D(y,\ Ty),\ D(x,\ Ty),\ D(y,\ Tx)\Big\}.\end{aligned}$$

Then End (T) is nonempty.

Example 2.10. Let $X = [0, \infty)$ be equipped with usual metric "d". Let $T : X \to B(X)$ be defined as follows:

$$Tx = \begin{cases} \left\{\frac{x}{8}\right\}, & \text{if} \quad 0 \leq x \leq 1 \\ \\ \left[x + \frac{1}{x} - \frac{1}{n},\ n\right], & \text{if} \quad n-1 < x \leq n, \quad \text{with} \quad n \geq 2. \end{cases}$$

Let $\alpha,\ \beta : X \to [0,\ \infty)$ be defined as

$$\alpha(x) = \begin{cases} e^x, & \text{if } \ 0 \le x \le 1 \\ \dfrac{1}{4}, & \text{if } \ x > 1 \end{cases}, \quad \beta(x) = \begin{cases} x+2, & \text{if } \ 0 \le x \le 1 \\ 0, & \text{if } \ x > 1. \end{cases}$$

Let $L \ge 0$ any real number.

(A)

Let $\psi : [0,\infty)^4 \to [0,\infty)$ be defined as $\psi(x_1,x_2,x_3,x_4) = \dfrac{x_1+x_2+x_3+x_4}{6}$. Then all the conditions of Theorem 2.3 are satisfied and End $(T) = \{0,\ 2,\ 3,\ 4,\ n, ...\}$.

(B)

Let $\varphi : [0,\ \infty)^3 \to [0,\ \infty)$ be defined as $\varphi(x_1,x_2,x_3) = \dfrac{x_1+x_2+x_3}{6}$. Then all the conditions of Theorem 2.8 are satisfied and End $(T) = \{0,\ 2,\ 3,\ 4,\ n, ...\}$.

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