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On k-Super Graceful Labeling of Graphs

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Abstract Let $G = (V(G), E(G))$ be a simple, finite and undirected graph of order p and size q. For $k \geq 1$, a bijection $f : V(G) \cup E(G) \to \{k, k+1, k+2, \ldots, k+p+q-1\}$ such that $f(uv) = |f(u) - f(v)|$ for every edge $uv \in E(G)$ is said to be a k-super graceful labeling of G. We say G is k-super graceful if it admits a k-super graceful labeling. In this paper, we study the k-super gracefulness of some standard graphs. Some general properties are obtained. Particularly, we found many sufficient conditions on k-super gracefulness for many families of (complete) bipartite and tripartite graphs. We show that some of the conditions are also necessary.

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1. Introduction

Let $G = (V(G), E(G))$ (or $G = (V, E)$ for short) be a simple, finite and undirected graph of order $|V| = p$ and size $|E| = q$. For integers a and b with $a \leq b$, let $[a, b]$ be the set of integers between a and b inclusively. All notation not defined in this paper can be found in [1]. An injective function $f : V \to [0, q]$ is called a graceful labeling of G if all the edge labels of G given by $f(uv) = |f(u) - f(v)|$ for every $uv \in E$ are distinct. In 1967, Rosa [2] published the conjecture of Kotzig that every nontrivial tree is graceful. Since then, there have been more than 2000 research papers on graph labelings with hundreds of graceful related results being published (see the dynamic survey by Gallian [3]).

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Definition 1.1. Given $k \geq 1$, a bijection $f : V \cup E \to [k, k+p+q-1]$ is called a *k-super graceful labeling* if $f(uv) = |f(u) - f(v)|$ for every edge uv in G. We say G is *k-super graceful* if it admits a k-super graceful labeling.

This is a generalization of super graceful labeling defined in [4, 5]. This was referred to as a k-sequential labeling in [6] that we are only aware of after the completion of this paper. For simplicity, 1-super graceful is also known as super graceful. In this paper, we study the k-super gracefulness of some standard graphs.

2. General Properties

By definition, we have

Theorem 2.1. *Let G be a (p, q)-graph with a k-super graceful labeling f. Suppose there exists vertex u_i with $f(u_i) = p + q - 1 + 2i$ for $1 \leq i \leq t \leq \lfloor k/2 \rfloor$. Join a new vertex v_i to u_i, then $G+\{v_1, \ldots, v_t\}$ is $(k-t)$-super graceful if we extend f to $f(v_i) = k+p+q-1+i$ and $f(u_iv_i) = k - i$. For $t = \lfloor k/2 \rfloor$, we can join at most $\lceil k/2 \rceil - 1$ new vertices to $G + \{v_1, \ldots, v_{\lfloor k/2 \rfloor}\}$ to get new r-super graceful graph for $r = \lceil k/2 \rceil - 1, \lceil k/2 \rceil - 2, \ldots, 1$ consecutively.*

Example 2.2. Take a k-super graceful graph G with $k = 10$, $p+q = 191$, $t = 5$, and that $f(u_i) = 190+2i, 1 \leq i \leq 5$ with $f(u_5) = k+p+q-1 = 200$ being the largest possible label. We join new vertex v_i to u_i and extend f with $f(v_i) = 200 + i$ and $f(u_iv_i) = k - i$. Now $G + \{v_1, \ldots, v_5\}$ is 5-super graceful. We can further join new vertices v_6 to v_2, v_7 to v_4, v_8 to v_6 and v_9 to v_8 with $f(v_6) = 206$, $f(v_7) = 207$, $f(v_8) = 208$ and $f(v_9) = 209$. After each addition, the new graph obtained is r-super graceful for $r = 4, 3, 2, 1$ consecutively.

Theorem 2.3. *Let G be a (p, q)-graph with a k-super graceful labeling f. Suppose there exist vertices u_i with (i) $f(u_i) = k - 1 + i$, or (ii) $f(u_i) = k + p + q - i$ for $1 \leq i \leq k$. Join the vertices u_1 and u_j by a new edge for $j = k, k-1, \ldots, 2$ consecutively gives a $(j-1)$-super graceful graph consecutively.*

Theorem 2.4. *Let $k, d \geq 1$. Suppose G is a (p, q)-graph with a vertex v of degree d. If G admits a k-super graceful labeling f such that $f(v) = k + p + q - 1$ and incident edge label(s) set is $[k, k+d-1]$, then $G - v$ is $(k+d)$-super graceful.*

Given $t \geq 3$ paths of length $n_j \geq 1$ with an end vertex $v_{j,1}$ ($1 \leq j \leq t$). A *spider graph* $SP(n_1, n_2, n_3, \ldots, n_t)$ is the one-point union of the t paths at vertex $v_{j,1}$. For simplicity, we shall use $a^{[n]}$ to denote a sequence of length n in which all items are a, where $a, n \geq 1$. Particularly, $SP(1^{[n]})$ is also known as a star graph $K_{1,n}$. Let $V(K_{1,n}) = \{v_i : 1 \leq i \leq n\} \cup \{u\}$ and $E(K_{1,n}) = \{uv_i : 1 \leq i \leq n\}$. We shall keep this notation throughout this paper.

We first consider $G - v$ that contains an isolated vertex. For $k = 1$, the star $K_{1,d}$ is such a graph for all possible $d \geq 1$ by having edge label(s) set $[1, d]$, end-vertex label(s) set $[d+1, 2d]$ and central vertex label $2d + 1$. The study of such graph with $k \geq 2$ is an interesting problem.

We now consider $G - v$ without an isolated vertex. Begin with a $K_{1,k+d}$, $k \geq 1$, by labeling the vertex u with $k+d$, v_i by $2k+2d+i$ and edge uv_i by $k+d+i$ for $1 \leq i \leq k+d$. Now, join a new vertex w to $v_i, 2 \leq i \leq d+1$. Label w by $3k + 3d + 1$ and edge wv_i by $k + d + 1 - i$ for $2 \leq i \leq d + 1$. The graph such obtained is k-super graceful and deleting the vertex w gives us a $(k+d)$-super graceful graph that has no isolated vertex.

Let $G + H$ be the disjoint union of graphs G and H. Let nG be the disjoint union of $n \geq 2$ copy of G.

Lemma 2.5. *Suppose H is super-graceful with edge label(s) set $[1, q]$. If G is $(q+1)$-super graceful, then $G + H$ is super graceful.*

Proof. Suppose t is the largest label of a $(q + 1)$-super graceful labeling of G. Keep all the labels of G and the edge labels of H. Add $t - q$ to each original vertex labels of H. We now have a super graceful labeling of $G + H$. ■

Theorem 2.6. *For $k \geq 1$, if a (p, q)-graph G admits a k-super graceful labeling, then the k largest integers in $[k, k + p + q - 1]$ must be vertex labels of k mutually non-adjacent vertices. Moreover, no two of the $k + 1$ smallest integers are vertex labels of adjacent vertices.*

Proof. The k largest integers are $p + q$ to $k + p + q - 1$. By definition, $k + p + q - 1$ must be a vertex label. If one of the integers in $[p + q, k + p + q - 2]$ is an edge label, then a corresponding end-vertex must be labeled with an integer less than k, a contradiction. Hence, all these k integers must be vertex labels. If there are two of these integers are labels of two adjacent vertices, then the corresponding edge label is an integer less than k, also a contradiction. Similarly, no two of the $k + 1$ smallest integers are vertex labels of adjacent vertices. ■

Corollary 2.7. *If G is k-super graceful, then $1 \leq k \leq \alpha$, where α is the independent number of G. Moreover, the upper bound is sharp.*

Proof. To prove the upper bound being sharp, we consider the star $K_{1,k}$. Label the central vertex by k and the remaining vertices by $2k + 1$ to $3k$ correspondingly. Clearly, it is a k-super graceful labeling. ■

In [7], we showed that the complete graph K_n is super graceful if and only if $n \leq 3$. The following result follows directly from Corollary 2.7.

Corollary 2.8. *The complete graph K_n is not k-super graceful for all $n, k \geq 2$.*

3. Trees and Cycles

Let T be a caterpillar. Suppose the central path of T is $P = a_1 b_1 a_2 b_2 \cdots a_r b_r$, $r \geq 1$. If $\deg(a_i) = n_i$ and $\deg(b_j) = m_j$, then rename a_i by $a_{i,0}$ and b_j by $b_{j,0}$, where $1 \leq i, j \leq r$. Let

$$\begin{aligned} N(a_{1,0}) &= \{b_{0,1}, \ldots, b_{0,n_1-1}\} \cup \{b_{1,0}\}; \\ N(b_{r,0}) &= \{a_{r,0}\} \cup \{a_{r,1}, \ldots, a_{r,m_r-1}\}; \\ N(a_{i,0}) &= \{b_{i-1,1}, \ldots, b_{i-1,n_i-2}\} \cup \{b_{i-1,0}, b_{i,0}\} \text{ for } 2 \leq i \leq r; \\ N(b_{j,0}) &= \{a_{j,0}, a_{j+1,0}\} \cup \{a_{j,1}, \ldots, a_{j,m_j-2}\} \text{ for } j \geq 2 \text{ if } r \geq 2. \end{aligned}$$

Now arrange $a_{i,s}$ as a sequence according to their subscripts under the lexicographic order. Let $A = \{u_1, u_2, \ldots, u_a\}$ be such ordered set. Similarly, we arrange $b_{j,t}$ by the same way and let $B = \{v_1, v_2, \ldots, v_b\}$ be the resulting ordered set. Hence (A, B) forms a bipartition of T. We shall denote T by $Ct(a, b)$. Note that different caterpillars may associate the same notation $Ct(a, b)$.

Note that $a = 1 - r + \sum_{j=1}^{r} m_j$, $b = 1 - r + \sum_{i=1}^{r} n_i$. Moreover,

$$a_1 = u_1,\ a_2 = u_{m_1},\ a_3 = u_{m_1+m_2-1}, \ldots,\ a_r = u_{m_1+m_2+\cdots+m_{r-1}-r+2};$$
$$b_1 = v_{n_1},\ b_2 = v_{n_1+n_2-1},\ \ldots,\ b_r = v_{n_1+n_2+\cdots+n_r-r+1},$$
$$a_i \text{ is adjacent to } v_j \text{ for } 2 - i + \sum_{l=1}^{i-1} n_l \le j \le 1 - i + \sum_{l=1}^{i} n_l, \text{ where } 1 \le i \le r;$$
$$b_j \text{ is adjacent to } u_i \text{ for } 2 - j + \sum_{l=1}^{j-1} m_l \le i \le 1 - j + \sum_{l=1}^{j} m_l, \text{ where } 1 \le j \le r.$$

Suppose the central path of T is $P = a_1b_1a_2b_2\cdots a_rb_ra_{r+1}$, $r \ge 1$. If $\deg(a_i) = n_i$ and $\deg(b_j) = m_j$, then rename a_i by $a_{i,0}$ and b_j by $b_{j,0}$, where $1 \le i \le r+1$, $1 \le j \le r$. Let

$$N(a_{1,0}) = \{b_{0,1}, \ldots, b_{0,n_1-1}\} \cup \{b_{1,0}\};$$
$$N(a_{r+1,0}) = \{b_{r,0}\} \cup \{b_{r,1}, \ldots, b_{r,n_{r+1}-1}\};$$
$$N(a_{i,0}) = \{b_{i-1,1}, \ldots, b_{i-1,n_i-2}\} \cup \{b_{i-1,0}, b_{i,0}\} \text{ for } 2 \le i \le r;$$
$$N(b_{j,0}) = \{a_{j,0}, a_{j+1,0}\} \cup \{a_{j,1}, \ldots, a_{j,m_j-2}\} \text{ for } j \ge 1.$$

By a similar rearrangement as above, we have vertices u's and v's, and a bipartition (A, B) of T.

Theorem 3.1. *A caterpillar $Ct(a, b)$ is k-super graceful for $k = a, b$.*

Proof. Suppose $k = a$.
Case 1: The length of the central path is $2r$. Define a labeling $f : V(Ct(a, b)) \to [a, 3a + 2b - 2]$ as follows:

(1) $f(u_i) = 3a + 2b - 1 - i, 1 \le i \le a$,
(2) $f(v_j) = a + j - 1, 1 \le j \le b$.

Now, the vertex labels set is $[a, a + b - 1] \cup [2a + 2b - 1, 3a + 2b - 2]$. For each edge u_iv_j, define $f(u_iv_j) = f(u_i) - f(v_j)$. Now
$f(a_1v_j) = f(u_1v_j) = 2a + 2b - 1 - j$ for $1 \le j \le n_1$,
$f(a_iv_j) = f(u_{m_1+\cdots+m_{i-1}-i+2}v_j) = 2a + 2b - 2 + i - j - \sum_{l=1}^{i-1} m_l$ for $2 - i + \sum_{l=1}^{i-1} n_l \le j \le 1 - i + \sum_{l=1}^{i} n_l$, $2 \le i \le r$ and
$f(u_ib_j) = f(u_iv_{n_1+\cdots+n_j-j+1}) = 2a + 2b - 1 - i + j - \sum_{l=1}^{j} n_l$ for $2 - j + \sum_{l=1}^{j-1} m_l \le i \le 1 - j + \sum_{l=1}^{j} m_l$, $2 \le j \le r$.

Thus

$$\{f(a_1v_j)\ :\ 1\le j\le n_1\}=[2a+2b-1-n_1,2a+2b-2];$$

$$\{f(a_iv_j)\ :\ 2-i+\sum_{l=1}^{i-1}n_l\le j\le 1-i+\sum_{l=1}^{i}n_l\}$$

$$=\left[2a+2b-3+2i-\sum_{l=1}^{i-1}m_l-\sum_{l=1}^{i}n_l,2a+2b-4+2i-\sum_{l=1}^{i-1}m_l-\sum_{l=1}^{i-1}n_l\right],$$

where $2\le i\le r$;

$$\{f(u_ib_j)\ :\ 2-j+\sum_{l=1}^{j-1}m_l\le i\le 1-j+\sum_{l=1}^{j}m_l\}$$

$$=\left[2a+2b-2+2j-\sum_{l=1}^{j}m_l-\sum_{l=1}^{j}n_l,2a+2b-3+2j-\sum_{l=1}^{j-1}m_l-\sum_{l=1}^{j}n_l\right],$$

where $1\le j\le r$.

One may check that these edge labels cover the interval $[a+b,2a+2b-2]$. Hence, f is an a-super graceful labeling.

Case 2: The length of the central path is $2r+1$. Using the same labeling method and a similar argument, we can show that an a-super graceful labeling also exists.

Suppose $k=b$. Let $\overline{f}=(3a+3b-2)-f:V(Ct(a,b)\to[b,2a+3b-2]$, where f is defined in the case $k=a$. Define $\overline{f}(u_iv_j)=|\overline{f}(u_i)-\overline{f}(v_j)|=f(u_i)-f(v_j)$ if u_iv_j is an edge. It is easy to check that $\overline{f}$ is a b-super graceful labeling of $Ct(a,b)$. ■

Corollary 3.2. *For each $k\ge 1$, there are infinite families of k-super graceful trees.*

Proof. We can construct infinitely many caterpillars $Ct(a,k)$ in which the central path $P_{2k+1}=a_1b_1\cdots a_kb_ka_{k+1}$ and $\deg(b_j)=2$ for $1\le j\le k$. ■

For $n\ge 2$, denote by $Ct(m_1,m_2,\dots,m_n)$ the caterpilar with central path $u_1u_2\cdots u_n$ such that there are m_i vertices attached to vertex u_i, $1\le i\le n$. For $n\ge 3$, the ringworm graph $RW(m_1,m_2,\dots,m_n)$ is then obtained from $Ct(m_1,m_2,\dots,m_n)$ by joining vertex u_1 and u_n by an edge. From the approach used in Theorem 3.1, we can get the following results on $RW(m_1,m_2,\dots,m_n)$.

Theorem 3.3. *(i) Suppose $n=2d+1$, $d\ge 1$, and $k=m_1+m_3+\cdots+m_n+d$ or $k=m_2+m_4+\cdots+m_{n-1}+d+1$, then $RW(m_1,m_2,\dots,m_n)$ is $(k-1)$-super graceful. (ii) Suppose $n=2d,d\ge 2$, and $k=m_1+m_3+\cdots+m_{n-1}+d$ or $k=m_2+m_4+\cdots+m_n+d$, then $RW(m_1,m_2,\dots,m_n-1,0)$ is $(k-1)$-super graceful.*

By Theorem 2.3, we also obtain many t-super graceful tripartite graphs, $1\le t\le k-2$, from each $(k-1)$-super graceful ringworm graph in Theorem 3.3.

Theorem 3.4. *There exists k-super graceful non-star graph for each $k\ge 1$.*

Proof. For $k\ge 1$, we begin with a $K_{1,k+1}$ by labeling the vertex u by $k+1$, v_i by $2k+2+i$ and edge uv_i by $k+1+i$ for $1\le i\le k+1$. Join a new vertex w to vertex v_2 and label edge wv_2 and vertex w by k and $3k+4$ respectively. The graph such obtained is k-super graceful spider $SP(1^{[k]},2)$. ■

In [5, Theorems 2.1 and 2.8], Perumal *et al.* proved that

Theorem 3.5. *All paths and cycles are super graceful.*

We now investigate the k-super gracefulness of paths and cycles. Let $P_n = u_1 \cdots u_n$ and $C_n = u_1 \cdots u_n u_1$ be the path and the cycle of order n, respectively. By Corollary 2.7 we have

Proposition 3.6. *If P_n and C_n are k-super graceful, then $k \le \lceil \frac{n}{2} \rceil$ and $k \le \lfloor \frac{n}{2} \rfloor$, respectively.*

Consider odd $n \ge 3$. Define $f(u_{2i-1}) = (3n+1)/2 - i$ for $1 \le i \le (n+1)/2$, $f(u_{2i}) = (3n-1)/2 + i$ for $1 \le i \le (n-1)/2$ and $f(u_i u_{i+1}) = i$ for $1 \le i \le n-1$. We have f is a super graceful labeling of P_n such that the edge label(s) set is $[1, n-1]$. In a similar approach, we see that for even $n \ge 2$, P_n also admits such a super graceful labeling. By Lemma 2.5, we have

Corollary 3.7. *If G is n-super graceful, then $G + P_n$ is super graceful.*

Applying Theorem 3.1, we have

Corollary 3.8. *The path P_n is k-super graceful for odd $n \ge 3$ with $k = (n \pm 1)/2$, and for even $n \ge 4$ with $k = n/2$.*

We now give some results showing that the necessary conditions in Proposition 3.6 may be sufficient.

Corollary 3.9. *The path P_2 and cycle C_3 are k-super graceful if and only if $k = 1$.*

Proof. It follows from Corollary 2.7 and Theorem 3.5. ∎

Proposition 3.10. *For $n = 3, 4, 5$, the path P_n is k-super graceful if and only if $k \le \lceil \frac{n}{2} \rceil$.*

Proof. It follows from Corollary 2.7, Theorem 3.5 and Corollary 3.8. ∎

The next result shows that a path P_n is k-super graceful for infinitely many k and n.

Proposition 3.11. *The paths P_{6k-3}, P_{6k-2}, P_{6k} and P_{6k+1} are k-super graceful for $k \ge 1$.*

Proof. For $P_{6k-3} = v_1 u_1 v_2 u_2 \ldots v_{3k-2} u_{3k-2} v_{3k-1}$, define a labeling $f : V(P_{6k-3}) \to [k, 13k-8]$ as follows:

(1) $f(v_i) = 13k - 7 - i$ for $1 \le i \le k$;
(2) $f(v_{k+i}) = 10k - 6 - i$ for $1 \le i \le 2k-1$;
(3) $f(u_i) = k + i - 1$ for $1 \le i \le 3k-2$;
(4) $f(u_i v_i) = 12k - 6 - 2i$ for $1 \le i \le k$;
(5) $f(u_i v_{i+1}) = 12k - 7 - 2i$ for $1 \le i \le k-1$;
(6) $f(u_{k+i} v_{k+i}) = 8k - 5 - 2i$ for $1 \le i \le 2k-2$;
(7) $f(u_{k-1+i} v_{k+i}) = 8k - 4 - 2i$ for $1 \le i \le 2k-1$.

It is easy to verify that f is a k-super graceful labeling for P_{6k-3}.

For $P_{6k-2} = v_1 u_1 v_2 u_2 \ldots v_{3k-1} u_{3k-1}$, define a labeling $f : V(P_{6k-2}) \to [k, 13k-6]$ as follows:

(1) $f(v_i) = 13k - 5 - i$ for $1 \le i \le k$;
(2) $f(v_{k+i}) = 10k - 4 - i$ for $1 \le i \le 2k-1$;

(3) $f(u_i) = k + i - 1$ for $1 \le i \le 3k - 1$;
(4) $f(u_iv_i) = 12k - 4 - 2i$ for $1 \le i \le k$;
(5) $f(u_iv_{i+1}) = 12k - 5 - 2i$ for $1 \le i \le k - 1$;
(6) $f(u_{k+i}v_{k+i}) = 8k - 3 - 2i$ for $1 \le i \le 2k - 1$;
(7) $f(u_{k-1+i}v_{k+i}) = 8k - 2 - 2i$ for $1 \le i \le 2k - 1$.

It is easy to verify that f is a k-super graceful labeling for P_{6k-2}.

For $P_{6k} = u_1v_1u_2v_2 \dots u_{3k}v_{3k}$, define a labeling $f : V(P_{6k}) \to [k, 13k - 2]$ as follows:

(1) $f(u_i) = k + i - 1$ for $1 \le i \le 3k$;
(2) $f(v_i) = 13k - 1 - i$ for $1 \le i \le k$;
(3) $f(v_{k+i}) = 10k - 1 - i$ for $1 \le i \le 2k$;
(4) $f(u_iv_i) = 12k - 2i$ for $1 \le i \le k$;
(5) $f(u_iv_{i+1}) = 12k - 1 - 2i$ for $1 \le i \le k$;
(6) $f(u_{k+i}v_{k+i}) = 8k - 2i$ for $1 \le i \le 2k$;
(7) $f(u_{k+1+i}v_{k+i}) = 8k - 1 - 2i$ for $1 \le i \le 2k - 1$.

It is easy to verify that f is a k-super graceful labeling for P_{6k}.

For $P_{6k+1} = u_1v_1u_2v_2 \dots u_{3k}v_{3k}u_{3k+1}$, define a labeling $f : V(P_{6k+1}) \to [k, 13k]$ as follows:

(1) $f(u_i) = k + i - 1$ for $1 \le i \le 3k + 1$;
(2) $f(v_i) = 13k + 1 - i$ for $1 \le i \le k$;
(3) $f(v_{k+i}) = 10k + 1 - i$ for $1 \le i \le 2k$;
(4) $f(u_iv_i) = 12k + 2 - 2i$ for $1 \le i \le k$;
(5) $f(u_{i+1}v_i) = 12k + 1 - 2i$ for $1 \le i \le k$;
(6) $f(u_{k+i}v_{k+i} = 8k + 2 - 2i$ for $1 \le i \le 2k$;
(7) $f(u_{k+1+i}v_{k+i}) = 8k + 1 - 2k$ for $1 \le i \le 2k$.

It is easy to verify that f is a k-super graceful labeling for P_{6k+1}. ■

We believe that P_n is 2-super graceful for all $n \ge 3$. Examples for $6 \le n \le 11$ with consecutive vertex and edge labels are given below.

(1) $n = 6 :$ 2, 10, 12, 8, 4, 7, 11, 5, 6, 3, 9.
(2) $n = 7 :$ 10, 8, 2, 12, 14, 3, 11, 5, 6, 7, 13, 9, 4.
(3) $n = 8 :$ 5, 8, 13, 7, 6, 10, 16, 14, 2, 9, 11, 4, 15, 3, 12.
(4) $n = 9 :$ 18, 16, 2, 15, 17, 14, ,3, 10, 13, 9, 4, 8, 12, 7, 5, 6, 11 or
14, 4, 18, 8, 10, 6, 16, 7, 9, 3, 12, 5, 17, 15, 2, 11, 13.
(5) $n = 10 :$ 20, 18, 2, 17, 19, 16, 3, 12, 15, 11, 4, 10, 14, 9, 5, 8, 13, 7, 6.
(6) $n = 11 :$ 12, 8, 4, 17, 21, 18, 3, 19, 22, 20, 2, 14, 16, 11, 5, 10, 15, 9, 6, 7, 13.

Moreover, we also obtained the following k-super graceful labeling for P_n with consecutive labels given below.

(1) $n = 8, k = 3$: 11, 4, 7, 8, 15, 9, 6, 10, 16, 13, 3, 14, 17, 12, 5.
(2) $n = 9, k = 3$: 18, 11, 7, 12, 19, 4, 15, 5, 10, 6, 16, 13, 3, 14, 17, 9, 8.
(3) $n = 10, k = 3$: 20, 17, 3, 18, 21, 7, 14, 8, 6, 13, 19, 9, 10, 5, 15, 11, 4, 12, 16.
(4) $n = 11, k = 3$: 20, 13, 7, 16, 23, 19, 4, 11, 15, 6, 9, 12, 21, 3, 18, 10, 8, 14, 22, 17, 5.
(5) $n = 10, k = 4$: 11, 8, 19, 9, 10, 12, 22, 18, 4, 17, 21, 16, 5, 15, 20, 14, 6, 7, 13.
(6) $n = 11, k = 4$: 20, 12, 8, 16, 24, 5, 19, 15, 4, 18, 22, 13, 9, 14, 23, 6, 17, 7, 10, 11, 21.

Note that P_{6k-4} and P_{6k-1} are k-super graceful for $k = 1, 2$. Thus, together with Corollary 3.8, we have shown that for $n \leq 11$, P_n is k-super graceful if and only if $1 \leq k \leq \lceil n/2 \rceil$.

Conjecture 1. *The path P_n is k-super graceful if and only if $1 \leq k \leq \lceil n/2 \rceil$.*

Proposition 3.12. *If C_n is k-super graceful such that k is an edge label of C_n, then P_n is $(k+1)$-super graceful.*

Proof. Given C_n and its k-super graceful labeling having k as an edge label, deleting the edge labeled with k gives a $(k+1)$-super graceful P_n. ∎

By Theorem 3.1, we see that the given $(k+1)$-super graceful labeling of P_{2k+1} gives a k-super graceful C_{2k+1} having k as an edge label.

Corollary 3.13. *For all $k \geq 1$, C_{2k+1} is k-super graceful.*

Problem 1. *Determine all values of k such that C_n is k-super graceful having k as an edge label.*

Proposition 3.14. *The cycles C_4 and C_5 are k-super graceful if and only if $k = 1, 2$.*

Proof. Theorem 2.8 in [5] shows that all cycles are 1-super graceful. We assume $k = 2$. We can label the vertices of C_4 and C_5 sequentially as follows: $2, 8, 5, 9$ and $3, 10, 4, 9, 11$. So, sufficiency holds. By Proposition 3.6, necessity holds. ∎

We now show that the $k \leq \lfloor n/2 \rfloor$ is not a sufficient condition on k-super gracefulness of cycles. Suppose f is a 3-super graceful labeling of C_6. By Corollary 2.7, without loss of generality, we may assume that $f(u_1) = 12$, $f(u_3) = 13$ and $f(u_5) = 14$. Since 1 is not available, 11 must be used to label an edge incident with the vertex u_5.

(1) Suppose $f(u_5u_6) = 11$. Then $f(u_6) = 3$ and $f(u_6u_1) = 9$. Since 9 is used, 4 is not a vertex label. Since 1 is not available and 9 is used, $f(u_1u_2) = 4$ or $f(u_4u_5) = 4$.

If $f(u_1u_2) = 4$, then $f(u_2) = 8$, $f(u_2u_3) = 5$. In this case, we can see that 6 cannot be used.

If $f(u_4u_5) = 4$, then $f(u_4) = 10$ and hence $f(u_3u_4) = 3$ which is impossible.

(2) Suppose $f(u_4u_5) = 11$. Then $f(u_4) = 3$, $f(u_3u_4) = 10$. Since 3 is used, 9 must be used to label an edge incident with the vertex u_3 or u_5.

If $f(u_2u_3) = 9$, then $f(u_2) = 4$, $f(u_1u_2) = 8$. Now 7 cannot be used to label u_6 or u_5u_6. So $f(u_6u_1) = 7$. This yields $f(u_6) = 5$ and $f(u_5u_6) = 9$ which is impossible.

If $f(u_5u_6) = 9$, then $f(u_6) = 5$, $f(u_6u_1) = 7$. Since 5 is used, 8 cannot be used to label u_2 or u_2u_3. So $f(u_1u_2) = 8$, $f(u_2) = 4$. This yields $f(u_2u_3) = 9$ which is impossible.

Thus C_6 is not 3-super graceful. ∎

Using a similar approach, we can also show that C_8 is not 4-super graceful. However, C_8 is 2-super graceful with consecutive vertex labels $17, 4, 14, 12, 15, 7, 16, 11$. The corresponding edge labels are $13, 10, 2, 3, 8, 9, 5, 6$. Deleting the edge with label 2, we have another 3-super graceful labeling for P_8.

A *tadpole graph* $T_{m,k}$ is a simple graph obtained from an m-cycle by attaching a path of length k, where $m \geq 3$ and $k \geq 1$.

Proposition 3.15. *For even $k \geq 2$, $T_{4,k}$ is $(k+2)/2$-super graceful.*

Proof. Begin with the $n/2$-super graceful labeling of P_n as in Theorem 3.1, where $n \geq 6$ is even. Exchange the labels of u_2 and u_1u_2. Add the edge u_1u_4. We get the graph $T_{4,n-4}$ which is $(n-2)/2$-super graceful. ∎

Proposition 3.16. *For odd $n \geq 5$, $T_{n,1}$ is $(n-1)/2$-super graceful.*

Proof. Begin with the $m/2$-super graceful labeling of P_m, where $m \geq 6$ is even. Add the edge u_1u_{m-1} to get $T_{m-1,1}$ which is clearly $(m-2)/2$-super graceful. Let $n = m-1$, the result follows. ∎

Using the labelings in Proposition 3.11, it is easy to get the following.

Proposition 3.17. *The graphs $T_{2k-1,4k-2}$, $T_{2k-1,4k-1}$, $T_{2k-1,4k+1}$ and $T_{2k-1,4k+2}$ are $(k-1)$-super graceful for $k \geq 2$.*

4. Some Complete Bipartite and Tripartite Graphs

Lemma 4.1. *Suppose f is a k-super graceful labeling of $K_{1,n}$, where $k \geq 2$. Let m be the largest integer such that the m largest integers in $[k, k+2n]$ are labeled at m mutually non-adjacent vertices, then $f(u) = m$. Moreover, $k + 2n \geq 3m$.*

Proof. Note that $m \geq k$. After renumbering, we may assume that $f(v_i) = k+2n+1-i$, $1 \leq i \leq m$. Now $f(u) = k+2n-m$ or $k+2n-m$ is an edge-label.

For the first case, we have $f(uv_m) = 1$, a contradiction. For the latter case, one of the end vertices of this edge has label greater than $k+2n-m$. So $f(uv_i) = k+2n-m = f(v_i)-f(u)$, where $1 \leq i \leq m$. Since $k+2n-m \geq f(v_1)-f(u) \geq f(v_i)-f(u) = k+2n-m$, $i = 1$ and hence $f(u) = m$. Since $\{f(v_i), f(uv_i) : 1 \leq i \leq m\} = [k+2n-2m+1, k+2n]$, $m < k+2n-2m+1$, i.e., $k+2n \geq 3m$. ∎

Theorem 4.2. *For $n, k \geq 1$, the star $K_{1,n}$ is k-super graceful if and only if $n \equiv 0 \pmod{k}$. Moreover, for $k \geq 2$, the central vertex must have label k.*

Proof. We first prove the sufficiency. Suppose $n = kt$, $t \geq 1$. We rewrite all vertices v_l as $v_{(j-1)k+i}$, where $1 \leq j \leq t$, $1 \leq i \leq k$. Define a labeling $f : V(K_{1,n}) \cup E(K_{1,n}) \to [k, k+2kt]$ as follows:

(1) $f(u) = k$;
(2) $f(v_{(j-1)k+i}) = (2j)k + i$;
(3) $f(uv_{(j-1)k+i}) = (2j-1)k + i$.

It is easy to verify that f is a k-super graceful labeling.

The necessity obviously holds for $k = 1$. We now assume that $k \geq 2$. Let f be a k-super graceful labeling of $K_{1,n}$. Suppose m is the largest integer such that the m largest integers in $[k, k+2n]$ are labeled at m mutually non-adjacent vertices. By Lemma 4.1 we have $f(u) = m$ and $k+2n \geq 3m$. Also we may assume $\{f(v_i), f(uv_i) : 1 \leq i \leq m\} = [k+2n-2m+1, k+2n]$.

If $m = n$, then $3m \geq k+2m = k+2n \geq 3m$. Hence $k = m$ and we have the result.

Suppose $n > m$. Consider $K_{1,n-m} \cong K_{1,n} - \{v_i : 1 \leq i \leq m\}$. The restriction of f on $K_{1,n-m}$ is still a k-super graceful labeling. So Lemma 4.1 can be applied on $K_{1,n-m}$. Repeating in this manner gives $n = mt$ for some t and $m = k$. ∎

Example 4.3. Take $k = 5$, $n = 15$, we can label u by 5, the edges uv_1 to uv_5 by 6 to 10, the vertices v_1 to v_5 by 11 to 15, the edges uv_6 to uv_{10} by 16 to 20, the vertices v_6 to v_{10} by 21 to 25, the edges uv_{11} to uv_{15} by 26 to 30, and the vertices v_{11} to v_{15} by 31 to 35.

Corollary 4.4. *For any finite set A of positive integers there is a graph that is k-super graceful for all $k \in A$.*

Proof. Let L be the least common multiple of all elements in A. The required graph is $K_{1,L}$ by Theorem 4.2. ∎

Theorem 4.5. *For $n \geq m \geq 1$, the complete bipartite graph $K_{m,n}$ is n- and m-super graceful.*

Proof. Let $V(K_{m,n}) = \{u_i, v_j \,|\, 1 \leq i \leq m, 1 \leq j \leq n\}$ and $E(K_{m,n}) = \{u_iv_j \,|\, 1 \leq i \leq m, 1 \leq j \leq n\}$. Define a labeling $f : V(K_{m,n}) \cup E(K_{m,n}) \to [n, 2n + m + mn - 1]$ as follows:

(1) $f(u_i) = ni + i - 1$ for $1 \leq i \leq m$;
(2) $f(v_j) = 2n + m(n + 1) - j$ for $1 \leq j \leq n$;
(3) $f(u_iv_j) = (n + 1)(m - i) + 2n - j + 1$ for $1 \leq i \leq m$ and $1 \leq j \leq n$.

It is easy to verify that f is an n-super graceful labeling. By swapping the roles of m and n, we have an m-super graceful labeling for $K_{n,m} \cong K_{m,n}$. ∎

Now, keep the notation in the proof of Theorem 4.5. For $n \geq 2$, $1 \leq k \leq n - 1$ and $m \geq 1$, let $G(1, m, n-k)$ be a tripartite graph obtained from the complete bipartite graph $K_{m,n}$ by adding $n - k$ edges v_1v_j, $k + 1 \leq j \leq n$. Note that $G(1, m, n - 1) = K_{1,m,n-1}$, the complete tripartite graph. We shall keep this notation for the following theorem.

Theorem 4.6. *The tripartite graph $G(1, m, n - k)$ is k-super graceful. In particular, $K_{1,m,r}$ is super graceful for all $r \geq 1$.*

Proof. Note that $G(1, m, n-k)$ has $m+n$ vertices and $mn+n-k$ edges. Observe that for the n-super graceful labeling f of $K_{m,n}$, $n \geq 2$, in Theorem 4.5, we have $f(v_1) - f(v_j) = j-1$, $k+1 \leq j \leq n$. Here f is extended to be a k-super graceful labeling for $G(1, m, n-k)$. Note that if $k = 1$, we obtain a super graceful complete tripartite graph $K_{1,m,n-1}$ as required. ∎

Also observe that for $2 \leq i \leq m$, $f(u_i) - f(u_{i-1}) = n + 1$ and $f(u_i) - f(u_1) = (n + 1)(i - 1) = f(u_{m-i+2}v_n)$. Hence,

(1) by adding edge u_1u_2 and deleting edge u_1v_n of $G(1, 2, n - 1)$, we get for $n \geq 3$, $K_{1,1,1,n-1} - e$ is super graceful where $K_{1,1,1,n-1}$ is a complete 4-partite graph and e is an edge with an end vertex of degree 3.
(2) for $2 \leq i \leq m$, if we add edge u_1u_i and delete edge u_iv_n of $G(1, m, n - k)$, we get infinitely many k-super graceful 4-partite graphs.

The following theorem in [7, Theorem 2.2] now follows directly from Theorem 4.6.

Theorem 4.7. *For $r \geq 1$, the complete tripartite graph $K_{1,1,r}$ is super graceful.*

Corollary 4.8. *There are infinitely many 2-super graceful $K_{1,1,r} - e$ for $r \geq 1$ and e is an edge with an end vertex of degree 2.*

Theorem 4.9. *For $1 \leq k \leq r$, $K_{1,1,r}$ is k-super graceful if and only if $k = 1$.*

Proof. The sufficiency follows from Theorem 4.7. We now prove the necessity. Let f be a k-super graceful labeling of $K_{1,1,r}$. Without loss of generality, we may assume that $\{f(v_i) : 1 \leq i \leq r\}$ is a strictly decreasing sequence, and that $f(u_1) < f(u_2)$. Let $c \leq r$ be the greatest integer such that $f(v_i) = k + 3r + 3 - i$, for $1 \leq i \leq c$. By Theorem 2.6, $k \leq c$. If $c = 1$, then $k = 1$ and we are done. So we assume that $c \geq 2$.

Now, we consider the assignment of the label $k + 3r + 2 - c$, which is the greatest undetermined label. If it is a vertex label, then according to the choice of c and $f(u_2) > f(u_1)$, $f(u_2) = k + 3r + 2 - c$. In this case, $f(u_2 v_c) = 1$. Hence $k = 1$ and we are done.

From now on, we assume that $k+3r+2-c$ is an edge label. That is, $k+3r+2-c = f(xy) = f(x) - f(y)$ for some edge xy. Now $f(x) = k+3r+2-c+f(y) \geq k+3r+3-c$. So $x = v_i$ for some i ($1 \leq i \leq c$). Moreover, $y = u_1$ or u_2. Since $k+3r+2-c \geq f(v_1 y) \geq f(v_i y)$, we have $i = 1$ and $f(y) = c$. Since $f(v_1 u_1) > f(v_1 u_2)$ and $k + 3r + 2 - c$ is the greatest undetermined label, $y = u_1$. Thus $f(u_1) = c$, and so $f(u_1 v_i) = k+3r+3-c-i$, for $1 \leq i \leq c$. Now the next greatest undetermined label is $k+3r+2-2c \geq c+k+1 \geq c+2$.

Let $\mathcal{P}(t) =$"*Either* $f(v_{tc+j}) = k+3r+3-2tc-j$ *for* $1 \leq j \leq c$ *and* $k+3r+3-2tc-2c \geq c+2$, *or* $k = 1$." be a statement on $t \geq 0$.

From the above discussion, we know that $\mathcal{P}(0)$ holds. Now we assume that $\mathcal{P}(s)$ holds for $0 \leq s \leq t-1$ and consider $\mathcal{P}(t)$. Up to now integers in $[k + 3r + 3 - 2tc, k + 3r + 2]$ are assigned. We consider the assignment of $k + 3r + 2 - 2tc$, which is the greatest undetermined label at this moment.

Remark 4.10. At each stage, we always examine the greatest undetermined label. Observe that any greatest undetermined label must belong to an unlabeled vertex, or edge with a larger incident vertex label.

For convenience of exposition, we divide the undetermined labels into five types:

(I) Edge label $f(u_2 u_1)$.
(II) Edge labels $f(v_i u_1)$, $tc + 1 \leq i \leq r$.
(III) Edge labels $f(v_i u_2)$, $1 \leq i \leq r$.
(IV) Vertex label $f(u_2)$.
(V) Vertex labels $f(v_i)$, $tc + 1 \leq i \leq r$.

By Remark 4.10, Types (I) and (II) are not possible.

For Type (III): $f(v_i u_2) = 3r + k + 2 - 2tc$ for some i. Since $f(u_2) < 3r + k + 2 - 2tc$, $f(v_i) > 3r+k+2-2tc$. This implies that $1 \leq i \leq tc$. Since $3r+k+2-2tc$ is the greatest undetermined label, $i = 1$. Here we have $f(u_2) = f(v_1)-(3r+k+2-2tc) = 2tc$. If $t = 1$, then $f(u_2 u_1) = c = f(u_1)$ which is impossible. So we only need to deal with $t \geq 2$. Note that $2tc \notin [k + 3r + 3 - 2tc, k + 3r + 2]$ and hence $f(v_{sc+j} u_2) = 3r + k + 3 - 2(s+t)c - j$ for $0 \leq s \leq t-1$ and $1 \leq j \leq c$. Now $3r + k + 2 - 2tc - c$ is the greatest undetermined label. By Remark 4.10, Types (II) to (IV) are not possible.

(A) Suppose Type (I) holds. Now we have $2tc - c = f(u_2 u_1) = 3r + k + 2 - 2tc - c$. Hence we have $f(v_{tc}) = 2tc + c + 1$. Now we consider the next greatest undetermined label $3r + k + 1 - 2tc - c = (2t-1)c - 1 \geq 3c - 1$. By remark 4.10, we only need to check Types (III) and (V).

(a) For Type (III), suppose $f(v_iu_2) = 2tc - c - 1$. Since $c \geq 2$, $2tc - c - 1 = 3r+k+1-2tc-c > 3r+k+2-2c-2tc = f(v_{c+1}u_2)$, $i > tc$ and $f(u_2) > f(v_i)$. But we will get $f(v_i) = c + 1 = f(v_{tc}u_2)$, a contradiction.

(b) For Type (V), if $f(v_i) = (2t - 1)c - 1$ for some i, then $f(u_2v_i) = c + 1 = f(v_{tc}u_2)$, a contradiction.

(B) Suppose Type (V) holds. By definition, $f(v_{tc+1}) = 3r + k + 2 - 2tc - c$. Now $f(v_{tc+1}u_1) = 3r + k + 2 - 2tc - 2c = f(v_{c+1}u_2)$, a contradiction.

For Type (IV): $f(u_2) = 3r + k + 2 - 2tc$. Now $3r + k + 1 - 2tc$ becomes the greatest undetermined label. If it is a vertex label, then together with $f(u_2)$ we get that 1 is an edge label and hence we are done. Now, we assume that $3r + k + 1 - 2tc$ is labeled to an edge. From $3r + k + 1 - 2tc \geq c$ or Remark 4.10, only Type (III) is possible. If $f(v_i) < f(u_2)$, then $f(v_i) = 1$. We are done. If $f(v_i) > f(u_2)$, then $1 \leq i \leq (t-1)c$. Since $3r+k+1-2tc$ is the greatest undetermined label and $i = 1$ and hence $3r+k+1-2tc = 2tc$. Now $f(u_2) = 2tc+1$. It implies that $f(v_cu_2) = (2t-1)c+1 = f(u_2u_1)$ which is impossible.

Therefore, $f(v_{tc+1}) = k + 3r + 2 - 2tc$ is the only possibility.

Let m be the greatest integer such that $f(v_{tc+j}) = k + 3r + 3 - 2tc - j$, for $1 \leq j \leq m$. Since $k + 3r + 2 - 2tc > c = f(u_1)$, the m consecutive integers are greater than c. Therefore, $k + 3r + 3 - 2tc - m > c$. Now we consider the greatest undetermined label $k+3r+2-2tc-m$. By the choice of m or Remark 4.10, Types (I) and (V) are impossible. If $k + 3r + 2 - 2tc - m$ is the label at u_1 or u_2, then 1 is an edge label. Hence we are done. So we only need to consider Types (II) and (III).

For Type (III): $f(v_iu_2) = k + 3r + 2 - 2tc - m$. In this case, since the integers in $[k + 3r + 2 - 2tc - m, k + 3r + 2]$ are occupied, $f(u_2) \leq k + 3r + 1 - 2tc - m$ and $f(v_i) \geq k + 3r + 3 - 2tc - m$. Hence $i \leq tc + m$. Since $k + 3r + 2 - 2tc - m$ is the greatest undetermined label, $i = 1$ and hence $f(u_2) = 2tc + m$. Now $f(v_cu_2) = k + 3r + 3 - c - 2tc - m = f(v_{tc+m}u_1)$, a contradiction.

So Type (II) is the only possibility. Since $k+3r+2-2tc-m$ is the greatest undetermined label, $f(v_{tc+1}u_1) = k+3r+2-2tc-m$. On the other hand, $f(v_{tc+1}u_1) = k+3r+2-2tc-c$. Thus, we obtain $m = c$. Therefore, $f(v_{tc+j}) = k + 3r + 3 - 2tc - j$ and $f(v_{tc+j}u_1) = k + 3r + 3 - (2t + 1)c - j$, for $1 \leq j \leq c$. Since $k + 3r + 2 - 2tc > c$ and $f(v_{tc+j})$ and $f(v_{tc+j}u_1)$ $(1 \leq j \leq c)$ are $2c$ consecutive integers, $k+3r+3-(2t+1)c-c > c$ and hence $k + 3r + 3 - (2t + 2)c \geq c + 1$. If $k + 3r + 3 - (2t + 2)c = c + 1$, then $f(u_2) < c = f(u_1)$, a contradiction. Thus, we have $k + 3r + 3 - (2t + 2)c \geq c + 2$, i.e., $\mathcal{P}(t)$ holds.

By mathematical induction $\mathcal{P}(t)$ holds for all $t \geq 0$. Since k, r and c are fixed, $k + 3r + 3 - 2tc - 2c \geq c + 2$ cannot hold for all t. Therefore, we conclude that $k = 1$. ■

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