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Some Common Fixed Point Theorems in Complex Valued Metric Spaces

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Abstract In this work, some common fixed point results for the mappings satisfying rational expressions on a closed ball in complex valued metric spaces will be proposed. Presented theorems can be realized as extensions of some well-known results in the literature. Further, our result is well supported by nontrivial example which shows that the improvement is a actual.

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1. Introduction

Metric space is one of the most useful and important spaces in mathematics. Its wide area provides a powerful tool to the study of variational inequalities, optimization and approximation theory, computer sciences and etc.

In 2011, Azam et al. [1] gave the concepts of new spaces called complex valued metric spaces and established existence of fixed point theorems under the contraction condition. Recently, several authors have studied the existence and uniqueness of the fixed points and common fixed points of self-mappings in view of contrasting contractive conditions. For example, Sintunavarat and Kumam [2] and Sintunavarat et al. [3] established the existence of common fixed point theorems in complete complex-valued metric spaces and presented some applications for these results. Kumam et al. [4] established fixed point results satisfying contractive conditions of rational type in the setting of complex valued metric spaces. Ali [5] proved some common fixed point theorems for a pair of weakly compatible mappings satisfying (CLRg) property and property (E.A) in complex valued metric spaces. Lately, Verma and Pathak [6] proved some common fixed point theorems for two pairs of weakly compatible mappings in a complex-valued metric space by using new property, known as common limit converging in the subset or (CLCS)-property (see also [2, 7–10]).

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The purpose of this work is to present some common fixed point results for the mappings satisfying rational expressions on a closed ball in complex valued metric spaces where these results are very useful in the sense that they require contractiveness of the mappings only on a closed ball instead of the whole space. Moreover, the obtained results are generalizations of recent results proved by Ahmad et al. [7], Azam et al. [1], Klin-eam and Suanoom [11], Rouzkard and Imdad [9], Sitthikul and Saejung [10] and Sintunavarat and Kumam [2].

2. Preliminaries

In this section, we recall some definitions and properties in complex valued metric spaces that will be used in this paper.

Let $\mathbb{C}$ be the set of complex numbers and $z_1, z_2 \in \mathbb{C}$. Define a partial order $\preceq$ on $\mathbb{C}$ as follows:

$z_1 \preceq z_2$ if and only if $Re(z_1) \leq Re(z_2)$, $Im(z_1) \preceq Im(z_2)$.

It follows that $z_1 \preceq z_2$ if and only if one of the following conditions is satisfied:

(i) $Re(z_1) = Re(z_2)$, $Im(z_1) < Im(z_2)$,
(ii) $Re(z_1) < Re(z_2)$, $Im(z_1) = Im(z_2)$,
(iii) $Re(z_1) < Re(z_2)$, $Im(z_1) < Im(z_2)$,
(iv) $Re(z_1) = Re(z_2)$, $Im(z_1) = Im(z_2)$.

In particular, we will write $z_1 \precnsim z_2$ if $z_1 \neq z_2$ and one of (i), (ii), and (iii) is satisfied and we will write $z_1 \prec z_2$ if only (iii) is satisfied. Note that it is obvious that the following statements hold:

(i) If $0 \preceq z_1 \precnsim z_2$, then $|z_1| < |z_2|$.
(ii) If $z_1 \preceq z_2$ and $z_2 \precnsim z_3$, then $z_1 \prec z_3$.

Definition 2.1. [1] Let X be a nonempty set. Suppose that the mapping $d : X \times X \to \mathbb{C}$, satisfies:

(a) $0 \preceq d(x, y)$, for all $x, y \in X$ and $d(x, y) = 0$ if and only if $x = y$;
(b) $d(x, y) = d(y, x)$ for all $x, y \in X$;
(c) $d(x, z) \preceq d(x, y) + d(y, z)$, for all $x, y, z \in X$.

Then d is called a complex valued metric on X, and (X, d) is called a complex valued metric space.

Definition 2.2. [1] Let (X, d) be a complex valued metric space.

(i) A point $x \in X$ is called interior point of a set $A \subseteq X$ whenever there exists $0 \prec r \in \mathbb{C}$ such that $B(x, r) = \{y \in X : d(x, y) \prec r\} \subseteq A$, where $B(x, r)$ is an open ball. Then $\overline{B(x, r)} = \{y \in X : d(x, y) \preceq r\}$ is a closed ball.
(ii) A point $x \in X$ is called a limit point of A whenever for every $0 \prec r \in \mathbb{C}$, $B(x, r) \cap (A - X) \neq \emptyset$.
(iii) A subset $A \subseteq X$ is called open whenever each element of A is an interior point of A. A subset $B \subseteq X$ is called closed whenever each limit point of B belongs to B. In this case, the family $F = \{B(x, r) : x \in X, 0 \prec r\}$, is a sub-basis for a Hausdorff topology τ on X.

Definition 2.3. [1] Let (X, d) be a complex valued metric space. Let $\{x_n\}$ be a sequence in X and $x \in X$.

(i) If for every $c \in \mathbb{C}$ with $0 \prec c$ there is $n_0 \in \mathbb{N}$ such that $d(x_n, x) \prec c$, for all $n > n_0$, then $\{x_n\}$ is said to be convergent to x. We denote this by $lim_n x_n = x$ or $x_n \to x$.

(ii) If for every $c \in \mathbb{C}$ with $0 \prec c$ there is $n_0 \in \mathbb{N}$ such that $d(x_n, x_m) \prec c$ for all $n, m > n_0$, then $\{x_n\}$ is called a Cauchy sequence.

(iii) If every Cauchy sequence is convergent in (X, d), then (X, d) is called a complete complex valued metric space.

Lemma 2.4. [1] *Let (X, d) be a complex valued metric space and let $\{x_n\}$ be a sequence in X. Then $\{x_n\}$ converges to x if and only if $|d(x_n, x)| \to 0$ as $n \to \infty$.*

Lemma 2.5. [1] *Let (X, d) be a complex valued metric space and let $\{x_n\}$ be a sequence in X. Then $\{x_n\}$ converges to x if and only if $|d(x_n, x_{n+m})| \to 0$ as $n \to \infty$, where $m \in \mathbb{N}$.*

Lemma 2.6. [10] *Let (X, d) be a complete complex valued metric space. Suppose $\{x_n\}$ be a sequence in X and $h \in [0, 1)$. If $a_n = |d(x_n, x_{n+1})|$ satisfies*

$$a_n \leq h a_{n-1}, \ \forall \ n \in \mathbb{N},$$

then $\{x_n\}$ is a Cauchy sequence.

Recently, many authors established some common fixed point results in complex valued metric.

Theorem 2.7. [1] *Let (X, d) be a complete complex valued valued metric space and let λ_1, λ_2 be nonnegative real numbers such that $\lambda_1 + \lambda_2 < 1$. Suppose $S, T : X \longrightarrow X$ are mappings satisfying*

$$d(Sx, Ty) \quad \preceq \quad \lambda_1 d(x, y) + \frac{\lambda_2 d(x, Sx) d(y, Ty)}{1 + d(x, y)},$$

for all $x, y \in X$. Then S and T have a unique common fixed point in X.

Theorem 2.8. [9] *If S and T are self-mappings defined on a complete complex valued metric space (X, d) satisfying the condition*

$$d(Sx, Ty) \quad \preceq \quad \lambda_1 d(x, y) + \frac{\lambda_2 d(x, Sx) d(y, Ty) + \lambda_3 d(y, Sx) d(x, Ty)}{1 + d(x, y)},$$

for all $x, y \in X$, where $\lambda_1, \lambda_2, \lambda_3$ are nonnegative reals with $\lambda_1 + \lambda_2 + \lambda_3 < 1$. Then S and T have a unique common fixed point in X.

Theorem 2.9. [11] *If S and T are self-mappings defined on a complete complex valued metric space (X, d) satisfying the condition*

$$\begin{aligned} d(Sx, Ty) \quad \preceq \quad & \lambda_1 d(x, y) + \frac{\lambda_2 d(x, Sx) d(y, Ty)}{1 + d(x, y)} \\ & + \frac{\lambda_3 d(y, Sx) d(x, Ty)}{1 + d(x, y)} + \frac{\lambda_4 d(x, Sx) d(x, Ty)}{1 + d(x, y)} \\ & + \frac{\lambda_5 d(y, Sx) d(y, Ty)}{1 + d(x, y)}, \end{aligned}$$

for all $x, y \in X$,where λ_i for $i = 1, 2, 3, 4, 5$ are nonnegative with $\lambda_1 + \lambda_2 + \lambda_3 + 2\lambda_4 + 2\lambda_5 < 1$. Then S and T have a unique common fixed point in X.

Theorem 2.10. [2] *Let (X,d) be a complete complex valued metric space and $S,T: X \longrightarrow X$. If there exists a mapping $\Lambda(x),\ \Xi(x): X \longrightarrow [0,1)$ such that for all $x \in X$:*

(i) $\Lambda(Sx) \leq \Lambda(x)$ *and* $\Xi(Sx) \leq \Xi(x)$;

(ii) $\Lambda(Tx) \leq \Lambda(x)$ *and* $\Xi(Tx) \leq \Xi(x)$;

(iii) $(\Lambda + \Xi)(x) < 1$;

(iv) $d(Sx,Ty) \preceq \Lambda(x)d(x,y) + \dfrac{\Xi(x)d(x,Sx)d(y,Ty)}{1+d(x,y)}$.

Then S and T have a unique common fixed point.

3. Main Results

The following theorem is the main result of this paper.

Theorem 3.1. *Let (X,d) be a complete complex valued metric space, $x_0 \in X$ and $S,T: X \longrightarrow X$. Suppose that there exist mappings $\lambda_i : X \times X \longrightarrow [0,1)$ where $i = 1,2,\cdots,7$ such that*

(a) $\lambda_i(TSx,y) \leq \lambda_i(x,y)$ *and* $\lambda_i(x,STy) \leq \lambda_i(x,y)$,

(b) $\lambda_1(x,y)+\lambda_2(x,y)+\lambda_3(x,y)+2\left(\lambda_4(x,y)+\lambda_5(x,y)+\lambda_6(x,y)+\lambda_7(x,y)\right) < 1$,

(c)

$$\begin{aligned} d(Sx,Ty) \preceq{}& \lambda_1(x,y)d(x,y) + \lambda_2(x,y)\tfrac{d(x,Sx)d(y,Ty)}{1+d(x,y)} + \lambda_3(x,y)\tfrac{d(y,Sx)d(x,Ty)}{1+d(x,y)} \\ &+\lambda_4(x,y)\tfrac{d(x,Sx)d(x,Ty)}{1+d(x,y)} + \lambda_5(x,y)\tfrac{d(y,Sx)d(y,Ty)}{1+d(x,y)} \\ &+\lambda_6(x,y)\tfrac{d(x,Ty)(1+d(x,Sx))}{1+d(x,y)} + \lambda_7(x,y)\tfrac{d(y,Sx)(1+d(y,Ty))}{1+d(x,y)}, \end{aligned} \tag{3.1}$$

for all $x,y \in \overline{B(x_0,r)}$. If

$$|d(x_0,Sx_0)| \leq (1-\lambda)|r|, \tag{3.2}$$

where

$$\begin{aligned} \lambda \;=\; & \max\{\frac{\lambda_1(x_0,Sx_0)+\lambda_4(x_0,Sx_0)+\lambda_6(x_0,Sx_0)}{1-\lambda_2(x_0,Sx_0)-\lambda_4(x_0,Sx_0)-\lambda_6(x_0,Sx_0)}, \\ & \frac{\lambda_1(x_0,Sx_0)+\lambda_5(x_0,Sx_0)+\lambda_7(x_0,Sx_0)}{1-\lambda_2(x_0,Sx_0)-\lambda_5(x_0,Sx_0)-\lambda_7(x_0,Sx_0)}\}, \end{aligned}$$

then there exists a unique point $u \in \overline{B(x_0,r)}$ such that $u = Su = Tu$.

Proof. Let x_0 be an arbitrary point in X. Define the sequence $\{x_n\}$ by

$$x_{2n+1} = Sx_{2n},\ x_{2n+2} = Tx_{2n+1},\ \forall\ n = 0,1,2,\cdots. \tag{3.3}$$

From condition (a), for $x,y \in X$, $n = 0,1,2,\cdots$ and $i = 1,2,\cdots,7$, we have

$$\begin{aligned} \lambda_i(x_{2n},y) &= \lambda_i(TSx_{2n-2},y) \leq \lambda_i(x_{2n-2},y) \\ &\leq \lambda_i(x_{2n-4},y) \leq \cdots \leq \lambda_i(x_0,y). \end{aligned} \tag{3.4}$$

Similarly, we obtain

$$\begin{aligned} \lambda_i(x,x_{2n+1}) &= \lambda_i(x,STx_{2n-1}) \leq \lambda_i(x,x_{2n-1}) \\ &\leq \lambda_i(x,x_{2n-3}) \leq \cdots \leq \lambda_i(x,x_1). \end{aligned} \tag{3.5}$$

We break the argument into four steps.

Step 1. $x_n \in \overline{B(x_0,r)}$ for all $n \in \mathbb{N}$.

Proof. By induction, let $n = 1$. From condition (b), we have $0 \leq \lambda < 1$. Using (3.2), we obtain $|d(x_0, Sx_0)| \leq (1-\lambda)|r| \leq |r|$. So $x_1 \in \overline{B(x_0, r)}$, and the result hold for $n = 1$. Let $x_2, x_3 ..., x_j \in \overline{B(x_0, r)}$. It is enough to show that $x_{j+1} \in \overline{B(x_0, r)}$. First suppose that $j = 2k$, then $j + 1 = 2k + 1$. Now from inequality (3.1), we have

$$\begin{aligned} d(x_{2k+1}, x_{2k}) =& d(STx_{2k-1}, Tx_{2k-1}) \preceq \lambda_1(Tx_{2k-1}, x_{2k-1}) d(Tx_{2k-1}, x_{2k-1}) \\ &+ \lambda_2(Tx_{2k-1}, x_{2k-1}) \frac{d(Tx_{2k-1}, STx_{2k-1}) d(x_{2k-1}, Tx_{2k-1})}{1 + d(Tx_{2k-1}, x_{2k-1})} \\ &+ \lambda_3(Tx_{2k-1}, x_{2k-1}) \frac{d(x_{2k-1}, STx_{2k-1}) d(Tx_{2k-1}, Tx_{2k-1})}{1 + d(Tx_{2k-1}, x_{2k-1})} \\ &+ \lambda_4(Tx_{2k-1}, x_{2k-1}) \frac{d(Tx_{2k-1}, STx_{2k-1}) d(Tx_{2k-1}, Tx_{2k-1})}{1 + d(Tx_{2k-1}, x_{2k-1})} \\ &+ \lambda_5(Tx_{2k-1}, x_{2k-1}) \frac{d(x_{2k-1}, STx_{2k-1}) d(x_{2k-1}, Tx_{2k-1})}{1 + d(Tx_{2k-1}, x_{2k-1})} \\ &+ \lambda_6(Tx_{2k-1}, x_{2k-1}) \frac{d(Tx_{2k-1}, Tx_{2k-1})(1 + d(Tx_{2k-1}, STx_{2k-1}))}{1 + d(Tx_{2k-1}, x_{2k-1})} \\ &+ \lambda_7(Tx_{2k-1}, x_{2k-1}) \frac{d(x_{2k-1}, STx_{2k-1})(1 + d(x_{2k-1}, Tx_{2k-1}))}{1 + d(Tx_{2k-1}, x_{2k-1})}, \end{aligned} \tag{3.6}$$

and hence

$$\begin{aligned} d(x_{2k+1}, x_{2k}) &= d(STx_{2k-1}, Tx_{2k-1}) \preceq \lambda_1(x_{2k}, x_{2k-1}) d(x_{2k}, x_{2k-1}) \\ &+\lambda_2(x_{2k}, x_{2k-1}) \frac{d(x_{2k}, x_{2k+1}) d(x_{2k-1}, x_{2k})}{1 + d(x_{2k}, x_{2k-1})} \\ &+\lambda_3(x_{2k}, x_{2k-1}) \frac{d(x_{2k-1}, x_{2k+1}) d(x_{2k}, x_{2k})}{1 + d(x_{2k}, x_{2k-1})} \\ &+\lambda_4(x_{2k}, x_{2k-1}) \frac{d(x_{2k}, x_{2k+1}) d(x_{2k}, x_{2k})}{1 + d(x_{2k}, x_{2k-1})} \\ &+\lambda_5(x_{2k}, x_{2k-1}) \frac{d(x_{2k-1}, x_{2k+1}) d(x_{2k-1}, x_{2k})}{1 + d(x_{2k}, x_{2k-1})} \\ &+\lambda_6(x_{2k}, x_{2k-1}) \frac{d(x_{2k}, x_{2k})(1 + d(x_{2k}, x_{2k+1}))}{1 + d(x_{2k}, x_{2k-1})} \\ &+\lambda_7(x_{2k}, x_{2k-1}) \frac{d(x_{2k-1}, x_{2k+1})(1 + d(x_{2k-1}, x_{2k}))}{1 + d(x_{2k}, x_{2k-1})}. \end{aligned} \tag{3.7}$$

Combining (3.4), (3.5) and (3.7), since $|d(x_{2k}, x_{2k-1})| \leq |1 + d(x_{2k}, x_{2k-1})|$, we get

$$\begin{aligned} |d(x_{2k+1}, x_{2k})| &= |d(STx_{2k-1}, Tx_{2k-1})| \leq \lambda_1(x_0, x_{2k-1}) |d(x_{2k}, x_{2k-1})| \\ &\quad +\lambda_2(x_0, x_{2k-1}) |d(x_{2k}, x_{2k+1})| + \lambda_5(x_0, x_{2k-1}) |d(x_{2k-1}, x_{2k+1})| \\ &\quad +\lambda_7(x_0, x_{2k-1}) |d(x_{2k-1}, x_{2k+1})| \\ &\leq \lambda_1(x_0, x_1) |d(x_{2k}, x_{2k-1})| + \lambda_2(x_0, x_1) |d(x_{2k}, x_{2k+1})| \\ &\quad +\lambda_5(x_0, x_1) [|d(x_{2k}, x_{2k-1})| + |d(x_{2k}, x_{2k+1})|] \\ &\quad +\lambda_7(x_0, x_1) [|d(x_{2k}, x_{2k-1})| + |d(x_{2k}, x_{2k+1})|], \end{aligned}$$

which implies that

$$|d(x_{2k+1}, x_{2k})| \leq \frac{\lambda_1(x_0, x_1) + \lambda_5(x_0, x_1) + \lambda_7(x_0, x_1)}{1 - \lambda_2(x_0, x_1) - \lambda_5(x_0, x_1) - \lambda_7(x_0, x_1)} |d(x_{2k}, x_{2k-1})|. \quad (3.8)$$

Similarly, if $j = 2k + 1$ then $j + 1 = 2k + 2$, and we have

$$\begin{aligned} d(x_{2k+2}, x_{2k+1}) &= d(x_{2k+1}, x_{2k+2}) = d(Sx_{2k}, TSx_{2k}) \\ &\leq \lambda_1(x_0, x_1)|d(x_{2k}, x_{2k+1})| + \lambda_2(x_0, x_1)|d(x_{2k+1}, x_{2k+2})| \\ &+ \lambda_4(x_0, x_1)[|d(x_{2k}, x_{2k+1})| + |d(x_{2k+1}, x_{2k+2})|] \\ &+ \lambda_6(x_0, x_1)[|d(x_{2k}, x_{2k+1})| + |d(x_{2k+1}, x_{2k+2})|], \end{aligned}$$

which implies that

$$|d(x_{2k+2}, x_{2k+1})| \leq \frac{\lambda_1(x_0, x_1) + \lambda_4(x_0, x_1) + \lambda_6(x_0, x_1)}{1 - \lambda_2(x_0, x_1) - \lambda_4(x_0, x_1) - \lambda_6(x_0, x_1)} |d(x_{2k+1}, x_{2k})|. \quad (3.9)$$

Since $\lambda = \max\{\frac{\lambda_1(x_0,x_1)+\lambda_4(x_0,x_1)+\lambda_6(x_0,x_1)}{1-\lambda_2(x_0,x_1)-\lambda_4(x_0,x_1)-\lambda_6(x_0,x_1)}, \frac{\lambda_1(x_0,x_1)+\lambda_5(x_0,x_1)+\lambda_7(x_0,x_1)}{1-\lambda_2(x_0,x_1)-\lambda_5(x_0,x_1)-\lambda_7(x_0,x_1)}\}$, then by (3.8) and (3.9), we conclude that $|d(x_{j+1}, x_j)| \leq \lambda |d(x_j, x_{j-1})|$ for all $j \in \mathbb{N}$. Therefore we have $|d(x_{j+1}, x_j)| \leq \lambda^j |d(x_1, x_0)|$ for all $j \in \mathbb{N}$ and

$$\begin{aligned} |d(x_{j+1}, x_0)| &\leq |d(x_{j+1}, x_j)| + \cdots + |d(x_1, x_0)| \\ &\leq \lambda^j |d(x_1, x_0)| + \cdots + |d(x_1, x_0)| \\ &= |d(x_1, x_0)|(\lambda^j + \lambda^{j-1} + \cdots + 1) \\ &\leq |(1 - \lambda)r| \frac{1 - \lambda^{j+1}}{1 - \lambda} \\ &\leq |r|. \end{aligned}$$

Hence, $x_{j+1} \in \overline{B(x_0, r)}$, and consequently $x_n \in \overline{B(x_0, r)}$ for all $n \in \mathbb{N}$ and

$$|d(x_{n+1}, x_n)| \leq \lambda |d(x_n, x_{n-1})|, \quad (3.10)$$

for all $n \in \mathbb{N}$. ∎

Step 2. $\{x_n\}$ is a Cauchy sequence.

Proof. From inequality (3.10) and applying Lemma 2.6, we get that $\{x_n\}$ is a Cauchy sequence in $\overline{B(x_0, r)}$. ∎

Step 3. T and S have a common fixed point.

Proof. Inasmuch as $\overline{B(x_0, r)}$ is complete, there exists $z \in \overline{B(x_0, r)}$ such that $lim_n x_n = z$. Now, we show that z is a fixed point of S. From rectangle inequality and (3.1), we have

$$
\begin{aligned}
d(z,Sz) &\preceq d(z,Tx_{2n+1}) + d(Tx_{2n+1},Sz) \\
&= d(z,x_{2n+2}) + d(Sz,Tx_{2n+1}) \\
&\preceq d(z,x_{2n+2}) + \lambda_1(z,x_{2n+1})d(z,x_{2n+1}) \\
&\quad +\lambda_2(z,x_{2n+1})\frac{d(z,Sz)d(x_{2n+1},Tx_{2n+1})}{1+d(z,x_{2n+1})} \\
&\quad +\lambda_3(z,x_{2n+1})\frac{d(x_{2n+1},Sz)d(z,Tx_{2n+1})}{1+d(z,x_{2n+1})} \\
&\quad +\lambda_4(z,x_{2n+1})\frac{d(z,Sz)d(z,Tx_{2n+1})}{1+d(z,x_{2n+1})} \\
&\quad +\lambda_5(z,x_{2n+1})\frac{d(x_{2n+1},Sz)d(x_{2n+1},Tx_{2n+1})}{1+d(z,x_{2n+1})} \\
&\quad +\lambda_6(z,x_{2n+1})\frac{d(z,Tx_{2n+1})(1+d(z,Sz))}{1+d(z,x_{2n+1})} \\
&\quad +\lambda_7(z,x_{2n+1})\frac{d(x_{2n+1},Sz)(1+d(x_{2n+1},Tx_{2n+1}))}{1+d(z,x_{2n+1})}.
\end{aligned}
$$

Consequently, from (3.5), we have

$$
\begin{aligned}
d(z,Sz) &\preceq d(z,x_{2n+2}) + \lambda_1(z,x_1)d(z,x_{2n+1}) \\
&\quad +\lambda_2(z,x_1)\frac{d(z,Sz)d(x_{2n+1},x_{2n+2})}{1+d(z,x_{2n+1})} \\
&\quad +\lambda_3(z,x_1)\frac{d(x_{2n+1},Sz)d(z,x_{2n+2})}{1+d(z,x_{2n+1})} \\
&\quad +\lambda_4(z,x_1)\frac{d(z,Sz)d(z,x_{2n+2})}{1+d(z,x_{2n+1})} \\
&\quad +\lambda_5(z,x_1)\frac{d(x_{2n+1},Sz)d(x_{2n+1},x_{2n+2})}{1+d(z,x_{2n+1})} \\
&\quad +\lambda_6(z,x_1)\frac{d(z,x_{2n+2})(1+d(z,Sz))}{1+d(z,x_{2n+1})} \\
&\quad +\lambda_7(z,x_1)\frac{d(x_{2n+1},Sz)(1+d(x_{2n+1},x_{2n+2}))}{1+d(z,x_{2n+1})}.
\end{aligned}
$$

Taking $n \to \infty$ results $|d(z,Sz)| \leq \lambda_7(z,x_1)|d(z,Sz)|$. By condition (b), $\lambda_7(z,x_1) < 1$. Thus $d(z,Sz) = 0$ and $z = Sz$. It follows similarly that $z = Tz$. Therefore, z is a common fixed point of S and T. ∎

Step 4. Common fixed point of S and T is unique.

Proof. Assume that $z* \in \overline{B(x_0, r)}$ is a second common fixed point of S and T. From (3.1), we have

$$\begin{aligned} d(z, z*) &= d(Sz, Tz*) \preceq \lambda_1(z, z*)d(z, z*) + \lambda_2(z, z*)\frac{d(z, Sz)d(z*, Tz*)}{1 + d(z, z*)} \\ &\quad + \lambda_3(z, z*)\frac{d(z*, Sz)d(z, z*)}{1 + d(z, z*)} + \lambda_4(z, z*)\frac{d(z, Sz)d(z, Tz*)}{1 + d(z, z*)} \\ &\quad + \lambda_5(z, z*)\frac{d(z*, Sz)d(z*, Tz*)}{1 + d(z, z*)} + \lambda_6(z, z*)\frac{d(z, Tz*)(1 + d(z, Sz))}{1 + d(z, z*)} \\ &\quad + \lambda_7(z, z*)\frac{d(z*, Sz)(1 + d(z*, Tz*))}{1 + d(z, z*)} \\ &= \lambda_1(z, z*)d(z, z*) + \lambda_3(z, z*)\frac{d(z*, z)d(z, z*)}{1 + d(z, z*)} \\ &\quad + \lambda_6(z, z*)\frac{d(z, z*)}{1 + d(z, z*)} + \lambda_7(z, z*)\frac{d(z*, z)}{1 + d(z, z*)}. \end{aligned}$$

Since $|1 + d(z, z*)| \geq |d(z, z*)|$ and $|1 + d(z, z*)| \geq 1$ we get,

$$|d(z, z*)| \leq [\lambda_1(z, z*) + \lambda_3(z, z*) + \lambda_6(z, z*) + \lambda_7(z, z*)]|d(z, z*)|.$$

From condition (b), $\lambda_1(z, z*) + \lambda_3(z, z*) + \lambda_6(z, z*) + \lambda_7(z, z*) < 1$, therefore we conclud that $|d(z, z*)| = 0$. Thus $z = z*$. This completes the proof. ∎

∎

Remark 3.2. Theorem 3.1 holds if the condition (3.2) is replaced by $|d(x_0, Tx_0)| \leq (1 - \lambda)|r|$.

Remark 3.3. Existence of the common fixed point of the mappings satisfying a contractive condition on the closed ball is a stronger condition instead of the whole space. The following example shows this fact.

Example 3.4. Let $X = \{(1, 2), (2, 3), (3, 4), (4, 5), (5, 6), (6, 7)\}$. Define the mapping $d : X \times X \to \mathbb{C}$ for all $z_1, z_2 \in X$, by

$$d(z_1, z_2) = |x_1 - x_2| + i|y_1 - y_2|,$$

where $z_1 = x_1 + iy_1$, $z_2 = x_2 + iy_2$. Then (X, d) is a complete complex valued metric space. Now, define two mappings $T, S : X \longrightarrow X$ by

$$Sz = Tz = \begin{cases} 6 + 7i & z = (5, 6) \\ \\ |\mathrm{x} - \mathrm{y}| + 2\mathrm{i}|\mathrm{x} - \mathrm{y}| & \mathrm{z} \in \{(1, 2), (2, 3), (3, 4), (4, 5), (6, 7)\} \end{cases}$$

for all $z = x + iy \in X$.

Take $z_0 = (2, 3) = 2 + 3i$ and $r = 2 + 2i$, so

$$\overline{B(z_0, r)} = \{(1, 2), (2, 3), (3, 4), (4, 5)\}.$$

Let $\lambda_i : X \times X \longrightarrow [0,1)$ where $i = 1, 2, \cdots, 7$ for all $z_1, z_2 \in \overline{B(z_0, r)}$ be defined as follows:

$$\lambda_1(z_1, z_2) = \frac{|x_1 - y_1||x_2 - y_2|}{10} + \frac{1}{24}$$
$$\lambda_i(z_1, z_2) = \frac{|x_1 - y_1||x_2 - y_2|}{24} \qquad \forall i = 2, 3, \cdots, 7.$$

Clearly, $\lambda_1(z_1, z_2)+\lambda_2(z_1, z_2)+\lambda_3(z_1, z_2)+2(\lambda_4(z_1, z_2)+\lambda_5(z_1, z_2)+\lambda_6(z_1, z_2)+\lambda_6(z_1, z_2)) < 1$ and also,

$$\begin{aligned} & \lambda_1(TSz_1, z_2) = \lambda_1(T(|x_1 - y_1| + 2i|x_1 - y_1|), z_2) \\ = & \lambda_1(||x_1 - y_1| - 2|x_1 - y_1|| + 2i||x_1 - y_1| - 2|x_1 - y_1||, z_2) \\ = & \lambda_1(|x_1 - y_1| + 2i|x_1 - y_1|, z_2) = \frac{||x_1 - y_1| - 2|x_1 - y_1|||x_2 - y_2|}{10} + \frac{1}{24} \\ = & \frac{|x_1 - y_1||x_2 - y_2|}{10} + \frac{1}{24} = \lambda_1(z_1, z_2) \end{aligned}$$

$$\begin{aligned} & \lambda_1(z_1, STz_2) = \lambda_1(z_1, S(|x_2 - y_2| + 2i|x_2 - y_2|)) \\ = & \lambda_1(z_1, ||x_2 - y_2| - 2|x_2 - y_2|| + 2i||x_2 - y_2| - 2|x_2 - y_2||) \\ = & \lambda_1(z_1, |x_2 - y_2| + 2i|x_2 - y_2|) = \frac{||x_2 - y_2| - 2|x_2 - y_2|||x_1 - y_1|}{10} + \frac{1}{24} \\ = & \frac{|x_1 - y_1||x_2 - y_2|}{10} + \frac{1}{24} = \lambda_2(z_1, z_2). \end{aligned}$$

Similarly for $i = 2, 3, \cdots, 7$ we will have

$$\lambda_i(TSz_1, z_2) \leq \lambda_i(z_1, z_2) \qquad and \qquad \lambda_i(z_1, STz_2) \leq \lambda_i(z_1, z_2).$$

Also we have,

$$\sqrt{2} = |d(z_0, Sz_0)| \leq (1 - 0.2571)2\sqrt{2},$$

where $\lambda = 0.2571$ and $|r| = 2\sqrt{2}$, so condition (3.2) holds.

We next verify inequality (3.1) of Theorem 3.1. For all $x, y \in \overline{B(z_0, r)}$ we have

$$0 \preceq d(x, y), \ \frac{d(x, Sx)d(y, Ty)}{1 + d(x, y)}, \ \frac{d(y, Sx)d(x, Ty)}{1 + d(x, y)}, \ \frac{d(x, Sx)d(x, Ty)}{1 + d(x, y)}$$
$$\frac{d(y, Sx)d(y, Ty)}{1 + d(x, y)}, \frac{d(x, Ty)(1 + d(x, Tx))}{1 + d(x, y)}, \frac{d(y, Tx)(1 + d(y, Ty))}{1 + d(x, y)}.$$

On the other hand, $d(Sz_1, Tz_2) = 0$ for all $z_1, z_2 \in \overline{B(z_0, r)}$. So we get inequality (3.1). Hence the required condition of Theorem 3.1 are satisfied and the point $(1, 2) \in \overline{B(z_0, r)}$ is a unique common fixed point of S and T.

Inequality (3.1) of Theorem 3.1 doesn't hold for the whole space X. Because for $z_1 = (1, 2)$ and $z_2 = (5, 6)$ where $z_2 \notin \overline{B(z_0, r)}$, we have

$$\begin{aligned} 5 + i5 &= d(S(1, 2), T(5, 6)) \succeq 0.8766 + 0.8604i \\ &= (\frac{1}{24} + \frac{1}{10})(4 + 4i) + 0 + \frac{1}{24}\frac{(4 + 4i)(5 + 5i)}{5 + 4i} + 0 + \frac{1}{24}\frac{(4 + 4i)(1 + i))}{5 + 4i} \\ &\quad + \frac{1}{24}\frac{(5 + 5i)(1 + 0)}{(5 + 4i)} + \frac{1}{24}\frac{(4 + 4i)(2 + i)}{(5 + 4i)}. \end{aligned}$$

If $S = T$ in Theorem 3.1, then we get the following result.

Theorem 3.5. *Let (X, d) be a complete complex valued metric space, $x_0 \in X$ and $T : X \longrightarrow X$. Suppose there exist mappings $\lambda_i : X \times X \longrightarrow [0, 1)$ where $i = 1, 2, \cdots, 7$ such that*

(a) $\lambda_i(Tx, y) \leq \lambda_i(x, y)$ *and* $\lambda_i(x, Ty) \leq \lambda_i(x, y)$,
(b) $\lambda_1(x, y) + \lambda_2(x, y) + \lambda_3(x, y) + 2\lambda_4(x, y) + 2\lambda_5(x, y) + 2\lambda_6(x, y) + 2\lambda_7(x, y) < 1$
(c)

$$\begin{aligned} d(Tx, Ty) \quad \preceq \quad & \lambda_1(x, y)d(x, y) + \lambda_2(x, y)\frac{d(x, Tx)d(y, Ty)}{1 + d(x, y)} \\ & +\lambda_3(x, y)\frac{d(y, Tx)d(x, Ty)}{1 + d(x, y)} + \lambda_4(x, y)\frac{d(x, Tx)d(x, Ty)}{1 + d(x, y)} \\ & +\lambda_5(x, y)\frac{d(y, Tx)d(y, Ty)}{1 + d(x, y)} + \lambda_6(x, y)\frac{d(x, Ty)(1 + d(x, Tx))}{1 + d(x, y)} \\ & +\lambda_7(x, y)\frac{d(y, Tx)(1 + d(y, Ty))}{1 + d(x, y)}. \end{aligned}$$

for all $x, y \in \overline{B(x_0, r)}$. If

$$|d(x_0, Tx_0)| \leq (1 - \lambda)|r|,$$

where $\lambda = \max\{\frac{\lambda_1(x_0,Tx_0)+\lambda_4(x_0,Tx_0)+\lambda_6(x_0,Tx_0)}{1-\lambda_2(x_0,Tx_0)-\lambda_4(x_0,Tx_0)-\lambda_6(x_0,Tx_0)}, \frac{\lambda_1(x_0,Tx_0)+\lambda_5(x_0,Tx_0)+\lambda_7(x_0,Tx_0)}{1-\lambda_2(x_0,Tx_0)-\lambda_5(x_0,Tx_0)-\lambda_7(x_0,Tx_0)}\}$, *then there exists a unique point $u \in \overline{B(x_0, r)}$ such that $u = Tu$.*

If we take fixed functions $\lambda_i(x, y) = \lambda_i$ where $i = 1, 2, \cdots, 7$, in Theorem 3.1, we conclude the following corollary.

Corollary 3.6. *Let (X, d) be a complete complex valued metric space and $x_0 \in X$. Let $S, T : X \longrightarrow X$ satisfying the following condition*

$$\begin{aligned} d(Sx, Ty) \quad \preceq \quad & \lambda_1 d(x, y) + \lambda_2\frac{d(x, Sx)d(y, Ty)}{1 + d(x, y)} + \lambda_3\frac{d(y, Sx)d(x, Ty)}{1 + d(x, y)} \\ & +\lambda_4\frac{d(x, Sx)d(x, Ty)}{1 + d(x, y)} + \lambda_5\frac{d(y, Sx)d(y, Ty)}{1 + d(x, y)} \\ & +\lambda_6\frac{d(y, Sy)d(x, Tx)}{1 + d(x, y)} + \lambda_7\frac{d(y, Sy)d(y, Tx)}{1 + d(x, y)}. \end{aligned}$$

for all $x, y \in \overline{B(x_0, r)}$, where λ_i for $i = 1, 2, ..., 7$ are nonnegative real numbers with $\lambda_1 + \lambda_2 + \lambda_3 + 2\lambda_4 + 2\lambda_5 + 2\lambda_6 + 2\lambda_7 < 1$.
If

$$|d(x_0, Sx_0)| \leq (1 - \lambda)|r|,$$

where $\lambda = \max\{\frac{\lambda_1+\lambda_4+\lambda_6}{1-\lambda_2-\lambda_4-\lambda_6}, \frac{\lambda_1+\lambda_5+\lambda_7}{1-\lambda_2-\lambda_5-\lambda_7}\}$, then there exists a unique point $u \in \overline{B(x_0, r)}$ such that $u = Su = Tu$.

By choosing $\lambda_i(x, y) = \lambda_i$ where $i = 1, 2, \cdots, 5$ and $\lambda_6(x, y) = \lambda_7(x, y) = 0$, in Theorem 3.1, we get the result obtained by Ahmad et al. in [7, Theorems 6].

Corollary 3.7. *Let (X, d) be a complete complex valued metric space and $S, T : X \longrightarrow X$ satisfying the following condition*

$$\begin{aligned} & d(Sx, Ty) \\ \preceq \ & \lambda_1 d(x, y) + \lambda_2 \frac{d(x, Sx)d(y, Ty)}{1 + d(x, y)} + \lambda_3 \frac{d(y, Sx)d(x, Ty)}{1 + d(x, y)} \\ & + \lambda_4 \frac{d(x, Sx)d(x, Ty)}{1 + d(x, y)} + \lambda_5 \frac{d(y, Sx)d(y, Ty)}{1 + d(x, y)}. \end{aligned}$$

for all $x, y \in \overline{B(x_0, r)}$, where λ_i for $i = 1, 2, ..., 5$ are nonnegative with $\lambda_1 + \lambda_2 + \lambda_3 + 2\lambda_4 + 2\lambda_5 < 1$. If

$$|d(x_0, Sx_0)| \leq (1 - \lambda)|r|,$$

where $\lambda = \max\{\frac{\lambda_1+\lambda_4}{1-\lambda_2-\lambda_4}, \frac{\lambda_1+\lambda_5}{1-\lambda_2-\lambda_5}\}$, then there exists a unique point $u \in \overline{B(x_0, r)}$ such that $u = Su = Tu$.

Remark 3.8. Setting $\lambda_i(x, y) = 0$ for $i = 4, 5$ in Theorem 3.1, we obtain an extension from the result obtained by Sitthikul and Saejung in [10, Theorems 2.4] to the closed ball.

Remark 3.9. By definition $\lambda_1(x, y) = \Lambda(x)$, $\lambda_2(x, y) = \Xi(x)$ and $\lambda_i(x, y) = 0$ for $i = 3, 4, 5$ in Theorem 3.1, we get an extension from Theorem 2.10 to the closed ball.

Remark 3.10. By choosing $\lambda_i(x, y) = \lambda_i$ for $i = 1, 2$, $\lambda_i(x, y) = 0$ for $i = 3, 4, 5$ in Theorem 3.1, we deduce an extension from Theorem 2.7 to the closed ball.

Remark 3.11. By letting $\lambda_i(x, y) = \lambda_i$ for $i = 1, 2, 3$, $\lambda_i(x, y) = 0$ for $i = 4, 5$ in Theorem 3.1, we get an extension from Theorem 2.8 to the closed ball.

Remark 3.12. By letting $\lambda_i(x, y) = \lambda_i$ for $i = 1, 2, 3, 4, 5$ in Theorem 3.1, we obtain an extension from Theorem 2.9 to the closed ball.

Remark 3.13. With given argument in Theorem 3.1 we can get an extension from the obtained results by Kumam et al. in [4] to the closed ball.

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