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$\beta-$Ideals of $\beta-$Subalgebras via Cubic Intuitionistic Set

Prakasam Muralikrishna[1,*], **Arsham Borumand Saeid**[2], **R. Vinodkumar**[3] **and Govindasamy Palani**[4]

[1] *Assistant Professor,* [3] *Research Scholar(PT), PG and Research Department of Mathematics, Muthurangam Government Arts College (Autonomus) (Affiliated to Thiruvalluvar University), Vellore, 632 002, India*
e-mail : pmkrishna@rocketmail.com

[2] *Department of Pure Mathematics, Faculty of Mathematics and Computer, Shahid Bahonar University of Kerman, Kerman, Iran*
e-mail : arsham@uk.ac.in

[3] *Assistant Professor, Department of Mathematics, Prathyusha Engineering College, Aranvoyalkuppam, Thiruvallur, 602 025, India*
e-mail : vinodmaths85@gmail.com

[4] *Assistant Professor, Department of Mathematics, Dr. Ambedkar Government Arts College, Chennai, 600 039, India*
e-mail : gpalani32@yahoo.co.in

Abstract Cubic intuitionistic fuzzy sets are an effective and versatile technique for encoding ambiguous data. In this paper, the notion of β-ideals have been merged with cubic intuitionistic set. The perception of cubic intuitionistic ideals of β-algebra is established with relavent results. Moreover, various properties on Cartesian product and the homomorphism of cubic intuitionistic ideals of β-algebra are studied. Further, multiplication of cubic intuitionistic $\beta-$ideals is introduced and few of its related results were investigated.

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1. Introduction

After Zadeh's[1] fuzzy set, Atanassov[2] proposed the notion of intuitionistic fuzzy sets with degrees of membership and non-membership. Aub Ayub Ansari and Chandramouleeswaran[22] established the concept of fuzzy β-subalgebras of β-algebra and discussed some of its analogous outcomes. Sujatha, Chandramouleeswaran and Muralikrishna[3] introduced the notion of intuitionistic Fuzzy $\beta-$sub algebras of $\beta-$algebras. The thought of β-algebra was explored by Neggers and Kim[4], where two operations were coupled. The notion of interval valued fuzzy β-ideals were presented by Hemavathi, Muralikrishna and Palanivel[5, 6] and also they have extended the idea of interval valued intuitionistic fuzzy β-subalgebras and dealt some fascinating results. Borumand Saeid, Muralikrishna and

*Corresponding author.

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Hemavathi[7] developed the notion of bi-normed intuitionistic fuzzy β-ideal. The idea of cubic intuitionistic structures of BCI-algebras has been initiated by Tapan Senapati, Young Bae Jun, Muhiddin and Shum [8].

The concept of cubic intuitionistic subalgebras and closed cubic intuitionistic ideals of B-algebras were discussed by Tapan Senapati, Young Bae Jun and Shum[9, 10]. Moreover, the authors initiated the conception of cubic intuitionistic structure of KU-algebras. In addition that, the Characterizations and relations of cubic intuitionistic KU-subalgebras and KU-ideals of KU-algebras are presented. Moshin Kalid [11] proposed the notion of multiplicative interpretation of neutrosophic cubic Set on B-Algebra. The conceptual interpretation of the cubic intuitionistic implicative ideals of BCK-algebras presented by Tapan Senapati [12]. Relationship between a cubic intuitionistic subalgebra, a cubic intuitionistic ideal and a cubic intuitionistic implicative ideal are also discussed. Senapati, Yager and Chen[13] were identified some impressive applications in multi-criteria decision-making based on cubic intuitionistic WASPAS technique. The idea of Cubic subalgebras and ideals have applied into the framework of BCK/BCI-algebras by Jun, Kim, Song and Kang[14]. Besides, they have presented a novel extension of cubic sets and its applications in BCK/BCI -algebras and provided various results based on their perception. Garg and Kaur[15] established the thought of Cubic Intuitionistic Fuzzy Sets and its Fundamental Properties.

Jun, Song and Kim[16] applied Cubic interval valued intuitionistic fuzzy sets into BCK and BCI- algebras. The authors discussed the relation between cubic interval valued intuitionistic fuzzy subalgebra and cubic intuitionistic fuzzy ideal in $BCK/BCI-$algebras. The novel idea of Cubic Intuitionistic q-ideals of BCI-algebras has been described by Senapati, Jana, Pal, Jun[17]. Relationship between a cubic intuitionistic subalgebra, a cubic intuitionistic ideal, and a cubic intuitionistic q-ideal is also discussed. Muralikrishna, Vinodkumar and Palani[18] have discussed some aspects on cubic fuzzy β-subalgebra of β-algebra. Recently, Muralikrishna, Davvaz, Vinodkumar and Palani[19] analysed the applications of cubic level set on $\beta-$subalgebras. Muralikrishna, Borumand Saeid, Vinodkumar and Palani[20] have provided an admirable overview of cubic intuitionistic $\beta-$subalgebras in which the conditions of $\beta-$algebra were enforced into the cubic intuitionistic fuzzy structure. Recently, Senapati, Jun, Iampan, Ronnason [21] have applied Cubic Intuitionistic Structure to Commutative Ideals of BCK-Algebras. The association between a cubic intuitionistic subalgebra, cubic intuitionistic ideal, and cubic intuitionistic commutative ideal is also considered. With all these inspiration, this paper provides the study of cubic intuitionistic β-ideals and some compelling results were presented.

2. Preliminaries

This section reveals the necessary definitions required for the work.

Definition 2.1. [4] A $\beta-$ algebra is a non-empty set $\mho$ with a constant 0 and two binary operations $+$ and $-$ satisfying the following axioms:

(i) $\mathfrak{p} - 0 = \mathfrak{p}$
(ii) $(0 - \mathfrak{p}) + \mathfrak{p} = 0$
(iii) $(\mathfrak{p} - \mathfrak{q}) - \mathfrak{r} = \mathfrak{p} - (\mathfrak{r} + \mathfrak{q})\ \forall\ \mathfrak{p}, \mathfrak{q}, \mathfrak{r} \in \mho$.

Example 2.2. The following Cayley table shows $(\mho = \{0, 1, 2, 3\}, +, -, 0)$ is a $\beta-$algebra.

$+$	0	1	2	3
0	0	1	2	3
1	1	3	0	2
2	2	0	3	1
3	3	2	1	0

$-$	0	1	2	3
0	0	2	1	3
1	1	0	3	2
2	2	3	0	1
3	3	1	2	0

Definition 2.3. [4] A non empty subset $\mathfrak{A}$ of a $\beta-$algebra $(\mho, +, -, 0)$ is called a β-subalgebra of $\mho$, if

(i) $\mathfrak{p} + \mathfrak{q} \in \mathfrak{A}$ and

(ii) $\mathfrak{p} - \mathfrak{q} \in \mathfrak{A} \quad \forall \mathfrak{p}, \mathfrak{q} \in \mathfrak{A}$.

Definition 2.4. [22] A non-empty subset $\mathfrak{I}$ of a $\beta-$algebra $(\mho, +, -, 0)$ is called a $\beta-$ideal of $\mho$, if

(i) $0 \in \mathfrak{I}$

(ii) $\mathfrak{p} + \mathfrak{q} \in \mathfrak{I}$

(iii) $\mathfrak{p} - \mathfrak{q} \& \mathfrak{q} \in \mathfrak{I}$ then $\mathfrak{p} \in \mathfrak{I} \quad \forall \mathfrak{p}, \mathfrak{q} \in \mho$.

Definition 2.5. [14] Let $\mho$ be a non empty set. By a cubic set in $\mho$ we mean a structure $C = \{\langle x, \overline{\Im}_C(\mathfrak{p}), \aleph_C(\mathfrak{p})\rangle : \ x \in \ \mho\}$
in which $\overline{\Im}_C$ is an interval valued fuzzy set in $\mho$ and $\aleph_C$ is a fuzzy set in $\mho$.

Definition 2.6. [8, 15, 16] Let $\mho$ be a non-empty set. By a Cubic intuitionistic set in $\mho$ we indicate a structure $\mathfrak{C} = \{\langle x, \Psi(\mathfrak{p}), \rho_(\mathfrak{p})\rangle : \mathfrak{p} \in \mho\}$ in which Ψ is an interval valued intuitionistic fuzzy set in $\mho$ and ρ is an intuitionistic fuzzy set in $\mho$. Since $\Psi = \{\langle \mathfrak{p}, \overline{\Im}_\Psi(\mathfrak{p}), \overline{\aleph}_\Psi(\mathfrak{p})\rangle : \mathfrak{p} \in \mho\}$ and $\rho = \{\langle \mathfrak{p}, \mathfrak{G}_\rho(\mathfrak{p}), \mathfrak{H}_\rho(\mathfrak{p})\rangle : \mathfrak{p} \in \mho\}$

Definition 2.7. [18] Let $\mathfrak{C} = \{\langle \mathfrak{p}, \overline{\Im}_\mathfrak{C}(\mathfrak{p}), \aleph_\mathfrak{C}(\mathfrak{p})\rangle : \ \mathfrak{p} \in \ \mho\}$ be a cubic set in a non empty set $\mho$. Then the set $\mathfrak{C}$ is a cubic $\beta-$ subalgebra if it satisfies the following conditions.

(i) $\overline{\Im}_\mathfrak{C}(\mathfrak{p} + \mathfrak{q}) \geq rmin\{\overline{\Im}_\mathfrak{C}(\mathfrak{p}), \overline{\Im}_\mathfrak{C}(\mathfrak{q})\} \& \overline{\Im}_\mathfrak{C}(\mathfrak{p} - \mathfrak{q}) \geq rmin\{\overline{\Im}_\mathfrak{C}(\mathfrak{p}), \overline{\Im}_\mathfrak{C}(\mathfrak{q})\}$

(ii) $\aleph_\mathfrak{C}(\mathfrak{p} + \mathfrak{q}) \leq max\{\aleph_\mathfrak{C}(\mathfrak{p}), \aleph_\mathfrak{C}(\mathfrak{q})\} \& \aleph_\mathfrak{C}(\mathfrak{p} - \mathfrak{q}) \leq max\{\aleph_\mathfrak{C}(\mathfrak{p}), \aleph_\mathfrak{C}(\mathfrak{q})\} \ \forall \mathfrak{p}, \mathfrak{q} \in \mho$

Definition 2.8. [18] Let $\mathfrak{C} = \{\langle \mathfrak{p}, \Psi(\mathfrak{p}), \rho_(\mathfrak{p})\rangle : \mathfrak{p} \in \mho\}$ be a cubic intuitionistic set in $\mho$, where Ψ is an interval valued intuitionistic fuzzy set in $\mho$ and ρ is an intuitionistic fuzzy set in $\mho$.Then the set $\mathfrak{C}$ is called a cubic intuitionistic $\beta-$subalgebra if it satisfies the following conditions:

(i) $\overline{\Im}_\Psi(\mathfrak{p} + \mathfrak{q}) \geq rmin\{\overline{\Im}_\Psi(\mathfrak{p}), \overline{\Im}_\Psi(\mathfrak{q})\} \& \overline{\Im}_\Psi(\mathfrak{p} - \mathfrak{q}) \geq rmin\{\overline{\Im}_\Psi(\mathfrak{p}), \overline{\Im}_\Psi(\mathfrak{q})\}$

(ii) $\overline{\aleph}_\Psi(\mathfrak{p} + \mathfrak{q}) \leq rmax\{\overline{\aleph}_\Psi(\mathfrak{p}), \overline{\aleph}_\Psi(\mathfrak{q})\} \& \overline{\aleph}_\Psi(\mathfrak{p} - \mathfrak{q}) \leq rmax\{\overline{\aleph}_\Psi(\mathfrak{p}), \overline{\aleph}_\Psi(\mathfrak{q})\}$

(iii) $\mathfrak{G}_\rho(\mathfrak{p} + \mathfrak{q}) \leq max\{\mathfrak{G}_\rho(\mathfrak{p}), \mathfrak{G}_\rho(\mathfrak{q})\} \& \mathfrak{G}_\rho(\mathfrak{p} - \mathfrak{q}) \leq max\{\mathfrak{G}_\rho(\mathfrak{p}), \mathfrak{G}_\rho(\mathfrak{q})\}$

(iv) $\mathfrak{H}_\rho(\mathfrak{p} + \mathfrak{q}) \geq min\{\mathfrak{H}_\rho(\mathfrak{p}), \mathfrak{H}_\rho(\mathfrak{q})\} \& \mathfrak{H}_\rho(\mathfrak{p} - \mathfrak{q}) \geq min\{\mathfrak{H}_\rho(\mathfrak{p}), \mathfrak{H}_\rho(\mathfrak{q})\} \quad \forall \mathfrak{p}, \mathfrak{q} \in \mho$

3. Cubic Intuitionistic $\beta-$Ideals

This section presents the definitions of cubic intuitionistic $\beta-$ideals of $\beta-$algebras and give some results.

Definition 3.1. Let $\mathfrak{C} = \{\mathfrak{p}, \Psi(\mathfrak{p}), \rho(\mathfrak{p}) : \mathfrak{p} \in \mho\}$ be a cubic intuitionistic set in $\mho$ is referred as a cubic intuitionistic $\beta-$ideal of $\mho$ if it satisfies the subsequent conditions for all $\mathfrak{p}, \mathfrak{q} \in \mho$

(i) $\overline{\Im}_\Psi(0) \geq \overline{\Im}_\Psi(\mathfrak{p}) \& \overline{\aleph}_\Psi(0) \leq \overline{\aleph}_\Psi(\mathfrak{p})$

(ii) $\mathfrak{G}_\rho(0) \leq \mathfrak{G}_\rho(\mathfrak{p}) \& \mathfrak{H}_\rho(0) \geq \mathfrak{H}_\rho(\mathfrak{p})$
(iii) $\overline{\Im}_\Psi(\mathfrak{p}+\mathfrak{q}) \geq rmin\{\overline{\Im}_\Psi(\mathfrak{p}), \overline{\Im}_\Psi(\mathfrak{q})\} \& \overline{\aleph}_\Psi(\mathfrak{p}+\mathfrak{q}) \leq rmax\{\overline{\aleph}_\Psi(\mathfrak{p}), \overline{\aleph}_\Psi(\mathfrak{q})\}$
(iv) $\mathfrak{G}_\rho(\mathfrak{p}+\mathfrak{q}) \leq max\{\mathfrak{G}_\rho(\mathfrak{p}), \mathfrak{G}_\rho(\mathfrak{q})\} \& \mathfrak{H}_\rho(\mathfrak{p}+\mathfrak{q}) \geq min\{\mathfrak{H}_\rho(\mathfrak{p}), \mathfrak{H}_\rho(\mathfrak{q})\}$
(v) $\overline{\Im}_\Psi(\mathfrak{p}) \geq rmin\{\overline{\Im}_\Psi(\mathfrak{p}-\mathfrak{q}), \overline{\Im}_\Psi(\mathfrak{q})\} \& \overline{\aleph}_\Psi(\mathfrak{p}) \leq rmax\{\overline{\aleph}_\Psi(\mathfrak{p}-\mathfrak{q}), \overline{\aleph}_\Psi(\mathfrak{q})\}$
(vi) $\mathfrak{G}_\rho(\mathfrak{p}) \leq max\{\mathfrak{G}_\rho(\mathfrak{p}-\mathfrak{q}), \mathfrak{G}_\rho(\mathfrak{q})\} \& \mathfrak{H}_\rho(\mathfrak{p}) \geq min\{\mathfrak{H}_\rho(\mathfrak{p}-\mathfrak{q}), \mathfrak{H}_\rho(\mathfrak{q})\}$

Example 3.2. Let $\mho = \{0, 1, 2, 3\}$ be a $\beta-$algebra with constant 0 and binary operations $+$ and $-$ are defined on $\mho$ as in the following cayley's table.

+	0	1	2	3
0	0	1	2	3
1	1	2	3	0
2	2	3	0	1
3	3	0	1	2

−	0	1	2	3
0	0	3	2	1
1	1	0	3	2
2	2	1	0	3
3	3	2	1	0

Define a Cubic intuitionistic set $\mathfrak{C} = \{\langle \mathfrak{p}, \Psi(\mathfrak{p}), \rho_(\mathfrak{p})\rangle : \mathfrak{p} \in \mho\}$ in $\mho$ as follows:

$\mathfrak{p}$	$\Psi = \langle \overline{\Im}_\Psi, \overline{\aleph}_\Psi \rangle$	$\rho = (\mathfrak{G}_\rho, \mathfrak{H}_\rho)$
0	$\langle [0.4, 0.6], [0.1, 0.4] \rangle$	$(0.4, 0.7)$
1	$\langle [0.2, 0.4], [0.3, 0.6] \rangle$	$(0.4, 0.3)$
2	$\langle [0.3, 0.5], [0.2, 0.5] \rangle$	$(0.4, 0.7)$
3	$\langle [0.2, 0.4], [0.3, 0.6] \rangle$	$(0.6, 0.3)$

Then $\mathfrak{C}$ is a Cubic intuitionistic $\beta-$ideal of $\mho$.

Theorem 3.3. *Let $\mathfrak{C} = \{\mathfrak{p}, \Psi(\mathfrak{p}), \rho(\mathfrak{p}) : \mathfrak{p} \in \mho\}$ be a Cubic intuitionistic $\beta-$ ideal of a $\beta-$algebra $\mho$. If $\mathfrak{p} \leq \mathfrak{q}$ then $\overline{\Im}_\Psi(\mathfrak{p}) \geq \overline{\Im}_\Psi(\mathfrak{q})$ & $\overline{\aleph}_\Psi(\mathfrak{p}) \leq \overline{\aleph}_\Psi(\mathfrak{q})$ and $\mathfrak{G}_\rho(\mathfrak{p}) \leq \mathfrak{G}_\rho(\mathfrak{q})$ & $\mathfrak{H}_\rho(\mathfrak{p}) \geq \mathfrak{H}_\rho(\mathfrak{q})$.*

Proof. For $\mathfrak{p}, \mathfrak{q} \in \mho, \mathfrak{p} \leq \mathfrak{q} \Rightarrow \mathfrak{p} - \mathfrak{q} = 0$ then

$$\begin{aligned} \overline{\Im}_\Psi(\mathfrak{p}) &\geq rmin\{\overline{\Im}_\Psi(\mathfrak{p}-\mathfrak{q}), \overline{\Im}_\Psi(\mathfrak{q})\} \\ &= rmin\{\overline{\Im}_\Psi(0), \overline{\Im}_\Psi(\mathfrak{q})\} \\ &= \overline{\Im}_\Psi(\mathfrak{q}) \\ Similarly, we\ can\ have\ \overline{\aleph}_\Psi(\mathfrak{p}) &\leq \overline{\aleph}_\Psi(\mathfrak{q}) \end{aligned}$$

$$\begin{aligned} \mathfrak{G}_\rho(\mathfrak{p}) &\leq max\{\mathfrak{G}_\rho(\mathfrak{p}-\mathfrak{q}), \mathfrak{G}_\rho(\mathfrak{q})\} \\ &= max\{\mathfrak{G}_\rho(0), \mathfrak{G}_\rho(\mathfrak{q})\} \\ &= \mathfrak{G}_\rho(\mathfrak{q}) \\ Similarly, we\ can\ have\ \mathfrak{H}_\rho(\mathfrak{p}) &\geq \mathfrak{H}_\rho(\mathfrak{q}) \end{aligned}$$

■

Theorem 3.4. *Let $\mathfrak{C}$ be a subset of $\mho$. Define a cubic intuitionistic set $\chi_{\mathfrak{C}} : \mho \to D[0, 1]$ such that*

$$\overline{\Im}_{\chi_{\mathfrak{C}}}(\mathfrak{p}) = \begin{cases} [t_0, t_1] & : \mathfrak{p} \in \mathfrak{C} \\ [t_2, t_3] & : \mathfrak{p} \notin \mathfrak{C} \end{cases} \qquad \overline{\aleph}_{\chi_{\mathfrak{C}}}(\mathfrak{p}) = \begin{cases} [s_0, s_1] & : \mathfrak{p} \in \mathfrak{C} \\ [s_2, s_3] & : \mathfrak{p} \notin \mathfrak{C} \end{cases}$$

$$\mathfrak{G}_{\chi_{\mathfrak{C}}}(\mathfrak{p}) = \begin{cases} k & : \ \mathfrak{p} \in \mathfrak{C} \\ l & : \ \mathfrak{p} \notin \mathfrak{C} \end{cases} \qquad \mathfrak{H}_{\chi_{\mathfrak{C}}}(\mathfrak{p}) = \begin{cases} m & : \ \mathfrak{p} \in \mathfrak{C} \\ n & : \ \mathfrak{p} \notin \mathfrak{C} \end{cases}$$

where $[t_0, t_1], [t_2, t_3], [s_0, s_1], [s_2, s_3] \in D[0, 1]$ *&* $k, l, m, n \in [0, 1]$ *with* $[t_0, t_1]$
$> [t_2, t_3], [s_0, s_1] < [s_2, s_3]$ & $k < l, m > n$.
Then $\chi_{\mathfrak{C}}$ *is a cubic intuitionistic* $\beta-$*ideal of* $\mho$ *,if and only if* $\mathfrak{C}$ *is a* $\beta-$*ideal of* $\mho$.

Proof. Suppose $\chi_{\mathfrak{C}}$ is cubic set on $\mho$.
(i) $\overline{\Im}_{\chi_{\mathfrak{C}}}(0) \geq \overline{\Im}_{\chi_{\mathfrak{C}}}(\mathfrak{p})$ $\forall \mathfrak{p} \in \mho$. Then $\overline{\Im}_{\chi_{\mathfrak{C}}}(0) = [t_0, t_1]$ or $[t_2, t_3]$ with $[t_0, t_1]$
$> [t_2, t_3]$. .... (1)
If $\overline{\Im}_{\chi_{\mathfrak{C}}}(0) = [t_0, t_1]$, then $[0, 0] \in \mathfrak{C}$ which gives $\overline{\Im}_{\chi_{\mathfrak{C}}}(0) = [t_2, t_3]$...... (2)
From (1) and (2), $[t_2, t_3] \geq \overline{\Im}_{\chi_{\mathfrak{C}}}(\mathfrak{p}) = [t_0, t_1]$, Which is a contradiction. Hence $\overline{\Im}_{\chi_{\mathfrak{C}}}(0) = [t_0, t_1]$, gives $[0, 0] \in \mathfrak{C}$.
(ii) For $\mathfrak{p}, \mathfrak{q} \in \mathfrak{C}$ we have $\overline{\Im}_{\chi_{\mathfrak{C}}}(\mathfrak{p}) = [t_0, t_1] = \overline{\Im}_{\chi_{\mathfrak{C}}}(\mathfrak{q})$. Also ,
$\overline{\Im}_{\chi_{\mathfrak{C}}}(\mathfrak{p} + \mathfrak{q}) \geq rmin\{\overline{\Im}_{\chi_{\mathfrak{C}}}(\mathfrak{p}), \overline{\Im}_{\chi_{\mathfrak{C}}}(\mathfrak{q})\} = rmin\{[t_0, t_1], [t_0, t_1]\} = [t_0, t_1]$.
Therefore $\overline{\Im}_{\chi_{\mathfrak{C}}}(\mathfrak{p} + \mathfrak{q}) = [t_0, t_1]$ yields $\mathfrak{p} + \mathfrak{q} \in C$.
(iii) For any $\mathfrak{p}, \mathfrak{q} \in \mho$ if $\mathfrak{p} - \mathfrak{q}$ & $\mathfrak{q} \in \mathfrak{C}$ then $\overline{\Im}_{\chi_{\mathfrak{C}}}(\mathfrak{p} - \mathfrak{q}) = [t_0, t_1] = \overline{\Im}_{\chi_{\mathfrak{C}}}(\mathfrak{q})$. Now, $\overline{\Im}_{\chi_{\mathfrak{C}}}(\mathfrak{p}) \geq rmin\{\overline{\Im}_{\chi_{\mathfrak{C}}}(\mathfrak{p} - \mathfrak{q}), \overline{\Im}_{\chi_{\mathfrak{C}}}(\mathfrak{q})\} = rmin\{[t_0, t_1], [t_0, t_1]\} = [t_0, t_1]$ hence $\mathfrak{p} \in \mathfrak{C}$.

(i) $\overline{\aleph}_{\chi_{\mathfrak{C}}}(0) \leq \overline{\aleph}_{\chi_{\mathfrak{C}}}(\mathfrak{p})$ $\forall \mathfrak{p} \in \mho$. We have $\overline{\aleph}_{\chi_{\mathfrak{C}}}(0) = [s_0, s_1]$ *or* $[s_2, s_3]$*with* $[s_0, s_1]$
$< [s_2, s_3]$. .... (3)
If $\overline{\aleph}_{\chi_{\mathfrak{C}}}(0) = [s_0, s_1]$, then $[1, 1] \in C$ gives $\overline{\aleph}_{\chi_{\mathfrak{C}}}(0) = [s_2, s_3]$.......... (4)
(3) and (4)$\Rightarrow$ $[s_2, s_3] \leq \overline{\aleph}_{\chi_{\mathfrak{C}}}(\mathfrak{p}) = [s_0, s_1]$, Which is a contradiction. Hence $\overline{\aleph}_{\chi_{\mathfrak{C}}}(0) = [s_0, s_1]$, gives $[1, 1] \in \mathfrak{C}$.
(ii) For $\mathfrak{p}, \mathfrak{q} \in \mathfrak{C}$ we have $\overline{\aleph}_{\chi_{\mathfrak{C}}}(\mathfrak{p}) = [s_0, s_1] = \overline{\aleph}_{\chi_{\mathfrak{C}}}(\mathfrak{q})$. Now ,
$\overline{\aleph}_{\chi_{\mathfrak{C}}}(\mathfrak{p} + \mathfrak{q}) \leq rmax\{\overline{\aleph}_{\chi_{\mathfrak{C}}}(\mathfrak{p}), \overline{\aleph}_{\chi_{\mathfrak{C}}}(\mathfrak{q})\} = rmax\{[s_0, s_1], [s_0, s_1]\} = [s_0, s_1]$.
Therefore $\overline{\aleph}_{\chi_{\mathfrak{C}}}(\mathfrak{p} + \mathfrak{q}) = [s_0, s_1]$ gives $\mathfrak{p} + \mathfrak{q} \in \mathfrak{C}$.
(iii) For any $\mathfrak{p}, \mathfrak{q} \in \mho$ if $\mathfrak{p} - \mathfrak{q}$ & $\mathfrak{q} \in \mathfrak{C}$ then $\overline{\aleph}_{\chi_{\mathfrak{C}}}(\mathfrak{p} - \mathfrak{q}) = [s_0, s_1] = \overline{\aleph}_{\chi_{\mathfrak{C}}}(\mathfrak{q})$. Also $\overline{\aleph}_{\chi_{\mathfrak{C}}}(\mathfrak{p}) \leq rmax\{\overline{\aleph}_{\chi_{\mathfrak{C}}}(\mathfrak{p} - \mathfrak{q}), \overline{\aleph}_{\chi_{\mathfrak{C}}}(\mathfrak{q})\} = rmax\{[s_0, s_1], [s_0, s_1]\} = [s_0, s_1]$ then $\mathfrak{p} \in \mathfrak{C}$.

(i) $\mathfrak{G}_{\chi_{\mathfrak{C}}}(0) \leq \mathfrak{G}_{\chi_{\mathfrak{C}}}(\mathfrak{p})$ $\forall \mathfrak{p} \in \mho$.*Then* $\mathfrak{G}_{\chi_{\mathfrak{C}}}(0) = k$ *or* l *with* $k < l$. ... (5)
If $\mathfrak{G}_{\chi_{\mathfrak{C}}}(0) = k$, then $1 \in \mathfrak{C}$. Therefore $\mathfrak{G}_{\chi_{\mathfrak{C}}}(0) = l$.......... (6)
(5) and (6) yields $l \leq \mathfrak{G}_{\chi_{\mathfrak{C}}}(\mathfrak{p}) = k$, Which is a contradiction. Hence $\mathfrak{G}_{\chi_{\mathfrak{C}}}(0) = k$, gives $1 \in \mathfrak{C}$.
(ii) For $\mathfrak{p}, \mathfrak{q} \in \mathfrak{C}$ $\Rightarrow \mathfrak{G}_{\chi_{\mathfrak{C}}}(\mathfrak{p}) = k = \mathfrak{G}_{\chi_{\mathfrak{C}}}(\mathfrak{q})$. Now ,$\mathfrak{G}_{\chi_{\mathfrak{C}}}(\mathfrak{p} + \mathfrak{q}) \leq max\{\mathfrak{G}_{\chi_{\mathfrak{C}}}(\mathfrak{p}), \mathfrak{G}_{\chi_{\mathfrak{C}}}(\mathfrak{q})\}$
$= max\{k, k\} = k$. Therefore, $\mathfrak{G}_{\chi_{\mathfrak{C}}}(\mathfrak{p}+)\mathfrak{q} = k$ then $\mathfrak{p} + \mathfrak{q} \in \mathfrak{C}$.
(iii) For any $\mathfrak{p}, \mathfrak{q} \in \mho$ if $\mathfrak{p} - \mathfrak{q}$ & $\mathfrak{q} \in \mathfrak{C}$ $\Rightarrow \mathfrak{G}_{\chi_{\mathfrak{C}}}(\mathfrak{p} - \mathfrak{q}) = k = \mathfrak{G}_{\chi_{\mathfrak{C}}}(\mathfrak{q})$.
Now $\mathfrak{G}_{\chi_{\mathfrak{C}}}(\mathfrak{p}) \leq max\{\mathfrak{G}_{\chi_{\mathfrak{C}}}(\mathfrak{p} - \mathfrak{q}), \mathfrak{G}_{\chi_C}(\mathfrak{q})\} = max\{k, k\} = k$ then $\mathfrak{p} \in \mathfrak{C}$.

(i) $\mathfrak{H}_{\chi_{\mathfrak{C}}}(0) \geq \mathfrak{H}_{\chi_{\mathfrak{C}}}(\mathfrak{p})$ then $\mathfrak{H}_{\chi_{\mathfrak{C}}}(0) = m$ *or* n *with* $m > n$......... (7). If $\mathfrak{H}_{\chi_C}(0) = m$, then $0 \in \mathfrak{C}$. Therefore $\mathfrak{H}_{\chi_{\mathfrak{C}}}(0) = n$.......... (8). From (7) and (8),$n \geq \mathfrak{H}_{\chi_{\mathfrak{C}}}(\mathfrak{p}) = m$, Which is a contradiction. Hence $\mathfrak{H}_{\chi_{\mathfrak{C}}}(0) = m$, gives $0 \in \mathfrak{C}$.
(ii) For $\mathfrak{p}, \mathfrak{q} \in \mathfrak{C}$ *then*$\mathfrak{H}_{\chi_{\mathfrak{C}}}(\mathfrak{p}) = m = \mathfrak{H}_{\chi_{\mathfrak{C}}}(\mathfrak{q})$. Now ,$\mathfrak{H}_{\chi_{\mathfrak{C}}}(x + \mathfrak{q}) \geq min\{\mathfrak{H}_{\chi_{\mathfrak{C}}}(\mathfrak{p}),$
$\mathfrak{H}_{\chi_{\mathfrak{C}}}(\mathfrak{q})\} = min\{m, m\} = m$. Therefore, $\mathfrak{H}_{\chi_{\mathfrak{C}}}(\mathfrak{p} + \mathfrak{q}) = m$ then $\mathfrak{p} + \mathfrak{q} \in \mathfrak{C}$.
(iii) For any $\mathfrak{p}, \mathfrak{q} \in \mho$ if $\mathfrak{p} - \mathfrak{q}$ & $\mathfrak{q} \in \mathfrak{C}$ $\Rightarrow \mathfrak{H}_{\chi_{\mathfrak{C}}}(\mathfrak{p} - \mathfrak{q}) = m = \mathfrak{H}_{\chi_{\mathfrak{C}}}(\mathfrak{q})$.

Now $\mathfrak{H}_{\chi_{\mathfrak{C}}}(\mathfrak{p}) \geq min\{\mathfrak{H}_{\chi_{\mathfrak{C}}}(\mathfrak{p}-\mathfrak{q}), \mathfrak{H}_{\chi_{\mathfrak{C}}}(\mathfrak{q})\} = min\{m, m\} = m \Rightarrow \mathfrak{p} \in \mathfrak{C}$. Hence $\mathfrak{C}$ is a cubic intuitionistic $\beta-$ideal of $\mho$.

Conversely, assume $\mathfrak{C}$ is cubic intuitionistic $\beta-$ideal of $\mho$, then $[0,0] \in \mathfrak{C}$ gives $\overline{\Im}_{\chi_{\mathfrak{C}}}(0) = [t_0, t_1]$. Also $Im(\overline{\Im}_{\chi_{\mathfrak{C}}}) = \{[t_0, t_1], [t_2, t_3]\}$ & $[t_0, t_1] > [t_2, t_3]$ thus $\overline{\Im}_{\chi_{\mathfrak{C}}}(0) \geq \overline{\Im}_{\chi_{\mathfrak{C}}}(\mathfrak{p})$ for all$\mathfrak{p} \in \mho$. For $\mathfrak{p}, \mathfrak{q} \in \mathfrak{C}$ we have $\mathfrak{p}+\mathfrak{q} \in \mathfrak{C}$ then $\overline{\Im}_{\chi_{\mathfrak{C}}}(\mathfrak{p}) = \overline{\Im}_{\chi_{\mathfrak{C}}}(\mathfrak{q}) = \overline{\Im}_{\chi_{\mathfrak{C}}}(\mathfrak{p}+\mathfrak{q}) = [t_0, t_1] \geq rmin\{\overline{\Im}_{\chi_{\mathfrak{C}}}(\mathfrak{p}), \overline{\Im}_{\chi_{\mathfrak{C}}}(\mathfrak{q})\}$. Hence $\overline{\Im}_{\chi_{\mathfrak{C}}}(\mathfrak{p}+\mathfrak{q}) \geq rmin\{\overline{\Im}_{\chi_{\mathfrak{C}}}(\mathfrak{p}), \overline{\Im}_{\chi_{\mathfrak{C}}}(\mathfrak{q})\}$. For $\mathfrak{p}, \mathfrak{q} \in \mho$ if $\mathfrak{p}-\mathfrak{q}$ and $\mathfrak{q} \in \mathfrak{C}$ then $\mathfrak{p} \in \mathfrak{C}$.Moreover, $\overline{\Im}_{\chi_{\mathfrak{C}}}(\mathfrak{p}) = [t_0, t_1] = rmin\{[t_0, t_1], [t_0, t_1]\} = rmin\{\overline{\Im}_{\chi_{\mathfrak{C}}}(\mathfrak{p}-\mathfrak{q}), \overline{\Im}_{\chi_{\mathfrak{C}}}(\mathfrak{q})\}$. For some $\mathfrak{p} \in \mho$ if $\mathfrak{p}-\mathfrak{q} \in \mathfrak{C}$ and $\mathfrak{q} \notin \mathfrak{C} \Rightarrow \mathfrak{p} \in \mathfrak{C}$. Then we have $\overline{\Im}_{\chi_{\mathfrak{C}}}(\mathfrak{p}) = [t_2, t_3] = rmin\{[t_0, t_1], [t_2, t_3]\} \geq rmin\{\overline{\Im}_{\chi_{\mathfrak{C}}}(\mathfrak{p}-\mathfrak{q}), \overline{\Im}_{\chi_{\mathfrak{C}}}(\mathfrak{q})\}$. Hence $\overline{\Im}_{\chi_{\mathfrak{C}}}(\mathfrak{p}) \geq rmin\{\overline{\Im}_{\chi_{\mathfrak{C}}}(\mathfrak{p}-\mathfrak{q}), \overline{\Im}_{\chi_{\mathfrak{C}}}(\mathfrak{q})\}$

$[1,1] \in \mathfrak{C}$ then $\overline{\aleph}_{\chi_{\mathfrak{C}}}(0) = [s_0, s_1]$. Also $Im(\overline{\aleph}_{\chi_{\mathfrak{C}}}) = \{[s_0, s_1], [s_2, s_3]\}$ & $[s_0, s_1] < [s_2, s_3] \Rightarrow \overline{\aleph}_{\chi_{\mathfrak{C}}}(0) \leq \overline{\aleph}_{\chi_{\mathfrak{C}}}(\mathfrak{p})$ $\forall \mathfrak{p} \in \mho$. For $\mathfrak{p}, \mathfrak{q} \in \mathfrak{C}$ gives $\mathfrak{p}+\mathfrak{q} \in \mathfrak{C}$. Then $\overline{\aleph}_{\chi_{\mathfrak{C}}}(\mathfrak{p}) = \overline{\aleph}_{\chi_{\mathfrak{C}}}(\mathfrak{q}) = \overline{\aleph}_{\chi_{\mathfrak{C}}}(\mathfrak{p}+\mathfrak{q}) = [s_0, s_1] \leq rmax\{\overline{\aleph}_{\chi_{\mathfrak{C}}}(\mathfrak{p}), \overline{\aleph}_{\chi_{\mathfrak{C}}}(\mathfrak{q})\}$.
Hence $\overline{\aleph}_{\chi_{\mathfrak{C}}}(\mathfrak{p}+\mathfrak{q}) \leq rmax\{\overline{\aleph}_{\chi_{\mathfrak{C}}}(\mathfrak{p}), \overline{\aleph}_{\chi_{\mathfrak{C}}}(\mathfrak{q})\}$. For $\mathfrak{p}, \mathfrak{q} \in \mho$ if $\mathfrak{p}-\mathfrak{q}$ and $\mathfrak{q} \in \mathfrak{C}$ then $\mathfrak{p} \in \mathfrak{C}$. Also, $\overline{\aleph}_{\chi_{\mathfrak{C}}}(\mathfrak{p}) = [s_0, s_1] = rmax\{[s_0, s_1], [s_0, s_1]\} = rmax\{\overline{\aleph}_{\chi_{\mathfrak{C}}}(\mathfrak{p}-\mathfrak{q}), \overline{\aleph}_{\chi_{\mathfrak{C}}}(\mathfrak{q})\}$. For some $\mathfrak{p} \in \mho$ if $\mathfrak{p}-\mathfrak{q} \in \mathfrak{C}$ and $\mathfrak{q} \notin \mathfrak{C}$ gives $\mathfrak{p} \in \mathfrak{C}$. Then $\overline{\aleph}_{\chi_{\mathfrak{C}}}(\mathfrak{p}) = [s_2, s_3] = rmax\{[s_0, s_1], [s_2, s_3]\} \leq rmax\{\overline{\aleph}_{\chi_{\mathfrak{C}}}(\mathfrak{p}-\mathfrak{q}), \overline{\aleph}_{\chi_{\mathfrak{C}}}(\mathfrak{q})\}$. Hence $\overline{\aleph}_{\chi_{\mathfrak{C}}}(\mathfrak{p}) \leq rmax\{\overline{\aleph}_{\chi_{\mathfrak{C}}}(\mathfrak{p}-\mathfrak{q}), \overline{\aleph}_{\chi_{\mathfrak{C}}}(\mathfrak{q})\}$

Now $1 \in \mathfrak{C}$ then $\mathfrak{G}_{\chi_{\mathfrak{C}}}(0) = k$. Also $Im(\mathfrak{G}_{\chi_{\mathfrak{C}}}) = \{k, l\}$ & $k < l$ yields $\mathfrak{G}_{\chi_{\mathfrak{C}}}(0) \leq \mathfrak{G}_{\chi_{\mathfrak{C}}}(\mathfrak{p})$ $\forall \mathfrak{p} \in \mho$. For $\mathfrak{p}, \mathfrak{q} \in \mathfrak{C} \Rightarrow \mathfrak{p}+\mathfrak{q} \in \mathfrak{C}$. Then $\mathfrak{G}_{\chi_{\mathfrak{C}}}(\mathfrak{p}) = \mathfrak{G}_{\chi_{\mathfrak{C}}}(\mathfrak{q}) = \mathfrak{G}_{\chi_{\mathfrak{C}}}(\mathfrak{p}+\mathfrak{q}) = k \leq max\{\mathfrak{G}_{\chi_{\mathfrak{C}}}(\mathfrak{p}), \mathfrak{G}_{\chi_{\mathfrak{C}}}(\mathfrak{q})\}$. Hence $\mathfrak{G}_{\chi_{\mathfrak{C}}}(\mathfrak{p}+\mathfrak{q}) \leq max\{\mathfrak{G}_{\chi_{\mathfrak{C}}}(\mathfrak{p}), \mathfrak{G}_{\chi_{\mathfrak{C}}}(\mathfrak{q}\}$. For $\mathfrak{p}, \mathfrak{q} \in \mho$ if $\mathfrak{p}-\mathfrak{q}$ and $\mathfrak{q} \in \mathfrak{C}$ implies $\mathfrak{p} \in \mathfrak{C}$. Then $\mathfrak{G}_{\chi_{\mathfrak{C}}}(\mathfrak{p}) = k = max\{k, k\} = max\{\mathfrak{G}_{\chi_{\mathfrak{C}}}(\mathfrak{p}-\mathfrak{q}), \mathfrak{G}_{\chi_{\mathfrak{C}}}(\mathfrak{q})\}$. For some $\mathfrak{p} \in \mho$ if $\mathfrak{p}-\mathfrak{q} \in \mathfrak{C}$ and $\mathfrak{q} \notin \mathfrak{C}$ then $\mathfrak{p} \in \mathfrak{C}$. Then we have $\mathfrak{G}_{\chi_{\mathfrak{C}}}(\mathfrak{p}) = l = max\{k, l\} \leq max\{\mathfrak{G}_{\chi_{\mathfrak{C}}}(\mathfrak{p}-\mathfrak{q}), \mathfrak{G}_{\chi_{\mathfrak{C}}}(\mathfrak{q})\}$. Hence $\mathfrak{G}_{\chi_{\mathfrak{C}}}(\mathfrak{p}) \leq max\{\mathfrak{G}_{\chi_{\mathfrak{C}}}(\mathfrak{p}-\mathfrak{q}), \mathfrak{G}_{\chi_{\mathfrak{C}}}(\mathfrak{q})\}$

If $0 \in \mathfrak{C}$ then $\mathfrak{H}_{\chi_{\mathfrak{C}}}(0) = k$. Also $Im(\mathfrak{H}_{\chi_{\mathfrak{C}}}) = \{m, n\}$ & $m > n$ gives $\mathfrak{H}_{\chi_{\mathfrak{C}}}(0) \geq \mathfrak{H}_{\chi_{\mathfrak{C}}}(\mathfrak{p})$ $\forall \mathfrak{p} \in \mho$.. For $\mathfrak{p}, \mathfrak{q} \in \mathfrak{C}$ we have $\mathfrak{p}+\mathfrak{q} \in \mathfrak{C}$. Then $\mathfrak{H}_{\chi_{\mathfrak{C}}}(\mathfrak{p}) = \mathfrak{H}_{\chi_{\mathfrak{C}}}(\mathfrak{q}) = \mathfrak{H}_{\chi_{\mathfrak{C}}}(\mathfrak{p}+\mathfrak{q}) = m \geq min\{\mathfrak{H}_{\chi_{\mathfrak{C}}}(\mathfrak{p}), \mathfrak{H}_{\chi_{\mathfrak{C}}}(\mathfrak{q})\}$. Hence $\mathfrak{H}_{\chi_{\mathfrak{C}}}(\mathfrak{p}+\mathfrak{q}) \geq min\{\mathfrak{H}_{\chi_{\mathfrak{C}}}(\mathfrak{p}), \mathfrak{H}_{\chi_{\mathfrak{C}}}(\mathfrak{q})\}$. For $\mathfrak{p}, \mathfrak{q} \in \mho$ if $\mathfrak{p}-\mathfrak{q}$ and $\mathfrak{q} \in \mathfrak{C} \Rightarrow \mathfrak{p} \in \mathfrak{C}$. Then $\mathfrak{H}_{\chi_{\mathfrak{C}}}(\mathfrak{p}) = m = min\{m, m\} = min\{\mathfrak{H}_{\chi_{\mathfrak{C}}}(\mathfrak{p}-\mathfrak{q}), \mathfrak{H}_{\chi_{\mathfrak{C}}}(\mathfrak{q})\}$. For some $\mathfrak{p} \in \mho$ if $\mathfrak{p}-\mathfrak{q} \in \mathfrak{C}$ and $\mathfrak{q} \notin \mathfrak{C} \Rightarrow \mathfrak{p} \in \mathfrak{C}$. Then $\mathfrak{H}_{\chi_{\mathfrak{C}}}(\mathfrak{p}) = n = min\{m, n\} \geq min\{\mathfrak{H}_{\chi_{\mathfrak{C}}}(\mathfrak{p}-\mathfrak{q}), \mathfrak{H}_{\chi_{\mathfrak{C}}}(\mathfrak{q})\}$. $\mathfrak{H}_{\chi_{\mathfrak{C}}}(\mathfrak{p}) \geq min\{\mathfrak{H}_{\chi_{\mathfrak{C}}}(\mathfrak{p}-\mathfrak{q}), \mathfrak{H}_{\chi_{\mathfrak{C}}}(\mathfrak{q})\}$. Hence$\chi_{\mathfrak{C}}$ is a cubic intuitionistic $\beta-$ideal of $\mho$. ∎

4. Product on Cubic Intuitionistic $\beta-$Ideals

This section discusses the product on cubic intuitionistic $\beta-$ideals on $\beta-$algebras and some related results.

Definition 4.1. Let $(\mho, +, -, 0)$ and $(\Theta, +, -, 0)$ be two sets.
Let $\mathfrak{A} = \{\langle \mathfrak{p}, \Psi_{\mathfrak{A}}(\mathfrak{p}), \rho_{\mathfrak{A}}(\mathfrak{p})\rangle : \mathfrak{p} \in \mho\}$ and $\mathfrak{B} = \{\langle \mathfrak{q}, \Psi_{\mathfrak{B}}(\mathfrak{q}), \rho_{\mathfrak{B}}(\mathfrak{q})\rangle : \mathfrak{q} \in \Theta\}$ be cubic intuitionistic sets in $\mho$ and Θ respectively. The Cartesian product of $\mathfrak{A}$ and $\mathfrak{B}$ denoted by $\mathfrak{A} \times \mathfrak{B}$ is defined to be the set
$\mathfrak{A} \times \mathfrak{B} = \{\langle (\mathfrak{p}, \mathfrak{q}), \Psi_{\mathfrak{A}\times\mathfrak{B}}(\mathfrak{p}, \mathfrak{q}), \rho_{\mathfrak{A}\times\mathfrak{B}}(\mathfrak{p}, \mathfrak{q})\rangle : (\mathfrak{p}, \mathfrak{q}) \in \mho \times \Theta\}$
where $\Psi_{\mathfrak{A}\times\mathfrak{B}} = [\overline{\Im}_{\mathfrak{A}\times\mathfrak{B}}, \overline{\aleph}_{\mathfrak{A}\times\mathfrak{B}}]$ & $\rho_{\mathfrak{A}\times\mathfrak{B}} = (\mathfrak{G}_{\mathfrak{A}\times\mathfrak{B}}, \mathfrak{H}_{\mathfrak{A}\times\mathfrak{B}})$ and
$\overline{\Im}_{\mathfrak{A}\times\mathfrak{B}} : \mho \times \Theta \to D[0,1]$ is given by $\overline{\Im}_{\mathfrak{A}\times\mathfrak{B}}(\mathfrak{p}, \mathfrak{q}) = rmin\{\overline{\Im}_{\mathfrak{A}}(\mathfrak{p}), \overline{\Im}(\mathfrak{q})\}$,

$\overline{\aleph}_{\mathfrak{A}\times\mathfrak{B}} : \mho \times \Theta \to D[0,1]$ is given by $\overline{\aleph}_{\mathfrak{A}\times\mathfrak{B}}(\mathfrak{p},\mathfrak{q}) = rmax\{\overline{\aleph}_{\mathfrak{A}}(\mathfrak{p}), \overline{\aleph}_{\mathfrak{B}}(\mathfrak{q})\}$,
$\mathfrak{G}_{\mathfrak{A}\times\mathfrak{B}} : \mho \times \Theta \to [0,1]$ is given by $\mathfrak{G}_{\mathfrak{A}\times\mathfrak{B}}(\mathfrak{p},\mathfrak{q}) = max\{\mathfrak{G}_{\mathfrak{A}}(\mathfrak{p}), \mathfrak{G}_{\mathfrak{B}}(\mathfrak{q})\}$ and $\mathfrak{H}_{\mathfrak{A}\times\mathfrak{B}} : \mho \times \Theta \to [0,1]$ is given by $\mathfrak{H}_{\mathfrak{A}\times\mathfrak{B}}(\mathfrak{p},\mathfrak{q}) = min\{\mathfrak{H}_{\mathfrak{A}}(\mathfrak{p}), \mathfrak{H}_{\mathfrak{B}}(\mathfrak{q})\}$

Theorem 4.2. *Let $\mathfrak{A} = \{\langle \mathfrak{p}, \Psi_{\mathfrak{A}}(\mathfrak{p}), \rho_{\mathfrak{A}}(\mathfrak{p})\rangle : \mathfrak{p} \in \mho\}$ and $\mathfrak{B} = \{\langle \mathfrak{q}, \Psi_{\mathfrak{B}}(\mathfrak{q}), \rho_{\mathfrak{B}}(\mathfrak{q})\rangle : \mathfrak{q} \in \Theta\}$ be two cubic intuitionistic $\beta-$ideals of $\mho$ and Θ respectively. Then $\mathfrak{A}\times\mathfrak{B}$ is also a cubic intuitionistic $\beta-$ ideal of $\mho \times \Theta$.*

Proof. Let $\mathfrak{A} = \{\langle \mathfrak{p}, \Psi_{\mathfrak{A}}(\mathfrak{p}), \rho_{\mathfrak{A}}(\mathfrak{p})\rangle : \mathfrak{p} \in \mho\}$ and $\mathfrak{B} = \{\langle \mathfrak{q}, \Psi_{\mathfrak{B}}(\mathfrak{q}), \rho_{\mathfrak{B}}(\mathfrak{q})\rangle : \mathfrak{q} \in \Theta\}$ be two cubic intuitionistic subsets in $\mho$ and Θ respectively.
Take $(\mathfrak{p},\mathfrak{q}) \in \mho \times \Theta$

$$\begin{aligned}
\overline{\Im}_{\mathfrak{A}\times\mathfrak{B}}(0,0) &\geq rmin\{\overline{\Im}_{\mathfrak{A}\times\mathfrak{B}}(0), \overline{\Im}_{\mathfrak{A}\times\mathfrak{B}}(0)\}\\
&= rmin\{\overline{\Im}_{\mathfrak{A}\times\mathfrak{B}}(\mathfrak{p}), \overline{\Im}_{\mathfrak{A}\times\mathfrak{B}}(\mathfrak{q})\}\\
&= \overline{\Im}_{\mathfrak{A}\times\mathfrak{B}}(\mathfrak{p},\mathfrak{q})
\end{aligned}$$

$$\begin{aligned}
\overline{\aleph}_{\mathfrak{A}\times\mathfrak{B}}(0,0) &\leq rmax\{\overline{\aleph}_{\mathfrak{A}\times\mathfrak{B}}(0), \overline{\aleph}_{\mathfrak{A}\times\mathfrak{B}}(0)\}\\
&= rmax\{\overline{\aleph}_{\mathfrak{A}\times\mathfrak{B}}(\mathfrak{p}), \overline{\aleph}_{\mathfrak{A}\times\mathfrak{B}}(\mathfrak{q})\}\\
&= \overline{\aleph}_{\mathfrak{A}\times\mathfrak{B}}(\mathfrak{p},\mathfrak{q})
\end{aligned}$$

$$\begin{aligned}
\mathfrak{G}_{\mathfrak{A}\times\mathfrak{B}}(0,0) &\leq max\{\mathfrak{G}_{\mathfrak{A}\times\mathfrak{B}}(0), \mathfrak{G}_{\mathfrak{A}\times\mathfrak{B}}(0)\}\\
&= max\{\mathfrak{G}_{\mathfrak{A}\times\mathfrak{B}}(\mathfrak{p}), \mathfrak{G}_{\mathfrak{A}\times\mathfrak{B}}(\mathfrak{q})\}\\
&= \mathfrak{G}_{\mathfrak{A}\times\mathfrak{B}}(\mathfrak{p},\mathfrak{q})
\end{aligned}$$

$$\begin{aligned}
\mathfrak{H}_{\mathfrak{A}\times\mathfrak{B}}(0,0) &\geq min\{\mathfrak{H}_{\mathfrak{A}\times\mathfrak{B}}(0), \mathfrak{H}_{\mathfrak{A}\times\mathfrak{B}}(0)\}\\
&= min\{\mathfrak{H}_{\mathfrak{A}\times\mathfrak{B}}(\mathfrak{p}), \mathfrak{H}_{A\times\mathfrak{B}}(\mathfrak{q})\}\\
&= \mathfrak{H}_{\mathfrak{A}\times\mathfrak{B}}(\mathfrak{p},\mathfrak{q})
\end{aligned}$$

Take $(a,b) \in \mho \times \Theta$, where $a = (\mathfrak{p}_1,\mathfrak{q}_1)$ & $b = (\mathfrak{p}_2,\mathfrak{q}_2)$. Then we have $\overline{\Im}_{\mathfrak{A}\times\mathfrak{B}}(a+b) \geq rmin\{\overline{\Im}_{\mathfrak{A}\times\mathfrak{B}}(a), \overline{\Im}_{\mathfrak{A}\times\mathfrak{B}}(b)\}$ & $\overline{\aleph}_{\mathfrak{A}\times\mathfrak{B}}(a+b) \leq rmax\{\overline{\aleph}_{\mathfrak{A}\times\mathfrak{B}}(a), \overline{\aleph}_{\mathfrak{A}\times\mathfrak{B}}(b)\}$ and $\mathfrak{G}_{\mathfrak{A}\times\mathfrak{B}}(a+b) \leq max\{\mathfrak{G}_{\mathfrak{A}\times\mathfrak{B}}(a), \mathfrak{G}_{\mathfrak{A}\times\mathfrak{B}}(b)\}$ & $\mathfrak{H}_{\mathfrak{A}\times\mathfrak{B}}(a+b) \geq min\{\mathfrak{H}_{\mathfrak{A}\times\mathfrak{B}}(a), \mathfrak{H}_{A\times\mathfrak{B}}(b)\}$. Now,

$$\begin{aligned}
\overline{\Im}_{\mathfrak{A}\times\mathfrak{B}}(a) &= \overline{\Im}_{\mathfrak{A}\times\mathfrak{B}}(\mathfrak{p}_1,\mathfrak{q}_1)\\
&= rmin\{\overline{\Im}_{\mathfrak{A}\times\mathfrak{B}}(\mathfrak{p}_1), \overline{\Im}_{\mathfrak{A}\times\mathfrak{B}}(\mathfrak{q}_1)\}\\
&\geq rmin\{rmin\{\overline{\Im}_{\mathfrak{A}}(\mathfrak{p}_1-\mathfrak{p}_2), \overline{\Im}_{\mathfrak{A}}(\mathfrak{p}_2)\}, rmin\{\overline{\Im}_{\mathfrak{B}}(\mathfrak{q}_1-\mathfrak{q}_2), \overline{\Im}_{\mathfrak{B}}(\mathfrak{q}_2)\}\}\\
&\geq rmin\{rmin\{\overline{\Im}_{\mathfrak{A}}(\mathfrak{p}_1-\mathfrak{p}_2), \overline{\Im}_{\mathfrak{B}}(\mathfrak{q}_1-\mathfrak{q}_2)\}, rmin\{\overline{\Im}_{\mathfrak{A}}(\mathfrak{p}_2), \overline{\Im}_{\mathfrak{B}}(\mathfrak{q}_2)\}\}\\
&= rmin\{\overline{\Im}_{\mathfrak{A}\times\mathfrak{B}}((\mathfrak{p}_1,\mathfrak{q}_1)-(\mathfrak{p}_2,\mathfrak{q}_2)), \overline{\Im}_{\mathfrak{A}\times\mathfrak{B}}(\mathfrak{p}_2,\mathfrak{q}_2)\}\\
&= rmin\{\overline{\Im}_{\mathfrak{A}\times\mathfrak{B}}(a-b), \overline{\Im}_{\mathfrak{A}\times\mathfrak{B}}(b)\}
\end{aligned}$$

$$\begin{aligned}
\overline{\aleph}_{\mathfrak{A}\times\mathfrak{B}}(a) &= \overline{\aleph}_{\mathfrak{A}\times\mathfrak{B}}(\mathfrak{p}_1,\mathfrak{q}_1)\\
&= rmax\{\overline{\aleph}_{\mathfrak{A}\times\mathfrak{B}}(\mathfrak{p}_1), \overline{\aleph}_{\mathfrak{A}\times\mathfrak{B}}(\mathfrak{q}_1)\}
\end{aligned}$$

$$\leq rmax\{rmax\{\overline{\aleph}_{\mathfrak{A}}(\mathfrak{p}_1-\mathfrak{p}_2),\overline{\aleph}_{\mathfrak{A}}(\mathfrak{p}_2)\},rmax\{\overline{\aleph}_{\mathfrak{B}}(\mathfrak{q}_1-\mathfrak{q}_2),\overline{\aleph}_{\mathfrak{B}}(\mathfrak{q}_2)\}\}$$
$$\leq rmax\{rmax\{\overline{\aleph}_{\mathfrak{A}}(\mathfrak{p}_1-\mathfrak{p}_2),\overline{\aleph}_{\mathfrak{B}}(\mathfrak{q}_1-\mathfrak{q}_2)\},rmax\{\overline{\aleph}_{\mathfrak{A}}(\mathfrak{p}_2),\overline{\aleph}_{\mathfrak{B}}(\mathfrak{q}y_2)\}\}$$
$$= rmax\{\overline{\aleph}_{\mathfrak{A}\times\mathfrak{B}}((\mathfrak{p}_1,\mathfrak{q}_1)-(\mathfrak{p}_2,\mathfrak{q}_2)),\overline{\aleph}_{\mathfrak{A}\times\mathfrak{B}}(\mathfrak{p}_2,\mathfrak{q}_2)\}$$
$$= rmax\{\overline{\aleph}_{\mathfrak{A}\times\mathfrak{B}}(a-b),\overline{\aleph}_{\mathfrak{A}\times\mathfrak{B}}(b)\}$$

$$\begin{aligned}
\mathfrak{G}_{\mathfrak{A}\times\mathfrak{B}}(a) &= \mathfrak{G}_{\mathfrak{A}\times\mathfrak{B}}(\mathfrak{p}_1,\mathfrak{q}_1)\\
&= max\{\mathfrak{G}_{\mathfrak{A}\times\mathfrak{B}}(\mathfrak{p}_1),\mathfrak{G}_{\mathfrak{A}\times\mathfrak{B}}(\mathfrak{q}_1)\}\\
&\leq max\{max\{\mathfrak{G}_{\mathfrak{A}}(\mathfrak{p}_1-\mathfrak{p}_2),\mathfrak{G}_{\mathfrak{A}}(\mathfrak{p}_2)\},max\{\mathfrak{G}_{\mathfrak{B}}(\mathfrak{q}_1-\mathfrak{q}_2),\mathfrak{G}_{\mathfrak{B}}(\mathfrak{q}_2)\}\}\\
&\leq max\{max\{\mathfrak{G}_{\mathfrak{A}}(\mathfrak{p}_1-\mathfrak{p}_2),\mathfrak{G}_{\mathfrak{B}}(\mathfrak{q}_1-\mathfrak{q}_2)\},max\{\mathfrak{G}_{\mathfrak{A}}(\mathfrak{p}_2),\mathfrak{G}_{\mathfrak{B}}(\mathfrak{q}_2)\}\}\\
&= max\{\mathfrak{G}_{\mathfrak{A}\times\mathfrak{B}}((\mathfrak{p}_1,\mathfrak{q}_1)-(\mathfrak{p}_2,\mathfrak{q}_2)),\mathfrak{G}_{\mathfrak{A}\times\mathfrak{B}}(\mathfrak{p}_2,\mathfrak{q}_2)\}\\
&= max\{\mathfrak{G}_{\mathfrak{A}\times\mathfrak{B}}(a-b),\mathfrak{G}_{\mathfrak{A}\times\mathfrak{B}}(b)\}.
\end{aligned}$$

$$\begin{aligned}
\mathfrak{H}_{\mathfrak{A}\times\mathfrak{B}}(a) &= \mathfrak{H}_{\mathfrak{A}\times\mathfrak{B}}(\mathfrak{p}_1,\mathfrak{q}_1)\\
&= min\{\mathfrak{H}_{\mathfrak{A}\times\mathfrak{B}}(\mathfrak{p}_1),\mathfrak{H}_{\mathfrak{A}\times\mathfrak{B}}(\mathfrak{q}_1)\}\\
&\geq min\{min\{\mathfrak{H}_{\mathfrak{A}}(\mathfrak{p}_1-\mathfrak{p}_2),\mathfrak{H}_{\mathfrak{A}}(\mathfrak{p}_2)\},min\{\mathfrak{H}_{\mathfrak{B}}(\mathfrak{q}_1-\mathfrak{q}_2),\mathfrak{H}_{\mathfrak{B}}(y_2)\}\}\\
&\geq min\{min\{\mathfrak{H}_{\mathfrak{A}}(\mathfrak{p}_1-\mathfrak{p}_2),\mathfrak{H}_{\mathfrak{B}}(\mathfrak{q}_1-\mathfrak{q}_2)\},min\{\mathfrak{H}_{\mathfrak{A}}(\mathfrak{p}_2),\mathfrak{H}_{\mathfrak{B}}(\mathfrak{q}_2)\}\}\\
&= min\{\mathfrak{H}_{\mathfrak{A}\times\mathfrak{B}}((\mathfrak{p}_1,\mathfrak{q}_1)-(\mathfrak{p}_2,\mathfrak{q}_2)),\mathfrak{H}_{\mathfrak{A}\times\mathfrak{B}}(\mathfrak{p}_2,\mathfrak{q}_2)\}\\
&= min\{\mathfrak{H}_{\mathfrak{A}\times\mathfrak{B}}(a-b),\mathfrak{H}_{\mathfrak{A}\times\mathfrak{B}}(b)\}.
\end{aligned}$$

$\mathfrak{A}\times\mathfrak{B}$ is a cubic intuitionistic $\beta-$ideal of $\mho\times\Theta$. ∎

Lemma 4.3. *Let $\mathfrak{A}$ and $\mathfrak{B}$ be two cubic intuitionistic subsets of $\mho$ and Θ respectively. Then $\mathfrak{A}\times\mathfrak{B}$ is a cubic intuitionistic $\beta-$ ideal of $\mho\times\Theta$ then $\overline{\Im}_{\mathfrak{A}}(0)\geq\overline{\Im}_{\mathfrak{A}}(\mathfrak{p}),\overline{\Im}_{\mathfrak{B}}(0)\geq\overline{\Im}_{\mathfrak{B}}(\mathfrak{q})$ & $\overline{\aleph}_{\mathfrak{A}}(0)\leq\overline{\aleph}_{\mathfrak{A}}(\mathfrak{p}),\overline{\aleph}_{\mathfrak{B}}(0)\leq\overline{\aleph}_{\mathfrak{B}}(\mathfrak{q})$ and $\mathfrak{G}_{\mathfrak{A}}(0)\leq\mathfrak{G}_{\mathfrak{A}}(\mathfrak{p}),\mathfrak{G}_{\mathfrak{B}}(0)\leq\mathfrak{G}_{\mathfrak{B}}(\mathfrak{q})$ & $\mathfrak{H}_{\mathfrak{A}}(0)\leq\mathfrak{H}_{\mathfrak{A}}(\mathfrak{p}),\mathfrak{H}_{\mathfrak{B}}(0)\leq\mathfrak{H}_{\mathfrak{B}}(\mathfrak{q})$.*

Proof. Let $\mathfrak{A}$ and $\mathfrak{B}$ be two cubic intuitionistic subsets of $\mho$ and Θ
Suppose $\overline{\Im}_{\mathfrak{A}}(\mathfrak{p})\geq\overline{\Im}_{\mathfrak{A}}(0)$ and $\overline{\Im}_{\mathfrak{B}}(\mathfrak{q})\geq\overline{\Im}_{\mathfrak{B}}(0)$
$\overline{\aleph}_{\mathfrak{A}}(0)\geq\overline{\aleph}_{\mathfrak{A}}(\mathfrak{p})$ and $\overline{\aleph}_{\mathfrak{B}}(0)\geq\overline{\aleph}_{\mathfrak{B}}(\mathfrak{q})$ for some $\mathfrak{p}\in\mho,\mathfrak{q}\in\Theta$. Then

$$\begin{aligned}
\overline{\Im}_{\mathfrak{A}\times\mathfrak{B}}(\mathfrak{p},\mathfrak{q}) &\geq rmin\{\overline{\Im}_{\mathfrak{A}}(\mathfrak{p}),\overline{\Im}_{\mathfrak{B}}(\mathfrak{q})\}\\
&= rmin\{\overline{\Im}_{\mathfrak{A}}(0),\overline{\Im}_{\mathfrak{B}}(0)\}\\
&= \overline{\Im}_{\mathfrak{A}\times\mathfrak{B}}(0,0)
\end{aligned}$$

Similarly, we can get $\overline{\aleph}_{\mathfrak{A}\times\mathfrak{B}}(\mathfrak{p},\mathfrak{q})\leq\overline{\aleph}_{\mathfrak{A}\times\mathfrak{B}}(0,0)$.
Suppose $\mathfrak{G}_{\mathfrak{A}}(0)\geq\mathfrak{G}_{\mathfrak{A}}(\mathfrak{p})$ and $\mathfrak{G}_{\mathfrak{B}}(0)\geq\mathfrak{G}_{\mathfrak{B}}(\mathfrak{q})$
$\mathfrak{H}_{\mathfrak{A}}(\mathfrak{p})\geq\mathfrak{H}_{\mathfrak{A}}(0)$ and $\mathfrak{H}_{\mathfrak{B}}(\mathfrak{q})\geq\mathfrak{H}_{\mathfrak{B}}(0)$ for some $\mathfrak{p}\in\mho,\mathfrak{q}\in\Theta$.Then

$$\begin{aligned}
\mathfrak{G}_{\mathfrak{A}\times\mathfrak{B}}(\mathfrak{p},\mathfrak{q}) &\leq max\{\mathfrak{G}_{\mathfrak{A}}(\mathfrak{p}),\aleph_{\mathfrak{B}}(\mathfrak{q})\}\\
&= max\{\mathfrak{G}_{\mathfrak{A}}(0),\mathfrak{G}_{\mathfrak{B}}(0)\}\\
&= \mathfrak{G}_{\mathfrak{A}\times\mathfrak{B}}(0,0)
\end{aligned}$$

Similarly, we can have $\mathfrak{H}_{\mathfrak{A}\times\mathfrak{B}}(\mathfrak{p},\mathfrak{q})\geq\mathfrak{H}_{\mathfrak{A}\times\mathfrak{B}}(0,0)$. Which is a contradiction, proving the result. ∎

Theorem 4.4. *Let $\mathfrak{A}$ and $\mathfrak{B}$ be two cubic intuitionistic subsets of $\mho$ and Θ such that $\mathfrak{A}\times\mathfrak{B}$ is also a cubic intuitionistic $\beta-$ ideal of $\mho\times\Theta$. Then either $\mathfrak{A}$ is a cubic intuitionistic $\beta-$ideal of $\mho$ or $\mathfrak{B}$ is a cubic intuitionistic $\beta-$ideal of Θ.*

Proof. Now by lemma 5.3,let us take
$\overline{\Im}_{\mathfrak{A}}(0)\geq\overline{\Im}_{\mathfrak{A}}(\mathfrak{p}),\overline{\Im}_{\mathfrak{B}}(0)\geq\overline{\Im}_{\mathfrak{B}}(\mathfrak{q})$ & $\overline{\aleph}_{\mathfrak{A}}(0)\leq\overline{\aleph}_{\mathfrak{A}}(\mathfrak{p}),\overline{\aleph}_{\mathfrak{B}}(0)\leq\overline{\aleph}_{\mathfrak{B}}(\mathfrak{q})$ (1)
$\mathfrak{G}_{\mathfrak{A}}(0)\leq\mathfrak{G}_{\mathfrak{A}}(\mathfrak{p}),\mathfrak{G}_{\mathfrak{B}}(0)\leq\mathfrak{G}_{\mathfrak{B}}(\mathfrak{q})$ & $\mathfrak{H}_{\mathfrak{A}}(0)\leq\mathfrak{H}_{\mathfrak{A}}(\mathfrak{p}),\mathfrak{H}_{\mathfrak{B}}(0)\leq\mathfrak{H}_{\mathfrak{B}}(\mathfrak{q})$ (2) then $\overline{\Im}_{\mathfrak{A}\times\mathfrak{B}}(0,\mathfrak{q})$ $=rmin\{\overline{\Im}_{\mathfrak{A}}(0),\overline{\Im}_{\mathfrak{B}}(\mathfrak{q})\}$ & $\overline{\aleph}_{\mathfrak{A}\times\mathfrak{B}}(0,\mathfrak{q})=rmax\{\overline{\aleph}_{\mathfrak{A}}(0),\overline{\aleph}_{\mathfrak{B}}(\mathfrak{q})\}$
and $\mathfrak{G}_{\mathfrak{A}\times\mathfrak{B}}(0,\mathfrak{q})=max\{\mathfrak{G}_{\mathfrak{A}}(0),\mathfrak{G}_{B}(\mathfrak{q})\}$ & $\mathfrak{H}_{\mathfrak{A}\times\mathfrak{B}}(0,\mathfrak{q})=min\{\mathfrak{H}_{\mathfrak{A}}(0),\mathfrak{H}_{\mathfrak{B}}(\mathfrak{q})\}$
Since $\mathfrak{A}\times\mathfrak{B}$ is a cubic intuitionistic $\beta-$ideals of $\mho\times\Theta$,

$$\overline{\Im}_{\mathfrak{A}\times\mathfrak{B}}((\mathfrak{p}_1,\mathfrak{q}_1),(\mathfrak{p}_2,\mathfrak{q}_2))\geq rmin\{\overline{\Im}_{\mathfrak{A}\times\mathfrak{B}}((\mathfrak{p}_1,\mathfrak{q}_1)-(\mathfrak{p}_2,\mathfrak{q}_2)),\overline{\Im}_{\mathfrak{A}\times\mathfrak{B}}(\mathfrak{p}_2,\mathfrak{q}_2)\}$$

and since

$$\overline{\Im}_{\mathfrak{A}\times\mathfrak{B}}((\mathfrak{p}_1,\mathfrak{q}_1),(\mathfrak{p}_2,\mathfrak{q}_2))\geq rmin\{\overline{\Im}_{\mathfrak{A}\times\mathfrak{B}}((\mathfrak{p}_1,\mathfrak{q}_1)-(\mathfrak{p}_2,\mathfrak{q}_2)),\overline{\Im}_{\mathfrak{A}\times\mathfrak{B}}((\mathfrak{p}_2,\mathfrak{q}_2)\}$$
$$\overline{\Im}_{\mathfrak{A}\times\mathfrak{B}}((\mathfrak{p}_1,\mathfrak{q}_1),(\mathfrak{p}_2,\mathfrak{q}_2))\geq rmin\{\overline{\Im}_{\mathfrak{A}\times\mathfrak{B}}((\mathfrak{p}_1-\mathfrak{p}_2),(\mathfrak{q}_1-y_2)),\overline{\Im}_{\mathfrak{A}\times\mathfrak{B}}(\mathfrak{p}_2,y_2)\}$$
$$\overline{\Im}_{\mathfrak{A}\times\mathfrak{B}}((\mathfrak{p}_1-\mathfrak{p}_2),(y_1-\mathfrak{q}_2))\geq rmin\{\overline{\Im}_{\mathfrak{A}\times\mathfrak{B}}((\mathfrak{p}_1,\mathfrak{q}_1),\overline{\Im}_{\mathfrak{A}\times\mathfrak{B}}(\mathfrak{p}_2,\mathfrak{q}_2)\}\ (3)$$

Putting $\mathfrak{p}_1=\mathfrak{p}_2=0$ in (3) Then
$\overline{\Im}_{\mathfrak{A}\times\mathfrak{B}}((0,\mathfrak{q}_1)\geq rmin\{\overline{\Im}_{\mathfrak{A}\times\mathfrak{B}}(0,(\mathfrak{q}_1-\mathfrak{q}_2)),\overline{\Im}_{\mathfrak{A}\times\mathfrak{B}}(0,\mathfrak{q}_2)\}$ and

$$\overline{\Im}_{\mathfrak{A}\times\mathfrak{B}}(0,(\mathfrak{q}_1-\mathfrak{q}_2))\geq rmin\{\overline{\Im}_{\mathfrak{A}\times\mathfrak{B}}(0,\mathfrak{q}_1),\overline{\Im}_{\mathfrak{A}\times\mathfrak{B}}(0,\mathfrak{q}_2)\}\ (4)$$

Using equations (1) in (4)

$$\Rightarrow\overline{\Im}_{\mathfrak{B}}(\mathfrak{q}_1)\geq rmin\{\{\overline{\Im}_{\mathfrak{B}}(\mathfrak{q}_1-\mathfrak{q}_2),\overline{\Im}_{\mathfrak{B}}(\mathfrak{q}_2)\}$$

and $\overline{\Im}_{\mathfrak{B}}(\mathfrak{q}_1-\mathfrak{q}_2)\geq rmin\{\overline{\Im}_{\mathfrak{B}}(\mathfrak{q}_1),\overline{\Im}_{\mathfrak{B}}(\mathfrak{q}_2)\}$
Similarly,
$\overline{\aleph}_{\mathfrak{B}}(\mathfrak{q}_1)\leq rmax\{\{\overline{\aleph}_{\mathfrak{B}}(\mathfrak{q}_1-\mathfrak{q}_2),\overline{\aleph}_{\mathfrak{B}}(\mathfrak{q}_2)\}$ and $\overline{\aleph}_{\mathfrak{B}}(\mathfrak{q}_1-\mathfrak{q}_2)\leq rmax\{\overline{\aleph}_{\mathfrak{B}}(\mathfrak{q}_1),\overline{\aleph}_{\mathfrak{B}}(\mathfrak{q}_2)\}$

Also, $\mathfrak{G}_{\mathfrak{A}\times\mathfrak{B}}((\mathfrak{p}_1,\mathfrak{q}_1),(\mathfrak{p}_2,\mathfrak{q}_2))\leq max\{\mathfrak{G}_{\mathfrak{A}\times\mathfrak{B}}((\mathfrak{p}_1,\mathfrak{q}_1)-(\mathfrak{p}_2,\mathfrak{q}_2)),\mathfrak{G}_{\mathfrak{A}\times\mathfrak{B}}(\mathfrak{p}_2,\mathfrak{q}_2)\}$ and

$$\mathfrak{G}_{\mathfrak{A}\times\mathfrak{B}}((\mathfrak{p}_1,\mathfrak{q}_1),(\mathfrak{p}_2,\mathfrak{q}_2))\leq max\{\mathfrak{G}_{\mathfrak{A}\times\mathfrak{B}}((\mathfrak{p}_1,\mathfrak{q}_1)-(\mathfrak{p}_2,\mathfrak{q}_2)),\mathfrak{G}_{\mathfrak{A}\times\mathfrak{B}}((\mathfrak{p}_2,\mathfrak{q}_2)\}$$
$$\mathfrak{G}_{\mathfrak{A}\times\mathfrak{B}}((\mathfrak{p}_1,\mathfrak{q}_1),(\mathfrak{p}_2,\mathfrak{q}_2))\leq max\{\mathfrak{G}_{\mathfrak{A}\times\mathfrak{B}}((\mathfrak{p}_1-\mathfrak{p}_2),(\mathfrak{q}_1-\mathfrak{q}_2)),\mathfrak{G}_{\mathfrak{A}\times\mathfrak{B}}(\mathfrak{p}_2,\mathfrak{q}_2)\}$$
$$\mathfrak{G}_{\mathfrak{A}\times}\mathfrak{B}((\mathfrak{p}_1-\mathfrak{p}_2),(\mathfrak{q}_1-\mathfrak{q}_2))\leq max\{\mathfrak{G}_{\mathfrak{A}\times\mathfrak{B}}((\mathfrak{p}_1,\mathfrak{q}_1),\mathfrak{G}_{\mathfrak{A}\times\mathfrak{B}}(\mathfrak{p}_2,\mathfrak{q}_2)\}\ (5)$$

Putting $\mathfrak{p}_1=\mathfrak{p}_2=0$ in (5) Then
$\mathfrak{G}_{\mathfrak{A}\times\mathfrak{B}}((0,\mathfrak{q}_1)\leq max\{\mathfrak{G}_{\mathfrak{A}\times\mathfrak{B}}(0,(\mathfrak{q}_1-\mathfrak{q}_2)),\mathfrak{G}_{\mathfrak{A}\times\mathfrak{B}}(0,\mathfrak{q}_2)\}$ and

$$\mathfrak{G}_{\mathfrak{A}\times\mathfrak{B}}(0,(\mathfrak{q}_1-\mathfrak{q}_2))\leq max\{\mathfrak{G}_{\mathfrak{A}\times\mathfrak{B}}(0,\mathfrak{q}_1),\mathfrak{G}_{\mathfrak{A}\times\mathfrak{B}}(0,\mathfrak{q}_2)\}\ (6)$$

Using equations (1) in (6) which gives $\Rightarrow\mathfrak{G}_{\mathfrak{B}}(\mathfrak{q}_1)\leq max\{\{\mathfrak{G}_{\mathfrak{B}}(\mathfrak{q}_1-\mathfrak{q}_2),\mathfrak{G}_{\mathfrak{B}}(\mathfrak{q}_2)\}$
and $\mathfrak{G}_{\mathfrak{B}}(\mathfrak{q}_1-\mathfrak{q}_2)\leq max\{\mathfrak{G}_{\mathfrak{B}}(\mathfrak{q}_1),\mathfrak{G}_{\mathfrak{B}}(\mathfrak{q}_2)\}$.
Similarly we have, $\mathfrak{H}_{\mathfrak{B}}(\mathfrak{q}_1)\geq min\{\{\mathfrak{H}_{\mathfrak{B}}(\mathfrak{q}_1-\mathfrak{q}_2),\mathfrak{H}_{\mathfrak{B}}(\mathfrak{q}_2)\}$ and $\mathfrak{H}_{\mathfrak{B}}(\mathfrak{q}_1-\mathfrak{q}_2)\geq min\{\mathfrak{H}_{\mathfrak{B}}(\mathfrak{q}_1),\mathfrak{H}_{\mathfrak{B}}(\mathfrak{q}_2)\}$. Hence $\mathfrak{B}$ is a cubic intuitionistic $\beta-$ideal of Θ. ■

5. Homomorphic Image(Inverse Image) of Cubic Intuitionistic $\beta-$Ideals

This section deals the properties on homomorphic image(inverse image) of Cubic intuitionistic $\beta-$ideals.

Definition 5.1. Let $f : \mho \to \Theta$ be a function. Let $\mathfrak{A}$ and $\mathfrak{B}$ be two cubic intuitionistic $\beta-$ideals in $\mho$ and Θ respectively. Then inverse image of $\mathfrak{B}$ under f is defined by $f^{-1}(\mathfrak{B}) = \{f^{-1}(\overline{\Im}_{\mathfrak{B}}(\mathfrak{p})), f^{-1}(\overline{\aleph}_{\mathfrak{B}}(\mathfrak{p})), f^{-1}(\mathfrak{G}_{\mathfrak{B}}(x)), f^{-1}(\mathfrak{H}_{\mathfrak{B}}(\mathfrak{p})) : \mathfrak{p} \in \mho\}$ such that $f^{-1}(\overline{\Im}_{\mathfrak{B}}(\mathfrak{p})) = (\overline{\Im}_{\mathfrak{B}}(f(\mathfrak{p}))$, $f^{-1}(\overline{\aleph}_{\mathfrak{B}}(\mathfrak{p})) = (\overline{\aleph}_{\mathfrak{B}}(f(\mathfrak{p}))$,$f^{-1}(\mathfrak{G}_{\mathfrak{B}}(\mathfrak{p})) = (\mathfrak{G}_{\mathfrak{B}}(f(\mathfrak{p}))$ and $f^{-1}(\mathfrak{H}_{\mathfrak{B}}(\mathfrak{p})) = (\mathfrak{H}_{\mathfrak{B}}(f(\mathfrak{p}))$.

Theorem 5.2. *Let $f : \mho \to \mho$ be an endomorphism on $\mho$ and $\mathfrak{C} = \{\mathfrak{p}, \Psi(\mathfrak{p}), \rho(x) : x \in \mho\}$ be a cubic intuitionistic $\beta-$ideal of $\mho$. Then $\mathfrak{C}_f = \{f(\mathfrak{p}), \{\overline{\Im}_f(\mathfrak{p}), \overline{\aleph}_f(\mathfrak{p})\}, \{\mathfrak{G}_f(\mathfrak{p}), \mathfrak{H}_f(\mathfrak{p})\} : \mathfrak{p} \in \mho\}$ where $\overline{\Im}_f : \mho \to D[0,1]$ & $\overline{\aleph}_f : \mho \to D[0,1]$ and $\mathfrak{G}_f : \mho \to [0,1]$ & $\mathfrak{H}_f : \mho \to [0,1]$ are defined by $\overline{\Im}_f(\mathfrak{p}) = \overline{\Im}(f(\mathfrak{p}))$, $\overline{\aleph}_f(\mathfrak{p}) = \overline{\aleph}(f(\mathfrak{p}))$, $\mathfrak{G}_f(\mathfrak{p}) = \mathfrak{G}(f(\mathfrak{p}))$ and $\mathfrak{H}_f(\mathfrak{p}) = \mathfrak{H}(f(\mathfrak{p}))$, $\forall \mathfrak{p} \in \mho$, is a cubic intuitionistic $\beta-$ideal of $\mho$.*

Proof. Let $\mathfrak{C}$ be a cubic intuitionistic $\beta-$ideal of $\mho$.
For $\mathfrak{p} \in \mho$,
$\overline{\Im}_f(0) = \overline{\Im}(f(0)) = \overline{\Im}(0) \leq \overline{\Im}(\mathfrak{p})$,
$\overline{\aleph}_f(0) = \overline{\aleph}(f(0)) = \overline{\aleph}(0) \geq \overline{\aleph}(\mathfrak{p})$ $\forall \mathfrak{p} \in \mho$.
Then

$$\begin{aligned}
\overline{\Im}_f(\mathfrak{p}+\mathfrak{q}) &= \overline{\Im}(f(\mathfrak{p}+\mathfrak{q})) \\
&\geq \overline{\Im}(f(\mathfrak{p}) + f(y)) \\
&= rmin\{\overline{\Im}(f(\mathfrak{p})), \overline{\Im}(f(\mathfrak{q}))\} \\
&= rmin\{\overline{\Im}_f(\mathfrak{p}), \overline{\Im}_f(\mathfrak{q})\}.
\end{aligned}$$

Similarly, we will have $\overline{\aleph}_f(\mathfrak{p}+\mathfrak{q}) \leq rmax\{\overline{\aleph}_f(\mathfrak{p}), \overline{\aleph}_f(\mathfrak{q})\}$ Also,

$$\begin{aligned}
\overline{\Im}_f(\mathfrak{p}) &= \overline{\Im}(f(\mathfrak{p})) \\
&\geq rmin\{\overline{\Im}(f(\mathfrak{p}) - f(\mathfrak{q})), \overline{\Im}(f(\mathfrak{q}))\} \\
&= rmin\{\overline{\Im}(f(\mathfrak{p}-\mathfrak{q})), \overline{\Im}(f(\mathfrak{q}))\} \\
&= rmin\{\overline{\Im}_f(\mathfrak{p}-\mathfrak{q}), \overline{\Im}_f(\mathfrak{q})\}.
\end{aligned}$$

Similarly, we can have $\overline{\aleph}_f(\mathfrak{p}) \leq rmax\{\overline{\aleph}_f(\mathfrak{p}-\mathfrak{q}), \overline{\aleph}_f(\mathfrak{q})\}$.

For $\mathfrak{p} \in \mho$, $\mathfrak{G}_f(0) = \mathfrak{G}(f(0)) = \mathfrak{G}(0) \geq \mathfrak{G}(\mathfrak{p})$; $\mathfrak{H}_f(0) = \mathfrak{H}(f(0)) = \mathfrak{H}(0) \leq \mathfrak{H}(\mathfrak{p})$ $\forall \mathfrak{p} \in \mho$.
Then

$$\begin{aligned}
\mathfrak{G}_f(\mathfrak{p}+\mathfrak{q}) &= \mathfrak{G}(f(\mathfrak{p}+\mathfrak{q})) \\
&\leq \mathfrak{G}(f(\mathfrak{p}) + f(\mathfrak{q})) \\
&= max\{\mathfrak{G}(f(\mathfrak{p})), \mathfrak{G}(f(\mathfrak{q}))\} \\
&= max\{\mathfrak{G}_f(\mathfrak{p}), \mathfrak{G}_f(\mathfrak{q})\}
\end{aligned}$$

$$\begin{aligned}
\mathfrak{H}_f(\mathfrak{p}+\mathfrak{q}) &= \mathfrak{H}(f(\mathfrak{p}+\mathfrak{q})) \\
&\geq \mathfrak{H}(f(\mathfrak{p}) + f(\mathfrak{q})) \\
&= min\{\mathfrak{H}(f(\mathfrak{p})), \mathfrak{H}(f(\mathfrak{q}))\} \\
&= min\{\mathfrak{H}_f(\mathfrak{p}), \mathfrak{H}_f(\mathfrak{q})\}.
\end{aligned}$$

Also,

$$\mathfrak{G}_f(\mathfrak{p}) = \mathfrak{G}(f(\mathfrak{p}))$$

$$\le max\{\mathfrak{G}(f(\mathfrak{p}) - f(\mathfrak{q})), \mathfrak{G}(f(\mathfrak{q}))\}$$
$$= max\{\mathfrak{G}(f(\mathfrak{p} - \mathfrak{q})), \mathfrak{G}(f(\mathfrak{q}))\}$$
$$= max\{\mathfrak{G}_f(\mathfrak{p} - \mathfrak{q}), \mathfrak{G}_f(y)\}.$$

$$\mathfrak{H}_f(\mathfrak{p}) = \mathfrak{H}(f(\mathfrak{p}))$$
$$\ge min\{\mathfrak{H}(f(\mathfrak{p}) - f(\mathfrak{q})), \mathfrak{H}(f(\mathfrak{q}))\}$$
$$= min\{\mathfrak{H}(f(\mathfrak{p} - \mathfrak{q})), \mathfrak{H}(f(\mathfrak{q}))\}$$
$$= min\mathfrak{H}_f(\mathfrak{p} - \mathfrak{q}), \mathfrak{H}_f(\mathfrak{q})\}.$$

Hence $\mathfrak{C}_f$ is a cubic intuitionistic $\beta-$ideal of $\mho$. ■

Theorem 5.3. *Let $f : \mho \to \Theta$ be an onto homomorphism of $\beta-$algebras. If $\mathfrak{C} = \{\mathfrak{p}, \Psi(\mathfrak{p}), \rho(\mathfrak{p}) : \mathfrak{p} \in \mho\}$ is a cubic intuitionistic $\beta-$ideal of Θ, then the preimage $f^{-1}(\mathfrak{C})$ is a cubic intuitionistic $\beta-$ideal of $\mho$.*

Proof. Let $\mathfrak{C}$ be a cubic intuitionistic $\beta-$ideal of Θ.
For $\mathfrak{p} \in \mho$,

$$f^{-1}(\overline{\Im}_\Psi(0)) = \overline{\Im}_\Psi(f(0)) = \overline{\Im}_\Psi(0) \ge \overline{\Im}_\Psi(\mathfrak{p})$$

For $\mathfrak{p}, \mathfrak{q} \in \mho$,

$$f^{-1}(\overline{\Im}_\Psi)(\mathfrak{p} + \mathfrak{q}) = \overline{\Im}_\Psi(f(\mathfrak{p} + \mathfrak{q}))$$
$$= \overline{\Im}_\Psi(f(\mathfrak{p}) + f(\mathfrak{q}\mathfrak{q})$$
$$\ge rmin\{\overline{\Im}_\Psi(f(\mathfrak{p})), \overline{\Im}_\Psi(f(\mathfrak{q}))\}$$
$$= rmin\{f^{-1}(\overline{\Im}_\Psi(\mathfrak{p})), f^{-1}(\overline{\Im}_\Psi(\mathfrak{q}))\}$$

$$f^{-1}(\overline{\Im}_\Psi)(\mathfrak{p}) = \overline{\Im}_\Psi(f(\mathfrak{p}))$$
$$\ge rmin\{\overline{\Im}_\Psi(f(\mathfrak{p}) - f(\mathfrak{q})), \overline{\Im}_\Psi(f(\mathfrak{q}))\}$$
$$= rmin\{\overline{\Im}_\Psi(f(\mathfrak{p} - \mathfrak{q})), \overline{\Im}_\Psi(f(\mathfrak{q}))\}$$
$$= rmin\{f^{-1}((\overline{\Im}_\Psi)(\mathfrak{p} - \mathfrak{q})), f^{-1}(\overline{\Im}_\Psi(\mathfrak{q}))\}$$

Similarly for $\mathfrak{p}, \mathfrak{q} \in \mho$, $f^{-1}(\overline{\aleph}_\Psi(0)) = \overline{\aleph}_\Psi(f(0)) = \overline{\aleph}_\Psi(0) \le \overline{\aleph}_\Psi(\mathfrak{p})$ & $f^{-1}(\overline{\aleph}_\Psi)(\mathfrak{p}+\mathfrak{q}) \le rmax\{f^{-1}(\overline{\aleph}_\Psi(\mathfrak{p})), f^{-1}(\overline{\aleph}_\Psi(\mathfrak{q}))\}$ and $f^{-1}(\overline{\aleph}_\Psi)(\mathfrak{p}) \le rmax\{f^{-1}((\overline{\aleph}_\Psi)(\mathfrak{p}- \mathfrak{q})), f^{-1}(\overline{\aleph}_\Psi(\mathfrak{q}))\}$

For $\mathfrak{p} \in \mho$,

$$f^{-1}(\mathfrak{G}_\rho(0)) = \mathfrak{G}_\rho(f(0)) = \mathfrak{G}_\rho(0) \le \mathfrak{G}_\rho(\mathfrak{p})$$

For $\mathfrak{p}, \mathfrak{q} \in \mho$,

$$f^{-1}(\mathfrak{G}_\rho)(\mathfrak{p} + \mathfrak{q}) = \mathfrak{G}_\rho(f(\mathfrak{p} + \mathfrak{q}))$$
$$= \mathfrak{G}_\rho(f(\mathfrak{p}) + f(\mathfrak{q}))$$
$$\le max\{\mathfrak{G}_\rho(f(\mathfrak{p})), \mathfrak{G}_\rho(f(\mathfrak{q}))\}$$
$$= max\{f^{-1}(\mathfrak{G}_\rho(\mathfrak{p})), f^{-1}(\mathfrak{G}_\rho(\mathfrak{q}))\}$$

$$f^{-1}(\mathfrak{G}_\rho)(\mathfrak{p}) = \mathfrak{G}_\rho(f(\mathfrak{p}))$$
$$\le max\{\mathfrak{G}_\rho(f(\mathfrak{p}) - f(\mathfrak{q})), \mathfrak{G}_\rho(f(\mathfrak{q}))\}$$

$$= max\{\mathfrak{G}_\rho(f(\mathfrak{p} - \mathfrak{q})), \mathfrak{G}_\rho(f(\mathfrak{q}))\}$$
$$= max\{f^{-1}((\mathfrak{G}_\rho)(\mathfrak{p} - \mathfrak{q})), f^{-1}(\mathfrak{G}_\rho(\mathfrak{q}))\}$$

Likewise for $\mathfrak{p}, \mathfrak{q} \in \mho$, $f^{-1}(\mathfrak{H}_\rho(0)) = \mathfrak{H}_\rho(f(0)) = \mathfrak{H}_\rho(0) \geq \mathfrak{H}_\rho(\mathfrak{p})$ & $f^{-1}(\mathfrak{H}_\rho)(\mathfrak{p} + \mathfrak{q}) \geq min\{f^{-1}(\mathfrak{H}_\rho(\mathfrak{p})), f^{-1}(\mathfrak{H}_\rho(\mathfrak{q}))\}$ and $f^{-1}(\mathfrak{H}_\rho)(\mathfrak{p}) \geq min\{f^{-1}((\mathfrak{H}_\rho)(\mathfrak{p} - \mathfrak{q})), f^{-1}(\mathfrak{H}_\rho(\mathfrak{q}))\}$. Hence $f^{-1}(\mathfrak{C})$ is a cubic intuitionistic $\beta-$ideal of $\mho$. ∎

6. Multiplications of Cubic Intuitionistic β-Ideals

This section gives the notion of multiplications of cubic intuitionistic β-ideal and some of its results are investigated.

Definition 6.1. Let $\mathfrak{C} = \{\mathfrak{p}, \Psi(\mathfrak{p}), \rho(\mathfrak{p}) : \mathfrak{p} \in \mho\}$ be a cubic fuzzy set of $\mho$ and $\mu \in (0, 1]$. An object having the form $\mathfrak{C}_\mu^M = \left\{(\Psi(\mathfrak{p}))_\mu^M, (\rho(\mathfrak{p})_\mu^M\right\}$ is said to be cubic μ-multiplication of $\mathfrak{C}$ if it satisfies $\left(\overline{\Im}_\Psi\right)_\mu^M(x) = \mu.\overline{\Im}_\Psi(x)$; $\left(\overline{\aleph}_\Psi\right)_\mu^M(x) = \mu.\overline{\aleph}_\Psi(x)$;$(\mathfrak{G}_\rho)_\mu^M(x) = \mu.\mathfrak{G}_\rho(x)$ and $(\mathfrak{H}_\rho)_\mu^M(x) = \mu.\mathfrak{H}_\rho(x)$ for all $x \in \mho$.

Theorem 6.2. *If $\mathfrak{C} = \{\mathfrak{p}, \Psi(\mathfrak{p}), \rho(\mathfrak{p}) : \mathfrak{p} \in \mho\}$ is a cubic β-ideal of $\mho$ and let $\mu \in [0, 1]$. Then the cubic μ-multiplication $\mathfrak{C}_\mu^M$ of $\mathfrak{C}$ is cubic β-ideal of X.*

Proof. Suppose $\mathfrak{C} = \{\mathfrak{p}, \Psi(\mathfrak{p}), \rho(\mathfrak{p}) : \mathfrak{p} \in \mho\}$ is a cubic β-ideal of $\mho$. Then

$$\begin{aligned}\left(\overline{\Im}_\Psi\right)_\mu^M(0) &= \mu.\overline{\Im}_\Psi(0)\\ &\geq \mu.\overline{\Im}_\Psi(\mathfrak{p})\\ &= \left(\overline{\Im}_\Psi\right)_\mu^M(\mathfrak{p})\end{aligned}$$

$(i.e)$ $\left(\overline{\Im}_\Psi\right)_\mu^M(0) \geq \left(\overline{\Im}_\Psi\right)_\mu^M(\mathfrak{p})$

In a simalar way, we can have $\left(\overline{\aleph}_\Psi\right)_\mu^M(0) \leq \left(\overline{\aleph}_\Psi\right)_\mu^M(\mathfrak{p})$

$$\begin{aligned}(\mathfrak{G}_\rho)_\mu^M(0) &= \mu.\mathfrak{G}_\rho(0)\\ &\leq \mu.\mathfrak{G}_\rho(\mathfrak{p})\\ &= (\mathfrak{G}_\rho)_\mu^M(\mathfrak{p})\end{aligned}$$

$(i.e)$ $(\mathfrak{G}_\rho)_\mu^M(0) \leq (\mathfrak{G}_\rho)_\mu^M(\mathfrak{p})$

In the same manner, we have $(\mathfrak{H}_\rho)_\mu^M(0) \geq (\mathfrak{H}_\rho)_\mu^M(\mathfrak{p})$

$$\begin{aligned}\left(\overline{\Im}_\Psi\right)_\mu^M(\mathfrak{p} + \mathfrak{q}) &= \mu.\overline{\Im}_\Psi(\mathfrak{p} + \mathfrak{q})\\ &\geq \mu.rmin\{\overline{\Im}_\Psi(\mathfrak{p}), \overline{\Im}_\Psi(\mathfrak{q})\}\\ &= rmin\{\mu.\overline{\Im}_\Psi(\mathfrak{p})\ , \mu.\overline{\Im}_\Psi(\mathfrak{q})\}\\ &= rmin\left\{\left(\overline{\Im}_\Psi\right)_\mu^M(\mathfrak{p}), \left(\overline{\Im}_\Psi\right)_\mu^M(\mathfrak{q})\right\}\end{aligned}$$

$(i.e)$ $\left(\overline{\Im}_\Psi\right)_\mu^M(\mathfrak{p} + \mathfrak{q}) \geq rmin\left\{\left(\overline{\Im}_\Psi\right)_\mu^M(\mathfrak{p}), \left(\overline{\Im}_\Psi\right)_\mu^M(\mathfrak{q})\right\}$

Likewise, we obtain $\left(\overline{\aleph}_\Psi\right)_\mu^M(\mathfrak{p} + \mathfrak{q}) \leq rmax\left\{\left(\overline{\aleph}_\Psi\right)_\mu^M(\mathfrak{p}), \left(\overline{\aleph}_\Psi\right)_\mu^M(\mathfrak{q})\right\}$

$$\begin{aligned}(\mathfrak{G}_\rho)^M_\mu(\mathfrak{p}+\mathfrak{q}) &= \mu.\mathfrak{G}_\rho(\mathfrak{p}+\mathfrak{q})\\ &\le \mu.max\{\mathfrak{G}_\rho(\mathfrak{p}),\mathfrak{G}_\rho(\mathfrak{q})\}\\ &= max\{\mu.\mathfrak{G}_\rho(\mathfrak{p}),\mu.\mathfrak{G}_\rho(\mathfrak{q})\}\\ &= max\left\{(\mathfrak{G}_\rho)^M_\mu(\mathfrak{p}),(\mathfrak{G}_\rho)^M_\mu(\mathfrak{q})\right\}\end{aligned}$$

$$(i.e)\ (\mathfrak{G}_\rho)^M_\mu(\mathfrak{p}+\mathfrak{q}) \le max\left\{(\mathfrak{G}_\rho)^M_\mu(\mathfrak{p}),(\mathfrak{G}_\rho)^M_\mu(\mathfrak{q})\right\}$$

In the similar way, we get $(i.e)\ (\mathfrak{H}_\rho)^M_\mu(\mathfrak{p}+\mathfrak{q}) \ge min\left\{(\mathfrak{H}_\rho)^M_\mu(\mathfrak{p}),(\mathfrak{H}_\rho)^M_\mu(\mathfrak{q})\right\}$

$$\begin{aligned}\left(\overline{\Im}_\Psi\right)^M_\mu(\mathfrak{p}) &= \mu.\overline{\Im}_\Psi(\mathfrak{p})\\ &\ge \mu.rmin\{\overline{\Im}_\Psi(\mathfrak{p}-\mathfrak{q}),\overline{\Im}_\Psi(y)\}\\ &= rmin\{\mu.\overline{\Im}_\Psi(\mathfrak{p}-\mathfrak{q}),\mu.\overline{\Im}_\Psi(\mathfrak{q})\}\\ &= rmin\left\{\mu.\overline{\Im}_\Psi(\mathfrak{p}-\mathfrak{q}),\mu.\overline{\Im}_\Psi(\mathfrak{q})\right\}\\ &= rmin\left\{\left(\overline{\Im}_\Psi\right)^M_\mu(\mathfrak{p}-\mathfrak{q}),\left(\overline{\Im}_\Psi\right)^M_\mu(\mathfrak{q})\right\}\end{aligned}$$

$$(i.e)\ \left(\overline{\Im}_\Psi\right)^M_\mu(\mathfrak{p}) \ge rmin\left\{\left(\overline{\Im}_\Psi\right)^M_\mu(\mathfrak{p}-\mathfrak{q}),\left(\overline{\Im}_\Psi\right)^M_\mu(\mathfrak{q})\right\}$$

By using the same process, we get $\left(\overline{\aleph}_\Psi\right)^M_\mu(\mathfrak{p}) \le rmax\left\{\left(\overline{\aleph}_\Psi\right)^M_\mu(\mathfrak{p}-\mathfrak{q}),\left(\overline{\aleph}_\Psi\right)^M_\mu(\mathfrak{q})\right\}$

$$\begin{aligned}(\mathfrak{G}_\rho)^M_\mu(\mathfrak{p}) &= \mu.\mathfrak{G}_\rho(\mathfrak{p})\\ &\le \mu.max\{\mathfrak{G}_\rho(\mathfrak{p}-\mathfrak{q}),\mathfrak{G}_\rho(\mathfrak{q})\}\\ &= max\{\mu.\mathfrak{G}_\rho(\mathfrak{p}-\mathfrak{q}),\mu.\mathfrak{G}_\rho(\mathfrak{q})\}\\ &= max\left\{(\mathfrak{G}_\rho)^M_\mu(\mathfrak{p}-\mathfrak{q}),(\mathfrak{G}_\rho)^M_\mu(\mathfrak{q})\right\}\end{aligned}$$

$$(i.e)\ (\mathfrak{G}_\rho)^M_\mu(\mathfrak{p}) \le max\left\{(\mathfrak{G}_\rho)^M_\mu(\mathfrak{p}-\mathfrak{q}),(\mathfrak{G}_\rho)^M_\mu(\mathfrak{q})\right\}$$

Similarly, we can have $(\mathfrak{H}_\rho)^M_\mu(\mathfrak{p}) \le max\left\{(\mathfrak{H}_\rho)^M_\mu(\mathfrak{p}-\mathfrak{q}),(\mathfrak{H}_\rho)^M_\mu(\mathfrak{q})\right\}$

For all $\mathfrak{p},\mathfrak{q} \in \mho$ and $\mu \in (0,1]$. Hence C^M_μ of C is cubic β-ideal of X. ■

Theorem 6.3. *If $\mathfrak{C}$ is a cubic set of X such that cubic μ-multiplication $\mathfrak{C}^M_\mu$ of $\mathfrak{C}$ is cubic β-ideal of $\mho$ and $\mu \in [0,1]$ then $\mathfrak{C}$ is cubic β-ideal of $\mho$.*

Proof. Assume that $\mathfrak{C}^M_\mu(x)$ of $\mathfrak{C}$ be a cubic β-ideal of $\mho$, $\mu \in (0,1]$
Then

$$\begin{aligned}\mu.\overline{\Im}_\Psi(0) &= \left(\overline{\Im}_\Psi\right)^M_\mu(0)\\ &\ge \left(\overline{\Im}_\Psi\right)^M_\mu(\mathfrak{p})\\ &= \mu.\overline{\Im}_\Psi(\mathfrak{p})\end{aligned}$$

$$(i.e)\ \overline{\Im}_\Psi(0) \ge \overline{\Im}_\Psi(\mathfrak{p})$$

In a similar way, we have $\overline{\aleph}_\Psi(0) \le \overline{\aleph}_\Psi(\mathfrak{p})$

$$\begin{aligned}\mu.\mathfrak{G}_\rho(0) &= (\mathfrak{G}_\rho)^M_\mu(0)\\ &\leq (\mathfrak{G}_\rho)^M_\mu(\mathfrak{p})\\ &= \mu.\mathfrak{G}_\rho(\mathfrak{p})\end{aligned}$$

$(i.e)$ $\mathfrak{G}_\rho(0) \leq \mathfrak{G}_\rho(\mathfrak{p})$

Likewise, we get $\mathfrak{H}_\rho(0) \geq \mathfrak{H}_\rho(\mathfrak{p})$

$$\begin{aligned}\mu.\overline{\Im}_\Psi(\mathfrak{p}+\mathfrak{q}) &= \left(\overline{\Im}_\Psi\right)^M_\mu(\mathfrak{p}+\mathfrak{q})\\ &\geq rmin\{\left(\overline{\Im}_\Psi\right)^M_\mu(\mathfrak{p}), \left(\overline{\Im}_\Psi\right)^M_\mu(\mathfrak{q})\}\\ &= rmin\{\mu.\overline{\Im}_\Psi(\mathfrak{p}), \mu.\overline{\Im}_\Psi(\mathfrak{q})\}\\ &= \mu.rmin\left\{\overline{\Im}_\Psi(\mathfrak{p}), \overline{\Im}_\Psi(\mathfrak{q})\right\}\end{aligned}$$

$(i.e)$ $\overline{\Im}_\Psi(\mathfrak{p}+\mathfrak{q}) \geq rmin\left\{\overline{\Im}_\Psi(\mathfrak{p}), \overline{\Im}_\Psi(\mathfrak{q})\right\}$

In the same manner, we can have $\overline{\aleph}_\Psi(\mathfrak{p}+\mathfrak{q}) \leq rmax\left\{\overline{\aleph}_\Psi(\mathfrak{p}), \overline{\aleph}_\Psi(\mathfrak{q})\right\}$

$$\begin{aligned}\mu.\mathfrak{G}_\rho(\mathfrak{p}+\mathfrak{q}) &= (\mathfrak{G}_\rho)^M_\mu(\mathfrak{p}+\mathfrak{q})\\ &\leq max\{(\mathfrak{G}_\rho)^M_\mu(\mathfrak{p}), (\mathfrak{G}_\rho)^M_\mu(\mathfrak{q})\}\\ &= max\{\mathfrak{G}_\rho(\mathfrak{p}), \mu.\mathfrak{G}_\rho(\mathfrak{q})\}\\ &= \mu.max\{\mathfrak{G}_\rho(\mathfrak{p}), \mathfrak{G}_\rho(\mathfrak{q})\}\end{aligned}$$

$(i.e)$ $\mathfrak{G}_\rho(\mathfrak{p}+\mathfrak{q}) \leq max\{\mathfrak{G}_\rho(\mathfrak{p}), \mathfrak{G}_\rho(\mathfrak{q})\}$

Similarly, we have $\mathfrak{H}_\rho(\mathfrak{p}+\mathfrak{q}) \geq min\{\mathfrak{H}_\rho(\mathfrak{p}), \mathfrak{H}_\rho(\mathfrak{q})\}$

$$\begin{aligned}\mu.\overline{\Im}_\Psi(\mathfrak{p}) &= \left(\overline{\Im}_\Psi\right)^M_\mu(\mathfrak{p})\\ &\geq rmin\{\left(\overline{\Im}_\Psi\right)^M_\mu(\mathfrak{p}-\mathfrak{q}), \left(\overline{\Im}_\Psi\right)^M_\mu(\mathfrak{q})\}\\ &= rmin\{\mu.\overline{\Im}_\Psi(\mathfrak{p}-\mathfrak{q}), \mu.\overline{\Im}_\Psi(\mathfrak{q})\}\\ &= \mu.rmin\left\{\overline{\Im}_\Psi(\mathfrak{p}-\mathfrak{q}), \overline{\Im}_\Psi(\mathfrak{q})\right\}\end{aligned}$$

$(i.e)$ $\overline{\Im}_\Psi(\mathfrak{p}) \geq rmin\left\{\overline{\Im}_\Psi(\mathfrak{p}-\mathfrak{q}), \overline{\Im}_\Psi(\mathfrak{q})\right\}$

In the same way, we have $\overline{\aleph}_\Psi(\mathfrak{p}) \leq rmax\left\{\overline{\aleph}_\Psi(\mathfrak{p}-\mathfrak{q}), \overline{\aleph}_\Psi(\mathfrak{q})\right\}$

$$\begin{aligned}\mu.\mathfrak{G}_\rho(\mathfrak{p}) &= (\mathfrak{G}_\rho)^M_\mu(\mathfrak{p})\\ &\leq max\{(\mathfrak{G}_\rho)^M_\mu(\mathfrak{p}-\mathfrak{q}), (\mathfrak{G}_\rho)^M_\mu(\mathfrak{q})\}\\ &= max\{\mu.\mathfrak{G}_\rho(\mathfrak{p}-\mathfrak{q}), \mu.\mathfrak{G}_\rho(\mathfrak{q})\}\\ &= \mu.max\{\mathfrak{G}_\rho(\mathfrak{p}-\mathfrak{q}), \mathfrak{G}_\rho(\mathfrak{q})\}\end{aligned}$$

$(i.e)$ $\mathfrak{G}_\rho(\mathfrak{p}) \leq max\{\mathfrak{G}_\rho(\mathfrak{p}-\mathfrak{q}), \mathfrak{G}_\rho(\mathfrak{q})\}$

Likewise, we obtain $\mathfrak{H}_\rho(\mathfrak{p}) \geq min\{\mathfrak{H}_\rho(\mathfrak{p}-\mathfrak{q}), \mathfrak{H}_\rho(\mathfrak{q})\}$

For all $\mathfrak{p}, \mathfrak{q} \in \mho$ and $\mu \in (0,1]$. Hence $\mathfrak{C}$ is cubic β-ideal of $\mho$. ■

Theorem 6.4. *Intersection of any two cubic μ-multiplication $\mathfrak{C}_{\mu}^{M}$ of a cubic β-ideal $\mathfrak{C}$ of $\mho$ is a cubic β-ideal of $\mho$.*

Proof. Suppose $\mathfrak{C}_{\mu}^{M}$ and $\mathfrak{C}_{\mu'}^{M}$ are two cubic μ-multiplication of cubic β-ideal $\mathfrak{C}$ of $\mho$, where $\mu, \mu' \in (0,1]$. Assume $\mu = \mu'$ Since $\mathfrak{C}_{\mu}^{M}$ and $\mathfrak{C}_{\mu'}^{M}$are cubic μ-multiplication of cubic β-ideal of $\mho$. So,

$$\begin{aligned}\left(\left(\overline{\Im}_{\Psi}\right)_{\mu}^{M} \cap \left(\overline{\Im}_{\Psi}\right)_{\mu'}^{M}\right)(\mathfrak{p}) &= rmin\left\{\left(\overline{\Im}_{\Psi}\right)_{\mu}^{M}(\mathfrak{p}), \left(\overline{\Im}_{\Psi}\right)_{\mu'}^{M}(\mathfrak{p})\right\}\\ &= rmin\{\mu.\overline{\Im}_{\Psi}(\mathfrak{p}), \mu'.\overline{\Im}_{\Psi}(\mathfrak{p})\}\\ &= \mu.\overline{\Im}_{\Psi}(\mathfrak{p})\\ &= \left(\overline{\Im}_{\Psi}\right)_{\mu}^{M}(\mathfrak{p})\end{aligned}$$

In the same way, we can have $\left(\left(\overline{\aleph}_{\Psi}\right)_{\mu}^{M} \cap \left(\overline{\aleph}_{\Psi}\right)_{\mu'}^{M}\right)(\mathfrak{p}) = rmax\left(\overline{\aleph}_{\Psi}\right)_{\mu}^{M}(\mathfrak{p})$

$$\begin{aligned}\left((\mathfrak{G}_{\rho})_{\mu}^{M} \cap (\mathfrak{G}_{\rho})_{\mu'}^{M}\right)(\mathfrak{p}) &= max\left\{(\mathfrak{G}_{\rho})_{\mu}^{M}(\mathfrak{p}), (\mathfrak{G}_{\rho})_{\mu'}^{M}(\mathfrak{p})\right\}\\ &= max\{\mu.\mathfrak{G}_{\rho}\ (\mathfrak{p}), \mu'.\mathfrak{G}_{\rho}(\mathfrak{p})\}\\ &= \mu.\mathfrak{G}_{\rho}(\mathfrak{p})\\ &= (\mathfrak{G}_{\rho})_{\mu}^{M}(\mathfrak{p})\end{aligned}$$

In the same manner, we have

$$\left((\mathfrak{H}_{\rho})_{\mu}^{M} \cap (\mathfrak{H}_{\rho})_{\mu'}^{M}\right)(\mathfrak{p}) = min(\mathfrak{H}_{\rho})_{\mu}^{M}(\mathfrak{p})$$

Hence, $\mathfrak{C}_{\mu}^{M} \cap \mathfrak{C}_{\mu'}^{M}$ is cubic β-ideal of $\mho$.

∎

7. Conclusion

In this work, the thought of cubic intuitionistic β-ideal is proposed and examined some of its engrossing associated outcomes. Moreover, the results on cartesian product and few properties on homomorphism of cubic $\beta-$ideal have been investigated. Furthermore, interesting results based on cubic $\mu-$ multiplication are also provided. In particular, we have proved that the intersection of cubic $\mu-$ multiplication is also a cubic $\beta-$ideal.In future work, this can be extended into other algebraic structures.

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