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On Commutativity and Centralizers of Prime Ring with Involution

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Abstract The main purpose of this paper is to study the commutativity of prime rings with involution satisfying certain $*$-identities involving left centralizers.

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1. Introduction

This paper deals with the commutativity of prime rings with involution involving left centralizers and was motivated by [1]. Throughout R will represent an associative ring with centre $Z(R)$. We denote $[x, y] = xy - yx$, the commutator of x and y and $x \circ y = xy + yx$, the anti-commutator of x and y. An additive map $x \mapsto x^*$ of R into itself is said to be an involution if (i) $(xy)^* = y^*x^*$ and (ii) $(x^*)^* = x$ holds for all $x, y \in R$. A ring R together with an involution is known as ring with involution or $*$-ring. An element x in a ring with involution is said to be hermitian if $x = x^*$ and skew-hermitian if $-x = x^*$. We will denote these sets by $H(R)$ and $S(R)$ respectively. Note that $H(R) = S(R)$ if $char(R) = 2$. Therefore, we consider R to be a prime ring with involution such that $char(R) \neq 2$. The involution is said to be of the first kind if $Z(R) \subseteq H(R)$, otherwise it is said to be of the second kind. In the later case $S(R) \cap Z(R) \neq (0)$. An example is the ring of quaternions. A description of such rings can be found in [2], where further references can be found.

Following [3], an additive mapping $T : R \to R$ is said to be a left (resp. right) centralizer (multiplier) of R if $T(xy) = T(x)y$ (resp. $T(xy) = xT(y)$) for all $x, y \in R$. If T is both left as well as the right centralizer of R, it is said to be the centralizer of R. From last so many years considerable work has been done in the direction of studying

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the relationship between the commutativity of the ring R and certain specific types of maps on R. The first result in this direction is due to Divinsky [4] who proved that a simple artinian ring is commutative if it has a commuting non-trivial automorphism. This result was subsequently refined and extended by a number of authors in various directions (viz., [5–8]). Recently S. Ali and N. A. Dar in [1], studied the commutativity of prime rings with involution involving left centralizers. For instance one of the results proved by them is as under. Let R be a prime ring with involution $*$ such that $char(R) \neq 2$ and $S(R) \cap Z(R) \neq (0)$. If R admits a nonzero left centralizer $T : R \to R$ such that $T([x, x^*]) = 0$ for all $x \in R$, then R is commutative. The purpose of this paper is to study all these result in more generalized form. Finally, we provide two examples to prove that the assumed restrictions on our main results are not superfluous.

We shall restrict our attention on left centralizers, since all results presented in this article are also true for right centralizers because of left-right symmetry.

2. Main Results

We begin with the following lemmas, which are essential to prove our main results.

Lemma 2.1. [9, Lemma 2.1] *Let R be a prime ring with involution $*$ of the second kind. Then $[x, x^*] \in Z(R)$ for all $x \in R$ if and only if R is commutative.*

Lemma 2.2. [9, Lemma 2.2] *Let R be a prime ring with involution $*$ of the second kind. Then $x \circ x^* \in Z(R)$ for all $x \in R$ if and only if R is commutative.*

Theorem 2.3. *Let R be a prime ring with involution of the second kind such that $char(R) \neq 2$. If R admits a left centralizer $T : R \to R$ such that $T([x, x^*]) \mp (x^*)^2 \in Z(R)$ for all $x \in R$, then R is commutative.*

Proof. We first consider the case

$$T([x, x^*]) - (x^*)^2 \in Z(R) \text{ for all } x \in R. \tag{2.1}$$

If T is zero then we have $-(x^*)^2 \in Z(R)$ for all $x \in R$. Replacing x by x^* we get $-x^2 \in Z(R)$ for all $x \in R$. On solving we get $x \in Z(R)$ for all $x \in R$. This implies that R is commutative. Now consider T to be nonzero. Linearizing (2.1), we obtain

$$T([x, y^*]) + T([y, x^*]) - (x^* \circ y^*) \in Z(R) \text{ for all } x, y \in R. \tag{2.2}$$

Replacing y by ky in (2.2) where $k \in S(R) \cap Z(R)$, we get $2T([y, x^*])k \in Z(R)$ for all $x, y \in R$. Since $char(R) \neq 2$ and $S(R) \cap Z(R) \neq (0)$, we arrive at

$$T([y, x]) \in Z(R) \text{ for all } x, y \in R. \tag{2.3}$$

Taking x^2 for x in (2.3), where $x \in R$, we get

$$T(x)[y, x] + T([y, x])x \in Z(R) \text{ for all } x, y \in R. \tag{2.4}$$

Substituting yx for y in (2.3), where $x, y \in R$ this implies that

$$T([y, x])x \in Z(R) \text{ for all } x, y \in R. \tag{2.5}$$

Using (2.5) in (2.4), we obtain

$$T(x)[y, x] \in Z(R) \text{ for all } x, y \in Z(R). \tag{2.6}$$

That is,

$$[T(x)[y, x], r] = 0 \text{ for all } x, y, r \in R.$$

This implies that

$$T(x)[[y,x],r]+[T(x),r][y,x]=0 \text{ for all } x,y,r\in R. \tag{2.7}$$

Putting yw for y in (2.7), where $w\in R$ we get

$$[T(x),r][yw,x]+T(x)[[yw,x],r]=0 \text{ for all } x,y,w,r\in R.$$

This further implies that

$$\begin{aligned}&[T(x),r]y[w,x]+[T(x),r][y,x]w+T(x)y[[w,x],r]+\\&T(x)[y,r][w,x]+T(x)[y,x][w,r]+T(x)[[y,x],r]w=0 \text{ for all } x,y,w,r\in R.\end{aligned} \tag{2.8}$$

Multiplying (2.7) by w on the right side, where $w\in R$ and subtracting from (2.8) we arrive at

$$[T(x),r]y[w,x]+T(x)y[[w,x],r]+T(x)[y,r][w,x]+T(x)[y,x][w,r]=0 \tag{2.9}$$

for all $x,y,w,r\in R$. Taking $w=x$ in (2.9) we get

$$T(x)[y,x][x,r]=0 \text{ for all } x,y,r\in R. \tag{2.10}$$

Replacing r by rt in (2.10), where $t\in R$.

$$T(x)[y,x]r[x,t]+T(x)[y,x][x,r]t=0 \text{ for all } x,y,r,t\in R. \tag{2.11}$$

Comparing (2.10) and (2.11) we arrive at

$$T(x)[y,x]r[x,t]=0 \text{ for all } x,y,r,t\in R.$$

Using the primeness of the ring R, for each fixed $x\in R$ we get either $[x,t]=0$ for all $t\in R$ or $T(x)[y,x]=0$ for all $y\in R$. Define $A=\{x\in R \mid [t,x]=0 \text{ for all } t\in R\}$ and $B=\{x\in R \mid T(x)[y,x]=0\}$. Clearly, A and B are additive subgroups of R whose union is R. Hence by Brauer's trick, either $A=R$ or $B=R$. If $A=R$, then $[t,x]=0$ for all $x,t\in R$. This implies that R is commutative. If $B=R$

$$T(x)[y,x]=0 \text{ for all } x,y\in R. \tag{2.12}$$

Replacing y by yt in (2.12) where $t\in R$ we get $T(x)y[t,x]+T(x)[y,x]t=0$ for all $x,y,t\in R$. Making use of (2.12), we obtain $T(x)y[t,x]=0$ for all $x,y,t\in R$. Then by the primeness of R, for each fixed $x\in R$, we get either $[t,x]=0$ for all $t\in R$ or $T(x)=0$. Applying the same Brauer's trick, we get either R is commutative or $T(x)=0$ for all $x\in R$. Since $T\neq 0$, we conclude that R is commutative. Second case can be proved in similar manner with necessary variations. ∎

Theorem 2.4. *Let R be a prime ring with involution $*$ of the second kind and let $char(R)\neq 2$. If R admits a left centralizer $T:R\to R$ such that $T(xx^*)\pm(x\circ x^*)\in Z(R)$ for all $x\in R$, then either R is commutative or T is the centralizer of R.*

Proof. We have

$$T(xx^*)\pm(x\circ x^*)\in Z(R) \text{ for all } x\in R. \tag{2.13}$$

If T is zero, then by Lemma 2.2 R is commutative. Now consider T is nonzero and linearizing (2.13)

$$T(xy^*)+T(yx^*)\pm(x\circ y^*)\pm(y\circ x^*)\in Z(R) \text{ for all } x,y\in R. \tag{2.14}$$

Replacing y by ky in (2.13) where $k\in S(R)\cap Z(R)$ we get

$$-T(xy^*)k+T(yx^*)k\mp(x\circ y^*)k\pm(y\circ x^*)k\in Z(R) \text{ for all } x,y\in R. \tag{2.15}$$

By (2.14) and (2.15) we obtain

$$2(T(yx^*) \pm (y \circ x^*))k \in Z(R) \text{ for all } x, y \in R.$$

Since $char(R) \neq 2$ and $S(R) \cap Z(R) \neq (0)$, this implies that

$$T(yx^*) \pm (y \circ x^*) \in Z(R) \text{ for all } x, y \in R. \tag{2.16}$$

Using h for x in (2.16) where $h \in H(R) \cap Z(R)$ we get

$$(T(y) \pm 2y)h \in Z(R) \text{ for all } y \in R. \tag{2.17}$$

This implies that

$$T(y) \pm 2y \in Z(R) \text{ for all } y \in R. \tag{2.18}$$

This can be further written as $[T(y), r] \pm [2y, r] = 0$ for all $y, r \in R$. Taking y for r we arrive at

$$[T(y), y] = 0 \text{ for all } y \in R. \tag{2.19}$$

Linearizing (2.19) we get

$$[T(y), x] + [T(x), y] = 0 \text{ for all } x, y \in R. \tag{2.20}$$

Replacing y by yr in (2.20) and combining with (2.20) we get

$$T(y)[r, x] + y[T(x), r] = 0 \text{ for all } x, y, r \in R. \tag{2.21}$$

Taking xw for x in (2.21) and with the help of (2.21) we get

$$(T(y)x - yT(x))[r, w] = 0 \text{ for all } x, y, r, w \in R. \tag{2.22}$$

Substituting rm for r in (2.22) and using (2.22) we get

$$(T(y)x - yT(x))r[m, w] = 0 \text{ for all } x, y, r, m, w \in R.$$

Then by the primeness of the ring R we get either R is commutative or T is centralizer of R. ∎

Theorem 2.5. *Let R be a prime ring with involution of the second kind such that $char(R) \neq 2$. If R admits a nonzero left centralizer $T : R \to R$ such that $T(x \circ x^*) \mp [T(x), x^*] \in Z(R)$ for all $x \in R$, then R is commutative.*

Proof. We first consider the case

$$T(x \circ x^*) - [T(x), x^*] \in Z(R) \text{ for all } x \in R. \tag{2.23}$$

Linearizing (2.23), we get

$$T(x \circ y^*) + T(y \circ x^*) - [T(x), y^*] - [T(y), x^*] \in Z(R) \text{ for all } x, y \in R. \tag{2.24}$$

Replacing y by ky in (2.24) and using it we get

$$2(T(y \circ x^*) - [T(y), x^*])k \in Z(R) \text{ for all } x, y \in R.$$

Since $char(R) \neq 2$ and $S(R) \cap Z(R) \neq (0)$, we obtain

$$T(y \circ x^*) - [T(y), x^*] \in Z(R) \text{ for all } x, y \in R. \tag{2.25}$$

Replacing x for $h \in H(R) \cap Z(R)$ in (2.25), we arrive at $2T(y)h \in Z(R)$ for all $y \in R$ and $h \in H(R) \cap Z(R)$. Since $char(R) \neq 2$ and $S(R) \cap Z(R) \neq (0)$, this implies that $T(y) \in Z(R)$ for all $y \in R$. This gives

$$[T(y), r] = 0 \text{ for all } y, r \in R. \tag{2.26}$$

Replacing y by yw in (2.26), we get $T(y)[w,r] = 0$ for all $y, w, r \in R$. Using yx for y, we get $T(y)x[w,r] = 0$ for all $y, x, w, r \in R$ then by the primeness of the ring R and the fact that $T \neq 0$, we have R is commutative. Taking positive sign, if we arguing in the same way as argued in the previous case we have the same conclusion.
Thus, finally we conclude that R is commutative. ∎

Theorem 2.6. *Let R be a prime ring with involution of the second kind such that $char(R) \neq 2$. If R admits a nonzero left centralizer $T : R \to R$ such that $[T(x), x^*] \mp T(x) \circ x^* \in Z(R)$ for all $x \in R$, then R is commutative.*

Proof. We have

$$[T(x), x^*] \mp T(x) \circ x^* \in Z(R) \text{ for all } x \in R. \tag{2.27}$$

Linearize (2.27), we get

$$[T(x), y^*] + [T(y), x^*] \mp T(x) \circ y^* \mp T(y) \circ x^* \in Z(R) \text{ for all } x, y \in R. \tag{2.28}$$

Replacing y by ky in (2.28) and using (2.28) we get

$$2([T(y), x^*] \mp T(y) \circ x^*)k \in Z(R) \text{ for all } x, y \in R. \tag{2.29}$$

This implies that

$$[T(y), x^*] \mp T(y) \circ x^* \in Z(R) \text{ for all } x, y \in R. \tag{2.30}$$

Substituting h for x in (2.30) where $h \in H(R) \cap Z(R)$ we obtain

$$2T(y)h \in Z(R) \text{ for all } y \in R. \tag{2.31}$$

This implies that $T(y) \in Z(R)$ for all $y \in R$. This can be further written as $[T(y), r] = 0$ for all $y, r \in R$. This is same as (2.26), so following the same technique as we used after (2.26), we get R is commutative. ∎

Theorem 2.7. *Let R be a prime ring with involution $*$ of the second kind such that $char(R) \neq 2$. If R admits a left centralizer $T : R \to R$ such that $T(xx^*) \pm [x, x^*] \in Z(R)$ for all $x \in R$, then R is commutative.*

Proof. We have

$$T(xx^*) \pm [x, x^*] \in Z(R) \text{ for all } x \in R. \tag{2.32}$$

If T is zero, then by Lemma 2.1, R is commutative. Now consider T is nonzero, linearizing (2.32) we get

$$T(x)y^* + T(y)x^* \pm [x, y^*] \pm [y, x^*] \in Z(R) \text{ for all } x, y \in R. \tag{2.33}$$

Replacing y by ky in (2.33), where $k \in S(R) \cap Z(R)$ and using (2.33) we get

$$2(T(y)x^* \pm [y, x^*])k \in Z(R) \text{ for all } x, y \in R.$$

Since $char(R) \neq 2$ and $S(R) \cap Z(R) \neq (0)$, this implies that

$$T(y)x^* \pm [y, x^*] \in Z(R) \text{ for all } x, y \in R.$$

This implies that

$$T(y)x \pm [y, x] \in Z(R) \text{ for all } x, y \in R.$$

Substituting z for x, where $z \in Z(R)$, we get $T(y)z \in Z(R)$ for all $y \in R$. Now using the primeness of the ring R we obtain either $T(y) \in Z(R)$ for all $y \in R$ or $z = 0$ for all

$z \in Z(R)$. Since $S(R) \cap Z(R) \neq (0)$, therefore we have $T(y) \in Z(R)$ for all $y \in R$. This can be further written as

$$[T(y), r] = 0 \text{ for all } y, r \in R. \tag{2.34}$$

Replacing y by yw in (2.34) and combining with (2.34), then we have $T(y)[w,r] = 0$ for all $y, w, r \in R$. Now substituting w by wm, we get

$$T(y)w[m, r] = 0 \text{ for all } y, w, m, r \in R.$$

Using the primeness of the ring R we get either $T(y) = 0$ for all $y \in R$ or $[m, r] = 0$ for all $m, r \in R$. Since T is nonzero, therefore we only have $[m, r] = 0$ for all $m, r \in R$. This implies that R is commutative. ∎

Theorem 2.8. *Let R be a prime ring with involution $*$ of the second kind such that $char(R) \neq 2$. If R admits a left centralizer $T : R \to R$ such that $T(x^2) \pm (x^*)^2 \in Z(R)$ for all $x \in R$, then R is commutative.*

Proof. We have

$$T(x^2) \pm (x^*)^2 \in Z(R) \text{ for all } x \in R. \tag{2.35}$$

If T is zero, then R is commutative. Now consider T is nonzero, linearizing (2.35) we get

$$T(x \circ y) \pm (x^* \circ y^*) \in Z(R) \text{ for all } x, y \in R. \tag{2.36}$$

Replacing y by ky in (2.36) where $k \in S(R) \cap Z(R)$ we get

$$T(x \circ y)k \mp (x^* \circ y^*)k \in Z(R) \text{ for all } x, y \in R. \tag{2.37}$$

Combining (2.36) and (2.37) we get

$$2T(x \circ y)k \in Z(R) \text{ for all } x, y \in R.$$

Since $char(R) \neq 2$ and $S(R) \cap Z(R) \neq (0)$, this implies that

$$T(x \circ y) \in Z(R) \text{ for all } x, y \in R. \tag{2.38}$$

Replacing y by nonzero $z \in Z(R)$ in (2.38) and using the given conditions $char(R) \neq 2$ and $S(R) \cap Z(R) \neq (0)$, this implies that $T(x) \in Z(R)$ for all $x \in R$. This can be further written as

$$[T(x), y] = 0 \text{ for all } x, y \in R. \tag{2.39}$$

Substituting xv for x and using (2.39), we get $T(x)[v, y] = 0$ for all $x, y, v \in R$. Taking xw for x, where $w \in R$ we obtain $T(x)w[v, y] = 0$ for all $x, w, v, y \in R$. Invoking the primeness of the ring R, we get either $T(x) = 0$ for all $x \in R$ or $[v, y] = 0$ for all $v, y \in R$. Since T is nonzero, therefore we only have $[v, y] = 0$ for all $v, y \in R$. This implies that R is commutative. ∎

The following example shows that the second kind involution assumption is essential in Theorems 2.4 2.5, 2.6 and 2.7.

Example 2.9. Let $R = \left\{ \begin{pmatrix} \beta_1 & \beta_2 \\ \beta_3 & \beta_4 \end{pmatrix} \middle| \beta_1, \beta_2, \beta_3, \beta_4 \in \mathbb{Z} \right\}$. Of course, R with matrix addition and matrix multiplication is a non commutative prime ring. Define mappings $*, T : R \longrightarrow R$ such that $\begin{pmatrix} \beta_1 & \beta_2 \\ \beta_3 & \beta_4 \end{pmatrix}^* = \begin{pmatrix} \beta_4 & -\beta_2 \\ -\beta_3 & \beta_1 \end{pmatrix}$ and $T\begin{pmatrix} \beta_1 & \beta_2 \\ \beta_3 & \beta_4 \end{pmatrix} = \begin{pmatrix} \alpha\beta_1 & \alpha\beta_2 \\ \alpha\beta_3 & \alpha\beta_4 \end{pmatrix}$, where α is a fixed number from $\mathbb{Z}$. Obviously, $Z(R) =$

$\left\{ \left(\begin{array}{cc} \beta_1 & 0 \\ 0 & \beta_1 \end{array} \right) \middle| \ \beta_1 \in \mathbb{Z} \right\}$, then $x^* = x$ for all $x \in Z(R)$, and hence $Z(R) \subseteq H(R)$, which shows that the involution $*$ is of the first kind. Moreover, T is a left centralizer of R such that it satisfies Theorems 2.4 2.5, 2.6 and 2.7. However, R is not commutative. Hence, the hypothesis of second kind involution is crucial in the said theorems.

Our next example shows that Theorems 2.4 2.5, 2.6 and 2.7 are not true for semiprime rings.

Example 2.10. Let $S = R \times \mathbb{C}$, where R is same as in Example 2.9 with involution $*$ and Left centralizer T same as in above example, $\mathbb{C}$ is the ring of complex numbers with conjugate involution τ. Hence, S is a noncommutative semiprime ring such that $char(R) \neq 2$. Now define an involution α on S, as $(x, y)^\alpha = (x^*, y^\tau)$. Clearly, α is an involution of the second kind. Further, we define a mappings T_1 on S as follows $T_1(x, y) = (T(x), 0)$ for all $(x, y) \in S$ where T is same as in above example. It can be easily checked that T_1 is a left centralizer on S and satisfying Theorems 2.4 2.5, 2.6 and 2.7, but S is not commutative. Hence, the hypothesis of primeness is essential in the Theorems 2.4 2.5, 2.6 and 2.7.

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