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Factorisable Monoid of Generalized Cohypersubstitutions of Type $\tau = (2)$

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Abstract Generalized cohypersubstitutions of type $\tau = (n_i)_{i\in I}$ are mappings which send n_i-ary co-operation symbols to coterms of type τ. Every generalized cohypersubstitution can be extended to a mapping on the set of all coterms. We define a binary operation on the set of all generalized cohypersubstitutions by using this extension. In this paper, we characterize all unit elements and determine the set of all unit-regular elements of this monoid of type $\tau = (2)$. Finally, a submonoid of the monoid of all generalized hypersubstitutions of type $\tau = (2)$ which is factorisable is presented.

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1. Introduction

The monoid of cohypersubstitutions of type τ is one topics on universal algebra that interested by many authors. The concept of cohypersubstitutions, the main tool used to study cohyperidentities, was firstly studied and introduced in 2009 by K. Denecke and K. Seangsura [1]. They defined terms for coalgebras, coidentities, cohyperidentities and applied all the concepts to construct the monoid of cohypersubstitutions of type τ. In 2013, S. Jermjitpornchai and N. Seangsura [2] generalized the concepts of [1] by studying on the generalized cohypersubstitutions. They introducde the coterms, generalized superpositions, some algebraic-structural properties and constructed the monoid of generalized cohypersubstitutions of type $\tau = (n_i)_{i\in I}$. There are many researchers interested in the monoid of generalized cohypersubstitutions of type $\tau = (2), \tau = (3)$ and $\tau = (n)$. They characterized the idempotent and regular elements of these structures and characterized other special elements. For studying on the factorisable monoid, in 2015, A. Boonmee and S. Leeratanavalee [3] focused on the monoid of generalized hypersubstitutions of type

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τ. They characterized the set of all unit elements of this monoid $[U(Hyp_G(n))]$ and determined the set of all unit-regular elements of this monoid of type $\tau = (2)$ $[UR(Hyp_G(2))]$. Moreover, they showed that $[UR(Hyp_G(2))]$ was a maximal unit-regular submonoid of this monoid which was factorisable. Afterwards, in 2016, the same authors [4] generalized the concepts of [3] by presenting the set of all unit-regular elements of the monoid of generalized hypersubstitutions of type $\tau = (n)$ $[UR(Hyp_G(n))]$ and showed that it was a maximal unit-regular submonoid of this monoid which was factorisable. More related topics on terms may be seen [5–7].

In this present paper, we focus on the monoid of generalized cohypersubstitutions of type τ. We apply the concepts of [3] for characterizations of the set of all unit elements of this monoid. We fix a type $\tau = (2)$, and determine the set of all unit-regular elements of this monoid. In the last of this study, we show that it is a maximal unit-regular submonoid of the monoid of generalized cohypersubstitutions which is factorisable.

2. Monoid of Generalized Cohypersubstitutions

In this section, we provide the basic concept of the monoid of set of all generalized cohypersubstitutions which is very useful to this research.

Let A be a non-empty set and $n \in \mathbb{N}$. Define the union of n disjoint copies of A by $A^{\sqcup n} := \underline{n} \times A$ where $\underline{n} = \{1, 2, \ldots, n\}$, so it is called the n-th copower of A. An element (i, a) in this copower corresponds to the element a in the i-th copy of A where $i \in \underline{n}$. A mapping $f^A : A \to A^{\sqcup n}$ is a co-operation on A; the natural number n is called the arity of the co-operation f^A. Every n-ary co-operation f^A on the set A can be uniquely expressed as the pair of mappings (f_1^A, f_2^A) where $f_1^A : A \to \underline{n}$ gives the labelling used by f^A in mapping elements to copies of A, and $f_2^A : A \to A$ tells us what element of A any element is mapped to, so $f^A(a) = (f_1^A(a), f_2^A(a))$. We denoted the set of all n-ary co-operations defined on A by $cO_A^{(n)} = \{f^A : A \to A^{\sqcup n}\}$.

Let $\tau = (n_i)_{i \in I}$ and let $(f_i)_{i \in I}$ be an indexed set of co-operation symbols which f_i has arity n_i for each $i \in I$. Let $\bigcup \{e_j^n \mid n \geq 1, n \in \mathbb{N}, 0 \leq j \leq n-1\}$ be a set of symbols which disjoint from $\{f_i \mid i \in I\}$ such that e_j^n has arity n for each $0 \leq j \leq n-1$. The *coterms* of type τ are defined as follows:

(i) For every $i \in I$, the co-operation symbol f_i is an n_i-ary coterm of type τ.
(ii) For every $n \geq 1$ and $0 \leq j \leq n-1$ the symbol e_j^n is an n-ary coterm of type τ.
(iii) If $t_1, \ldots, t_{n_i}$ are n-ary coterms of type τ, then $f_i[t_1, \ldots, t_{n_i}]$ is an n-ary coterm of type τ for every $i \in I$, and if $t_0, \ldots, t_{n-1}$ are m-ary coterms of type τ, then $e_j^n[t_0, \ldots, t_{n-1}]$ is an m-ary coterm of type τ for every $0 \leq j \leq n-1$.

Let $CT_\tau^{(n)}$ be the set of all n-ary coterms of type τ, and $CT_\tau := \bigcup_{n \geq 1} CT_\tau^{(n)}$ the set of all coterms of type τ.

Definition 2.1. [2] Let $m \in \mathbb{N}^+ = \mathbb{N} \cup \{0\}$. A generalized superposition of coterms $S^m : CT_\tau^{m+1} \to CT_\tau$ is defined inductively by the following steps:

(i) If $t = e_i^n$ and $0 \leq i \leq m-1$, then $S^m(e_i^n, t_0, \ldots, t_{m-1}) = t_i$, where $t_0, \ldots, t_{m-1} \in CT_\tau$.
(ii) If $t = e_i^n$ and $0 < m \leq i \leq n-1$, then $S^m(e_i^n, t_0, \ldots, t_{m-1}) = e_i^n$, where $t_0, \ldots, t_{m-1} \in CT_\tau$.

(iii) If $t = f_i[s_1, \ldots, s_{n_i}]$, then
$S^m(t, t_1, \ldots, t_m) = f_i(S^m(s_1, t_1, \ldots, t_m), \ldots, S^m(s_{n_i}, t_1, \ldots, t_m))$,
where $S^m(s_1, t_1, \ldots, t_m), \ldots, S^m(s_{n_i}, t_1, \ldots, t_m) \in CT_\tau$.

The above definition can be written as the following forms:

(i) If $t = e_i^n$ and $0 \leq i \leq m-1$, then $e_i^n[t_0, \ldots, t_{m-1}] = t_i$,
where $t_0, \ldots, t_{m-1} \in CT_\tau$.

(ii) If $t = e_i^n$ and $0 < m \leq i \leq n-1$, then $e_i^n[t_0, \ldots, t_{m-1}] = e_i^n$,
where $t_0, \ldots, t_{m-1} \in CT_\tau$.

(iii) If $t = f_i[s_1, \ldots, s_{n_i}]$, then
$(f_i[s_1, \ldots, s_{n_i}])[t_1, \ldots, t_m] = f_i(s_1[t_1, \ldots, t_m], \ldots, s_{n_i}[t_1, \ldots, t_m])$,
where $s_1[t_1, \ldots, t_m], \ldots, s_{n_i}[t_1, \ldots, t_m] \in CT_\tau$.

Definition 2.2. [2] A *generalized cohypersubstitution* of type τ is a mapping $\sigma : \{f_i \,|\, i \in I\} \to CT_\tau$. The extension of σ is a mapping $\hat{\sigma} : CT_\tau \to CT_\tau$ which is inductively defined by the following steps :

(i) $\hat{\sigma}(e_j^n) := e_j^n$ for every $n \geq 1$ and $0 \leq j \leq n-1$,

(ii) $\hat{\sigma}(f_i) := \sigma(f_i)$ for every $i \in I$,

(iii) $\hat{\sigma}(f_i[t_1, \ldots, t_{n_i}]) := \sigma(f_i)[\hat{\sigma}(t_1), \ldots, \hat{\sigma}(t_{n_i})$ for $t_1, \ldots t_{n_i} \in CT_\tau^{(n)}$.

Let $Cohyp_G(\tau)$ be the set of all generalized cohypersubstitutions of type τ.

Proposition 2.3. [2] *If $t, t_1, \ldots, t_n \in CT_\tau$ and $\sigma \in Cohyp_G(\tau)$, then*

$$\hat{\sigma}(t[t_1, \ldots, t_n]) = \hat{\sigma}(t)[\hat{\sigma}(t_1), \ldots, \hat{\sigma}(t_n)].$$

On the set $Cohyp_G(\tau)$ of all generalized cohypersubstitutions of type τ we may define a function $\circ_{CG} : Cohyp_G(\tau) \times Cohyp_G(\tau) \to Cohyp_G(\tau)$ by $\sigma_1 \circ_{CG} \sigma_2 := \hat{\sigma_1} \circ \sigma_2$ for all $\sigma_1, \sigma_2 \in Cohyp_G(\tau)$ where $\circ$ is the usual composition of mappings. Let σ_{id} be the generalized cohypersubstitution such that $\sigma_{id}(f_i) := f_i[e_0^n, e_1^n, \ldots, e_{n_i-1}^n]$ for all $i \in I$. Then σ_{id} is an identity element in $Cohyp_G(\tau)$. Thus $\underline{Cohyp_G(\tau)} := (Cohyp_G(\tau), \circ_{CG}, \sigma_{id})$ is a monoid and called the *monoid of generalized cohypersubstitutions of type* τ. The algebraic structural-properties of the monoid $\underline{Cohyp_G(\tau)}$ can be found in [2].

Throughout this paper, we denote:

$\sigma_t :=$ the generalized cohypersubstitution σ of type τ which maps f to the coterm t,
$e_j^n :=$ the injection symbol for all $0 \leq j \leq n-1, n \in \mathbb{N}$,
$E :=$ the set of all injection symbols, i.e. $E := \{e_j^n \mid n, j \in \mathbb{N}\}$,
$E(t) :=$ the set of all injection symbols occuring in the coterm t,
$leftmost_{inj}(t) :=$ the first injection symbol (from the left) occuring in the coterm t,
$rightmost_{inj}(t) :=$ the last injection symbol occuring in the coterm t.

3. Unit Elements in the Monoid $\underline{Cohyp_G(n)}$

In this section, we focus on the type $\tau = (n)$ and characterize the unit elements of the monoid $\underline{Cohyp_G(n)}$. Firstly, we recall the definition of unit element as follow.

Definition 3.1. Let $\mathcal{M}$ be a monoid. An element $u \in M$ is call *unit* if there exists $u^{-1} \in \mathcal{M}$ such that $uu^{-1} = e = u^{-1}u$ where e is an identity element of $\mathcal{M}$. The set of all unit elements of $\mathcal{M}$ is denoted by $\mathcal{U}(\mathcal{M})$.

To give a characterization of unit elements in $\underline{Cohyp_G(n)}$, the following two lemmas are needed.

Lemma 3.2. *Let $\sigma_t \in Cohyp_G(n)$, where $t = f[t_0, \ldots, t_{n-1}] \in CT_{(n)}$. If $t_i \in CT_{(n)} \setminus E$ for some $i \in \{0, 1, \ldots, n-1\}$, then σ_t is not unit element.*

Proof. Let $t = f[t_0, \ldots, t_{n-1}] \in CT_{(n)}$ where $t_i \in CT_{(n)} \setminus E$ for some $i \in \{0, 1, \ldots, n-1\}$. Let $\sigma_s \in Cohyp_G(n)$ and $s = f[s_0, \ldots, s_{n-1}] \in CT_{(n)}$ where $s_i \in CT_{(n)}$ for all $i \in \{0, 1, \ldots, n-1\}$. Then

$$\begin{aligned}(\sigma_t \circ_{CG} \sigma_s)(f) &= \hat{\sigma}_t(f[s_0, \ldots, s_{n-1}]) \\ &= (\sigma_t(f))[\hat{\sigma}_t(s_0), \ldots, \hat{\sigma}_t(s_{n-1})] \\ &= (f[t_0, \ldots, t_i, \ldots, t_{n-1}])[\hat{\sigma}_t(s_0), \ldots, \hat{\sigma}_t(s_{n-1})] \\ &= f[t_0[\hat{\sigma}_t(s_0), \ldots, \hat{\sigma}_t(s_{n-1})], \ldots, t_i[\hat{\sigma}_t(s_0), \ldots, \hat{\sigma}_t(s_{n-1})], \ldots, \\ &\quad t_{n-1}[\hat{\sigma}_t(s_0), \ldots, \hat{\sigma}_t(s_{n-1})]].\end{aligned}$$

Since $t_i \notin E$. Then $t_i[\hat{\sigma}_t(s_0), \ldots, \hat{\sigma}_t(s_{n-1})] \neq e_i^n; 0 \leq i \leq n-1$. Therefore $(\sigma_t \circ_{CG} \sigma_s)(f) \neq f[e_0^n, \ldots, e_i^n, \ldots, e_{n-1}^n] = \sigma_{id}(f)$ for all $\sigma_s \in Cohyp_G(n)$. ∎

Lemma 3.3. *Let $\sigma_t \in Cohyp_G(n)$, where $t = f[e_{j_0}^n, \ldots, e_{j_{n-1}}^n] \in CT_{(n)}$. If $j_i \geq n$ for some $i \in \{0, 1, \ldots, n-1\}$, then σ_t is not unit element.*

Proof. Let $t = f[e_{j_0}^n, \ldots, e_{j_{n-1}}^n]$, where $j_i \geq n$ for some $i \in \{0, 1, \ldots, n-1\}$ and let $\sigma_s \in Cohyp_G(n)$, where $s = f[s_0, \ldots, s_{n-1}]$. Then

$$\begin{aligned}(\sigma_t \circ_{CG} \sigma_s)(f) &= \hat{\sigma}_t(f[s_0, \ldots, s_{n-1}]) \\ &= (\sigma_t(f))[\hat{\sigma}_t(s_0), \ldots, \hat{\sigma}_t(s_{n-1})] \\ &= (f[e_{j_0}^n, \ldots, e_{j_i}^n, e_{j_{n-1}}^n])[\hat{\sigma}_t(s_0), \ldots, \hat{\sigma}_t(s_{n-1})] \\ &= f[e_{j_0}^n[\hat{\sigma}_t(s_0), \ldots, \hat{\sigma}_t(s_{n-1})], \ldots, e_{j_i}^n[\hat{\sigma}_t(s_0), \ldots, \hat{\sigma}_t(s_{n-1})], \ldots, \\ &\quad e_{j_{n-1}}^n[\hat{\sigma}_t(s_0), \ldots, \hat{\sigma}_t(s_{n-1})]].\end{aligned}$$

Since $j_i \geq n$, then $e_{j_i}^n[\hat{\sigma}_t(s_0), \ldots, \hat{\sigma}_t(s_{n-1})] \neq e_i^n; 0 \leq i \leq n-1.$. So, that means $(\sigma_t \circ_{CG} \sigma_s)(f) \neq \sigma_{id}(f)$ for all $\sigma_s \in Cohyp_G(n)$. Hence σ_t is not unit in $Cohyp_G(n)$. ∎

Theorem 3.4. *Let $n \in \mathbb{N}^+$ and let S be a set of all permutation on the set $\{0, 1, \ldots, n-1\}$. An element $\sigma_t \in \mathcal{U}(Cohyp_G(n))$ if and only if $t = f[e_{\pi(0)}^n, \ldots, e_{\pi(n-1)}^n]$, where $\pi \in S_n$.*

Proof. Assume that $\sigma_t \in \mathcal{U}(Cohyp_G(n))$. Then there exists $\sigma_s \in Cohyp_G(n)$ such that $\sigma_t \circ_{CG} \sigma_s = \sigma_{id} = \sigma_s \circ_{CG} \sigma_t$. By Lemma 3.2 and Lemma 3.3, if $t = f[t_0, \ldots, t_{n-1}]$ and $s = f[s_0, \ldots, s_{n-1}]$, then $t_i, s_j \in \{e_0^n, \ldots, e_{n-1}^n\}$ for all $i, j \in \{0, 1, \ldots, n-1\}$. Let $\pi, \pi' \in S$ and $s = f[e_{\pi'(0)}^n, \ldots, e_{\pi'(n-1)}^n]$. Consider

$$\begin{aligned}\sigma_{id}(f) &= (\sigma_t \circ_{CG} \sigma_s)(f) \\ &= \hat{\sigma}_t(f[e_{\pi'(0)}^n, e_{\pi'(1)}^n, \ldots, e_{\pi'(n-1)}^n]) \\ f[e_0^n, e_1^n, \ldots, e_{n-1}^n] &= (f[e_{\pi(0)}^n, e_{\pi(1)}^n, \ldots, e_{\pi(n-1)}^n])[e_{\pi'(0)}^n, e_{\pi'(1)}^n, \ldots, e_{\pi'(n-1)}^n] \\ &= f[e_{\pi'(\pi(0))}^n, e_{\pi'(\pi(1))}^n, \ldots, e_{\pi'(\pi(n-1))}^n] \\ &= f[e_{(\pi' \circ \pi)(0)}^n, e_{(\pi' \circ \pi)(1)}^n, \ldots, e_{(\pi' \circ \pi)(n-1)}^n].\end{aligned}$$

And

$$\begin{aligned}\sigma_{id}(f) &= (\sigma_s \circ_{CG} \sigma_t)(f)\\ &= \hat{\sigma}_s(f[e^n_{\pi(0)}, e^n_{\pi(1)}, \ldots, e^n_{\pi(n-1)}])\\ f[e^n_0, e^n_1, \ldots, e^n_{n-1}] &= (f[e^n_{\pi'(0)}, e^n_{\pi'(1)}, \ldots, e^n_{\pi'(n-1)}])[e^n_{\pi(0)}, e^n_{\pi(1)}, \ldots, e^n_{\pi(n-1)}]\\ &= f[e^n_{\pi(\pi'(0))}, e^n_{\pi(\pi'(1))}, \ldots, e^n_{\pi(\pi'(n-1))}]\\ &= f[e^n_{(\pi\circ\pi')(0)}, e^n_{(\pi\circ\pi')(1)}, \ldots, e^n_{(\pi\circ\pi')(n-1)}].\end{aligned}$$

Then $\pi\circ\pi' = (1) = \pi'\circ\pi$ and $\pi\circ\pi', \pi'\circ\pi$ are bijective. Next, we will show that $\pi \in S_n$. Let $\pi(i) = \pi(j)$ for some $i, j \in \{0, \ldots, n-1\}$. Then $(\pi' \circ \pi)(i) = \pi'(\pi(i)) = \pi'(\pi(j)) = (\pi'\circ\pi)(j)$. Since $\pi'\circ\pi$ is one-to-one, so $i = j$. Thus π is one-to-one. Let $i \in \{0, \ldots, n-1\}$. Since $\pi \circ \pi'$ is onto, there exists $j \in \{0, \ldots, n-1\}$ such that $(\pi \circ \pi')(j) = i = \pi(\pi'(j))$ for some $\pi'(j) \in \{0, \ldots, n-1\}$. Hence π is onto, so $\pi \in S_n$.

Conversely, let $\sigma_t \in Cohyp_G(n)$ where $t = f[e^n_{\pi(0)}, e^n_{\pi(1)}, \ldots, e^n_{\pi(n-1)}]$ and $\pi \in S_n$. Since $(S_n, \circ)$ is a group, there exists $\pi' \in S_n$ such that $\pi \circ \pi' = (1) = \pi' \circ \pi$.

Let $\sigma_s \in Cohyp_G(n)$ where $s = f[e^n_{\pi'(0)}, e^n_{\pi'(1)}, \ldots, e^n_{\pi'(n-1)}]$. Then

$$\begin{aligned}(\sigma_t \circ_{CG} \sigma_s)(f) &= \hat{\sigma}_t(f[e^n_{\pi'(0)}, e^n_{\pi'(1)}, \ldots, e^n_{\pi'(n-1)}])\\ &= (f[e^n_{\pi(0)}, e^n_{\pi(1)}, \ldots, e^n_{\pi(n-1)}])[e^n_{\pi'(0)}, e^n_{\pi'(1)}, \ldots, e^n_{\pi'(n-1)}]\\ &= f[e^n_{\pi'(\pi(0))}, e^n_{\pi'(\pi(1))}, \ldots, e^n_{\pi'(\pi(n-1))}]\\ &= f[e^n_{(\pi'\circ\pi)(0)}, e^n_{(\pi'\circ\pi)(1)}, \ldots, e^n_{(\pi'\circ\pi)(n-1)}]\\ &= f[e^n_0, e^n_1, \ldots, e^n_{n-1}]\\ &= \sigma_{id}(f).\end{aligned}$$

Similarly, we have $\sigma_s \circ_{CG} \sigma_t = \sigma_{id}$. Hence $\sigma_t \in \mathcal{U}(Cohyp_G(n))$. ∎

Corollary 3.5. $|\mathcal{U}(Cohyp_G(n))| = n!$.

Example 3.6. Let $\tau = (2)$, we have $\mathcal{U}(Cohyp_G(2)) = \{\sigma_{f[e^2_0,e^2_1]} = \sigma_{id}, \sigma_{f[e^2_1,e^2_0]}\}$.

4. Unit Regular Elements and Factorisable Monoid

In this section, we fix a type $\tau = (2)$ and characterize the set of all unit-regular elements. We show that it is a factorisable. We first recall the definitions and the sets of elements using in this section. Let S be a semigroup and an element $e \in S$ is called *idempotent* if $e^2 = ee = e$, and we denote the set of all idempotent elements in S by $\mathcal{E}(S)$. An element $a \in S$ is called *regular* if there exists $x \in S$ such that $axa = a$. A semigroup S is called regular if all its elements are regular.

Definition 4.1. An element a of a monoid S is called *unit-regular* if there exists $u \in \mathcal{U}(S)$ such that $aua = a$. A monoid S is called *unit-regular* if all its elements are unit-regular.

Next, we fix type $\tau = (2)$ with a binary cooperation symbol f and denote the set of element of the monoid $Cohyp_G(2)$ as follows.

Let $\sigma_t \in Cohyp_G(2)$, we denote

$E_0 := \{\sigma_{e^2_0}, \sigma_{e^2_1}, \sigma_{id}, \sigma_{f[e^2_1,e^2_0]}\}$,
$E_1 := \{\sigma_t \mid E(t) \cap \{e^2_0, e^2_1\} = \emptyset\}$,
$E_2 := \{\sigma_t \mid t = f[e^2_0, s] \text{ where } E(t) \cap \{e^2_0, e^2_1\} = \{e^2_0\}, s \in CT_{(2)}\}$,
$E_3 := \{\sigma_t \mid t = f[s, e^2_1] \text{ where } E(t) \cap \{e^2_0, e^2_1\} = \{e^2_1\}, s \in CT_{(2)}\}$,

$E_4 := \{\sigma_t \mid t = f[s, e_0^2] \text{ where } E(t) \cap \{e_0^2, e_1^2\} = \{e_0^2\}, s \in CT_{(2)}\},$
$E_5 := \{\sigma_t \mid t = f[e_1^2, s] \text{ where } E(t) \cap \{e_0^2, e_1^2\} = \{e_1^2\}, s \in CT_{(2)}\}.$

In 2013 [8], N. Seangsura and S. Jermjitpornchai showed that : $\bigcup_{n=0}^{5} E_n$ is a set of all regular elements in $\underline{Cohyp_G(2)}$ and $(\bigcup_{n=0}^{3} E_n) \setminus \{\sigma_{f[e_1^2,e_0^2]}\} = \mathcal{E}(Cohyp_G(2))$.

Lemma 4.2. $\bigcup_{n=0}^{5} E_n$ *is a set of all unit-regular elements in* $\underline{Cohyp_G(2)}$.

Proof. Let $\sigma_t \in \bigcup_{n=0}^{5} E_n$. Then $\sigma_t \in (\bigcup_{n=0}^{3} E_n) \setminus \{\sigma_{f[e_1^2,e_0^2]}\}$ or $\sigma_t \in E_4$ or $\sigma_t \in E_5$ or $\sigma_t = \sigma_{f[e_1^2,e_0^2]}$. From Corollary 3.5, we have $\mathcal{U}(Cohyp_G(2)) = \{\sigma_{f[e_0^2,e_1^2]} = \sigma_{id}, \sigma_{f[e_1^2,e_0^2]}\}$.

Case 1: $\sigma_t \in (\bigcup_{n=0}^{3} E_n) \setminus \{\sigma_{f[e_1^2,e_0^2]}\} = \mathcal{E}(Cohyp_G(2))$. Then there exists $\sigma_{id} \in \mathcal{U}(Cohyp_G(2))$ such that $(\sigma_t \circ_{CG} \sigma_{id} \circ_{CG} \sigma_t)(f) = (\sigma_t \circ_{CG} \sigma_t)(f) = \sigma_t(f)$.

Case 2: $\sigma_t \in E_4$. Then $t = f[s, e_0^2]$ where $s \in CT_{(2)}$ and $E(t) \cap \{e_0^2, e_1^2\} = \{e_0^2\}$. There exists $\sigma_{f[e_1^2,e_0^2]} \in \mathcal{U}(Cohyp_G(2))$ such that

$$\begin{aligned}(\sigma_t \circ_{CG} \sigma_{f[e_1^2,e_0^2]} \circ_{CG} \sigma_t)(f) &= \hat{\sigma}_t(\hat{\sigma}_{f[e_1^2,e_0^2]}(f[s, e_0^2]))\\ &= \hat{\sigma}_t((f[e_1^2, e_0^2])[\hat{\sigma}_{f[e_1^2,e_0^2]}(s), e_0^2])\\ &= \hat{\sigma}_t(f[e_0^2, \hat{\sigma}_{f[e_1^2,e_0^2]}(s)])\\ &= (f[s, e_0^2])[e_0^2, \hat{\sigma}_t(\hat{\sigma}_{f[e_1^2,e_0^2]}(s))]\\ &= f[s, e_0^2] = \sigma_t(f) \text{ since } E(t) \cap \{e_0^2, e_1^2\} = \{e_0^2\}.\end{aligned}$$

Thus $\sigma_t \circ_{CG} \sigma_{f[e_1^2,e_0^2]} \circ_{CG} \sigma_t = \sigma_t$.

Case 3 : $\sigma_t \in E_5$. Then $t = f[e_1^2, s]$ where $s \in CT_{(2)}$ and $E(t) \cap \{e_0^2, e_1^2\} = \{e_1^2\}$. There exists $\sigma_{f[e_1^2,e_0^2]} \in \mathcal{U}(Cohyp_G(2))$ such that

$$\begin{aligned}(\sigma_t \circ_{CG} \sigma_{f[e_1^2,e_0^2]} \circ_{CG} \sigma_t)(f) &= \hat{\sigma}_t(\hat{\sigma}_{f[e_1^2,e_0^2]}(f[e_1^2, s]))\\ &= \hat{\sigma}_t((f[e_1^2, e_0^2])[e_1^2, \hat{\sigma}_{f[e_1^2,e_0^2]}(s)])\\ &= \hat{\sigma}_t(f[\hat{\sigma}_{f[e_1^2,e_0^2]}(s), e_1^2])\\ &= (f[e_1^2, s])[\hat{\sigma}_t(\hat{\sigma}_{f[e_1^2,e_0^2]}(s)), e_1^2]\\ &= f[e_1^2, s] = \sigma_t(f), \text{ since } E(t) \cap \{e_0^2, e_1^2\} = \{e_1^2\}.\end{aligned}$$

Thus $\sigma_t \circ_{CG} \sigma_{f[e_1^2,e_0^2]} \circ_{CG} \sigma_t = \sigma_t$.

Case 4 : $\sigma_t = \sigma_{f[e_1^2,e_0^2]}$. Then there exists $\sigma_{f[e_1^2,e_0^2]} \in \mathcal{U}(Cohyp_G(2))$ such that $(\sigma_{f[e_1^2,e_0^2]} \circ_{CG} \sigma_{f[e_1^2,e_0^2]} \circ_{CG} \sigma_{f[e_1^2,e_0^2]})(f) = (\sigma_{id} \circ_{CG} \sigma_{f[e_1^2,e_0^2]})(f) = \sigma_{f[e_1^2,e_0^2]}(f)$.

Hence, for any $\sigma_t \in \bigcup_{n=0}^{5} E_n$, there exists $\sigma_u \in \mathcal{U}(Cohyp_G(2))$ such that $\sigma_t \circ_{CG} \sigma_u \circ_{CG} \sigma_t = \sigma_t$. Therefore $\bigcup_{n=0}^{5} E_n$ is a set of all unit-regular elements in $Cohyp_G(2)$. ∎

We see that $\bigcup\limits_{n=0}^{5} E_n$ is not a submonoid of $\underline{Cohyp_G(2)}$ as the following example.

Example 4.3. (1) Let $\sigma_t \in E_2$ and $\sigma_r \in E_3$ such that $t = f[e_0^2, f[e_2^2, e_0^2]]$ and $r = f[f[e_1^2, e_3^2], e_1^2]$. Consider

$$\begin{aligned}(\sigma_t \circ_{CG} \sigma_r)(f) &= \hat{\sigma}_t(f[f[e_1^2, e_3^2], e_1^2]) \\ &= (\sigma_t(f))[\hat{\sigma}_t(f[e_1^2, e_3^2]), e_1^2] \\ &= (f[e_0^2, f[e_2^2, e_0^2]])[f[e_1^2, f[e_2^2, e_1^2]], e_1^2] \\ &= f[f[e_1^2, f[e_2^2, e_1^2]], f[e_2^2, f[e_1^2, f[e_2^2, e_1^2]]]].\end{aligned}$$

So $\sigma_t \circ_{CG} \sigma_r \notin \bigcup\limits_{n=0}^{5} E_n$.

(2) Let $\sigma_t \in E_4$ such that $t = f[f[e_0^2, e_5^2], e_0^2]$. Then

$$\begin{aligned}(\sigma_t \circ_{CG} \sigma_t)(f) &= \hat{\sigma}_t(f[f[e_0^2, e_5^2], e_0^2]) \\ &= (\sigma_t(f))[\hat{\sigma}_t(f[e_0^2, e_5^2]), e_0^2] \\ &= (f[f[e_0^2, e_5^2], e_0^2])[f[f[e_0^2, e_5^2], e_0^2], e_0^2] \\ &= f[f[f[f[e_0^2, e_5^2], e_0^2], e_5^2], f[f[e_0^2, e_5^2], e_0^2]].\end{aligned}$$

So $\sigma_t \circ_{CG} \sigma_t \notin \bigcup\limits_{n=0}^{5} E_n$.

(3) Let $\sigma_t \in E_5$ such that $t = f[e_1^2, f[e_4^2, e_1^2]]$. Then

$$\begin{aligned}(\sigma_t \circ_{CG} \sigma_t)(f) &= \hat{\sigma}_t(f[e_1^2, f[e_4^2, e_1^2]]) \\ &= (\sigma_t(f))[e_1^2, \hat{\sigma}_t(f[e_4^2, e_1^2])] \\ &= (f[e_1^2, f[e_4^2, e_1^2]])[e_1^2, f[e_1^2, f[e_4^2, e_1^2]]] \\ &= f[f[e_1^2, f[e_4^2, e_1^2]], f[e_4^2, f[e_1^2, f[e_4^2, e_1^2]]]].\end{aligned}$$

So $\sigma_t \circ_{CG} \sigma_t \notin \bigcup\limits_{n=0}^{5} E_n$.

Therefore, $\bigcup\limits_{n=0}^{5} E_n$ is not a submonoid of $\underline{Cohyp_G(2)}$.

Let $\sigma_t \in Cohyp_G(2)$, we denote

$E_2' := \{\sigma_t \mid t = f[e_0^2, s]$ where $E(t) \cap \{e_0^2, e_1^2\} = \{e_0^2\}, s \in CT_{(2)}$ and $rightmost_{inj}(s) \neq e_0^2\}$,

$E_3' := \{\sigma_t \mid t = f[s, e_1^2]$ where $E(t) \cap \{e_0^2, e_1^2\} = \{e_1^2\}, s \in CT_{(2)}$ and $leftmost_{inj}(s) \neq e_1^2\}$,

$E_4' := \{\sigma_t \mid t = f[s, e_0^2]$ where $E(t) \cap \{e_0^2, e_1^2\} = \{e_0^2\}, s \in CT_{(2)}$ and $leftmost_{inj}(s) \neq e_0^2\}$,

$E_5' := \{\sigma_t \mid t = f[e_1^2, s]$ where $E(t) \cap \{e_0^2, e_1^2\} = \{e_1^2\}, s \in CT_{(2)}$ and $rightmost_{inj}(s) \neq e_1^2\}$.

In 2020, N. Chansuriya [9] presented the following proposition.

Proposition 4.4. [9] *$E_2' \cup \{\sigma_{id}\}$ and $E_3' \cup \{\sigma_{id}\}$ are submonoids of $\underline{Cohyp_G(2)}$.*

Now, we see that $E'_4 \subset E_4, E'_5 \subset E_5$ and also have the following proposition.

Proposition 4.5. *$E'_4 \cup \{\sigma_{id}\}$ and $E'_5 \cup \{\sigma_{id}\}$ are submonoids of $\underline{Cohyp_G(2)}$.*

Proof. It is easy to see that $E'_4 \subset Cohyp_G(2)$ Next, we will show that E'_4 closed under the binary operation $\circ_{CG}$. Let $\sigma_t, \sigma_r \in E'_4$. Then $t = f[s, e^2_0]$ where $E(t) \cap \{e^2_0, e^2_1\} = \{e^2_0\}, s \in CT_{(2)}, leftmost_{inj}(s) \neq e^2_0$ and $r = f[s', e^2_0]$ where $E(r) \cap \{e^2_0, e^2_1\} = \{e^2_0\}, s' \in CT_{(2)}, leftmost_{inj}(s') \neq e^2_0$. Consider

$$\begin{aligned}(\sigma_t \circ_{CG} \sigma_r)(f) &= \hat{\sigma}_t(f[s', e^2_0]) \\ &= (\sigma_t(f))[\hat{\sigma}_t(s'), e^2_0] \\ &= (f[s, e^2_0])[\hat{\sigma}_t(s'), e^2_0] \\ &= f[s, e^2_0], \text{ since } E(t) \cap \{e^2_0, e^2_1\} = \{e^2_0\}.\end{aligned}$$

By the same way, we have $(\sigma_r \circ_{CG} \sigma_t)(f) = \sigma_{f[s', e^2_0]}(f)$. Hence $\sigma_t \circ_{CG} \sigma_r, \sigma_r \circ_{CG} \sigma_t \in E'_4$. Therefore, $E'_4 \cup \{\sigma_{id}\}$ is a submonoid of $\underline{Cohyp_G(2)}$.

Similarly, we can prove that $E'_5 \cup \{\sigma_{id}\}$ is a submonoid of $\underline{Cohyp_G(2)}$. ∎

To determine the unit-regular submonoid of $\underline{Cohyp_G(2)}$, we denote the set of all unit-regular elements of $\underline{Cohyp_G(2)}$ by $\mathcal{UR}(Cohyp_G(2)) = E_0 \cup E_1 \cup (\bigcup_{n=2}^{5} E'_n)$.

Theorem 4.6. *$\mathcal{UR}(Cohyp_G(2))$ is the unit-regular submonoid of $\underline{Cohyp_G(2)}$.*

Proof. It is easy to see that $\mathcal{UR}(Cohyp_G(2)) \subset Cohyp_G(2)$ and every elements in $\mathcal{UR}(Cohyp_G(2))$ is unit-regular. Then we will prove that $\mathcal{UR}(Cohyp_G(2))$ is a submonoid of $Cohyp_G(2)$. We have the following cases.

Case 1: $\sigma_t \in E'_2$. Then $t = f[e^2_0, s]$ where $E(t) \cap \{e^2_0, e^2_1\} = \{e^2_0\}, s \in CT_{(2)}$ and $rightmost_{inj}(s) \neq e^2_0$. Let $\sigma_r \in \mathcal{UR}(Cohyp_G(2))$. We consider the six subcases:

Case 1.1: $\sigma_r \in E_0$, so $r \in \{e^2_0, e^2_1, f[e^2_0, e^2_1], f[e^2_1, e^2_0]\}$.

If $r = e^2_0$, then $(\sigma_t \circ_{CG} \sigma_r)(f) = \hat{\sigma}_t(e^2_0) = e^2_0$ and $(\sigma_r \circ_{CG} \sigma_t)(f) = \hat{\sigma}_r(f[e^2_0, s]) = e^2_0[e^2_0, \hat{\sigma}_{e^2_0}(s)] = e^2_0$.

If $r = e^2_1$, then $(\sigma_t \circ_{CG} \sigma_r)(f) = \hat{\sigma}_t(e^2_1) = e^2_1$ and $(\sigma_r \circ_{CG} \sigma_t)(f) = \hat{\sigma}_r(f[e^2_0, s]) = e^2_1[e^2_0, \hat{\sigma}_{e^2_1}(s)]$. Since $E(t) \cap \{e^2_0, e^2_1\} = \{e^2_0\}$ and $rightmost_{inj}(s) \neq e^2_0$, we have $e^2_1[e^2_0, \hat{\sigma}_{e^2_1}(s)] = e^2_i; i \geq 2$.

If $r = f[e^2_0, e^2_1]$, then $(\sigma_t \circ_{CG} \sigma_r)(f) = \sigma_t(f) = (\sigma_r \circ_{CG} \sigma_t)(f)$.

If $r = f[e^2_1, e^2_0]$, then

$$(\sigma_t \circ_{CG} \sigma_r)(f) = \hat{\sigma}_t(f[e^2_1, e^2_0]) = (f[e^2_0, s])[e^2_1, e^2_0] = f[e^2_1, s[e^2_1, e^2_0]].$$

Since $E(t) \cap \{e^2_0, e^2_1\} = \{e^2_0\}$ and $rightmost_{inj}(s) \neq e^2_0$, this force that $E(s[e^2_1, e^2_0]) \cap \{e^2_0, e^2_1\} = \{e^2_1\}$ and $rightmost_{inj}(s[e^2_1, e^2_0]) \neq e^2_1$. Thus $\sigma_t \circ_{CG} \sigma_r \in E'_5 \subset \mathcal{UR}(Cohyp_G(2))$. Consider

$$(\sigma_r \circ_{CG} \sigma_t)(f) = \hat{\sigma}_r(f[e^2_0, s]) = (f[e^2_1, e^2_0])[e^2_0, \hat{\sigma}_r(s)] = f[\hat{\sigma}_r(s), e^2_0].$$

Since $E(t) \cap \{e^2_0, e^2_1\} = \{e^2_0\}$ and $rightmost_{inj}(s) \neq e^2_0$, then $E(\hat{\sigma}_r(s)) \cap \{e^2_0, e^2_1\} = \{e^2_0\}$ and $leftmost_{inj}(\hat{\sigma}_r(s)) \neq e^2_0$. Thus we get $\sigma_r \circ_{CG} \sigma_t \in E'_4 \subset \mathcal{UR}(Cohyp_G(2))$.

Case 1.2: $\sigma_r \in E_1$. Then $r = f[r_1, r_2]$ where $E(r) \cap \{e_0^2, e_1^2\} = \emptyset$. Consider

$$\begin{aligned}(\sigma_r \circ_{CG} \sigma_t)(f) &= \hat{\sigma}_r(f[e_0^2, s])\\ &= (\sigma_r(f))[e_0^2, \hat{\sigma}_r(s)]\\ &= (f[r_1, r_2])[e_0^2, \hat{\sigma}_r(s)]\\ &= f[r_1, r_2] \text{ since } E(r) \cap \{e_0^2, e_1^2\} = \emptyset.\end{aligned}$$

Thus $\sigma_r \circ_{CG} \sigma_t \in E_1 \subset \mathcal{UR}(Cohyp_G(2))$. Consider

$$\begin{aligned}(\sigma_t \circ_{CG} \sigma_r)(f) &= \hat{\sigma}_t(f[r_1, r_2])\\ &= (\sigma_t(f))[\hat{\sigma}_t(r_1), \hat{\sigma}_t(r_2)]\\ &= (f[e_0^2, s])[\hat{\sigma}_t(r_1), \hat{\sigma}_t(r_2)]\\ &= f[e_0^2[\hat{\sigma}_t(r_1), \hat{\sigma}_t(r_2)], s[\hat{\sigma}_t(r_1), \hat{\sigma}_t(r_2)]].\end{aligned}$$

Since $E(r) \cap \{e_0^2, e_1^2\} = \emptyset$, we have $e_0^2, e_1^2 \notin E(\hat{\sigma}_t(r_1)) \cup E(\hat{\sigma}_t(r_2))$. So $e_0^2, e_1^2 \notin E(\hat{\sigma}_t(r_1), \hat{\sigma}_t(r_2))$. Thus $\sigma_t \circ_{CG} \sigma_r \in E_1 \subset \mathcal{UR}(Cohyp_G(2))$.

Case 1.3: $\sigma_r \in E_2'$. Then, by Proposition 4.4, we have $\sigma_t \circ_{CG} \sigma_r, \sigma_r \circ_{CG} \sigma_t \in E_2' \subset \mathcal{UR}(Cohyp_G(2))$.

Case 1.4: $\sigma_r \in E_3'$. Then $r = f[s', e_1^2]$ where $E(r) \cap \{e_0^2, e_1^2\} = \{e_1^2\}, s' \in CT_{(2)}$ and $leftmost_{inj}(s') \neq e_1^2$. Compute

$$\begin{aligned}(\sigma_t \circ_{CG} \sigma_r)(f) &= \hat{\sigma}_t(f[s', e_1^2])\\ &= (\sigma_t(f))[\hat{\sigma}_t(s'), e_1^2]\\ &= (f[e_0^2, s])[\hat{\sigma}_t(s'), e_1^2]\\ &= f[e_0^2[\hat{\sigma}_t(s'), e_1^2], s[\hat{\sigma}_t(s'), e_1^2]].\end{aligned}$$

Since $E(r) \cap \{e_0^2, e_1^2\} = \{e_1^2\}$ and $leftmost_{inj}(s') \neq e_1^2$, we have $e_0^2, e_1^2 \notin E(\hat{\sigma}_t(s'))$. Since $E(t) \cap \{e_0^2, e_1^2\} = \{e_0^2\}$ and $e_0^2, e_1^2 \notin E(\hat{\sigma}_t(s'))$, so we have $e_0^2, e_1^2 \notin E(s[\hat{\sigma}_t(s'), e_1^2])$. Thus $\sigma_t \circ_{CG} \sigma_r \in E_1 \subset \mathcal{UR}(Cohyp_G(2))$.

Similarly, we have $\sigma_r \circ_{CG} \sigma_t \in E_1 \subset \mathcal{UR}(Cohyp_G(2))$.

Case 1.5: $\sigma_r \in E_4'$. Then $r = f[s', e_0^2]$ where $E(r) \cap \{e_0^2, e_1^2\} = \{e_0^2\}, s' \in CT_{(2)}$ and $leftmost_{inj}(s') \neq e_0^2$. Consider

$$\begin{aligned}(\sigma_t \circ_{CG} \sigma_r)(f) &= \hat{\sigma}_t(f[s', e_0^2])\\ &= (\sigma_t(f))[\hat{\sigma}_t(s'), e_0^2]\\ &= (f[e_0^2, s])[\hat{\sigma}_t(s'), e_0^2]\\ &= f[e_0^2[\hat{\sigma}_t(s'), e_0^2], s[\hat{\sigma}_t(s'), e_0^2]].\end{aligned}$$

Since $E(r) \cap \{e_0^2, e_1^2\} = \{e_0^2\}$ and $leftmost_{inj}(s') \neq e_0^2$, we have $e_0^2, e_1^2 \notin E(\hat{\sigma}_t(s'))$. Since $E(t) \cap \{e_0^2, e_1^2\} = \{e_0^2\}$ and $e_0^2, e_1^2 \notin E(\hat{\sigma}_t(s'))$, then we obtain $e_0^2, e_1^2 \notin E(s[\hat{\sigma}_t(s'), e_0^2])$. Thus $\sigma_t \circ_{CG} \sigma_r \in E_1 \subset \mathcal{UR}(Cohyp_G(2))$.

Similarly, we have $\sigma_r \circ_{CG} \sigma_t \in E_1 \subset \mathcal{UR}(Cohyp_G(2))$.

Case 1.6: $\sigma_r \in E'_5$. Then $r = f[e_1^2, s']$ where $E(r) \cap \{e_0^2, e_1^2\} = \{e_1^2\}, s' \in CT_{(2)}$ and $rightmost_{inj}(s') \neq e_1^2$. Consider

$$\begin{aligned}(\sigma_t \circ_{CG} \sigma_r)(f) &= \hat{\sigma}_t(f[e_1^2, s'])\\ &= (\sigma_t(f))[e_1^2, \hat{\sigma}_t(s')]\\ &= (f[e_0^2, s])[e_1^2, \hat{\sigma}_t(s')]\\ &= f[e_0^2[e_1^2, \hat{\sigma}_t(s')], s[e_1^2, \hat{\sigma}_t(s')]].\end{aligned}$$

So $e_0^2 \in E(s[e_1^2, \hat{\sigma}_t(s')])$. Since $rightmost_{inj}(s) \neq e_0^2$, we have $rightmost_{inj}(s[e_1^2, \hat{\sigma}_t(s')]) \neq e_1^2$. Thus $\sigma_t \circ_{CG} \sigma_r \in E'_5 \subset \mathcal{UR}(Cohyp_G(2))$. Consider

$$\begin{aligned}(\sigma_r \circ_{CG} \sigma_t)(f) &= \hat{\sigma}_r(f[e_0^2, s])\\ &= (\sigma_r(f))[e_0^2, \hat{\sigma}_r(s)]\\ &= (f[e_1^2, s'])[e_0^2, \hat{\sigma}_r(s)]\\ &= f[e_1^2[e_0^2, \hat{\sigma}_r(s)], s'[e_0^2, \hat{\sigma}_r(s)]].\end{aligned}$$

Since $E(t) \cap \{e_0^2, e_1^2\} = \{e_0^2\}$ and $rightmost_{inj}(s) \neq e_0^2$, then $e_0^2, e_1^2 \notin E(\hat{\sigma}_r(s))$. Since $E(r) \cap \{e_0^2, e_1^2\} = \{e_1^2\}$ and $e_0^2, e_1^2 \notin E(\hat{\sigma}_r(s))$, so $e_0^2, e_1^2 \notin E(s'[e_0^2, \hat{\sigma}_t(s)])$. Thus $\sigma_r \circ_{CG} \sigma_t \in E_1 \subset \mathcal{UR}(Cohyp_G(2))$.

Case 2: $\sigma_t \in E'_3$ and $\sigma_r \in E_0 \cup E_1 \cup E'_4 \cup E'_5$. We can prove similarly to Case 1.

Case 3: $\sigma_t \in E'_4$. Then $t = f[s, e_0^2]$ where $E(t) \cap \{e_0^2, e_1^2\} = \{e_0^2\}, s \in CT_{(2)}$ and $leftmost_{inj}(s) \neq e_0^2$. Let $\sigma_r \in E_0 \cup E_1 \cup E'_4 \cup E'_5$. We cansider the four subcases:

Case 3.1: $\sigma_r \in E_0$, we can prove similarly to Case 1.1.

Case 3.2: $\sigma_r \in E_1$, the proceed proof similarly Case 1.1. We have that $\sigma_r \circ_{CG} \sigma_t$ and $\sigma_t \circ_{CG} \sigma_r$ are in $\mathcal{UR}(Cohyp_G(2))$.

Case 3.3: $\sigma_r \in E'_4$. By Proposition 4.5, we obtain that $\sigma_r \circ_{CG} \sigma_t$ and $\sigma_t \circ_{CG} \sigma_r$ are in $\mathcal{UR}(Cohyp_G(2))$.

Case 3.4: $\sigma_r \in E'_5$. Then $r = f[e_1^2, s']$ where $E(r) \cap \{e_0^2, e_1^2\} = \{e_1^2\}, s' \in CT_{(2)}$ and $rightmost_{inj}(s') \neq e_1^2$. Consider

$$\begin{aligned}(\sigma_t \circ_{CG} \sigma_r)(f) &= \hat{\sigma}_t(f[e_1^2, s'])\\ &= (f[s, e_0^2])[e_1^2, \hat{\sigma}_t(s')]\\ &= f[s[e_1^2, \hat{\sigma}_t(s')], e_1^2].\end{aligned}$$

So, $e_0^2 \notin E(s[e_1^2, \hat{\sigma}_t(s')])$. Since $leftmost_{inj}(s) \neq e_0^2$, this force that $leftmost_{inj}(s[e_1^2, \hat{\sigma}_t(s')]) \neq e_1^2$. Thus $\sigma_t \circ_{CG} \sigma_r \in E'_3 \subset \mathcal{UR}(Cohyp_G(2))$.
Consider

$$\begin{aligned}(\sigma_r \circ_{CG} \sigma_t)(f) &= \hat{\sigma}_r(f[s, e_1^2])\\ &= (f[e_1^2, s'])[\hat{\sigma}_r(s), e_0^2]\\ &= f[e_0^2, s'[\hat{\sigma}_r(s), e_0^2]].\end{aligned}$$

So, $e_1^2 \notin E(s'[\hat{\sigma}_r(s), e_0^2])$. Since $rightmost_{inj}(s') \neq e_1^2$, this force that $rightmost_{inj}(s'[\hat{\sigma}_r(s), e_0^2]) \neq e_0^2$. Thus $\sigma_r \circ_{CG} \sigma_t \in E'_2 \subset \mathcal{UR}(Cohyp_G(2))$.

Case 4: $\sigma_t \in E'_5$ and $\sigma_r \in E_0 \cup E_1 \cup E'_5$. We can prove similarly to Case 3.

Case 5: $\sigma_t \in E_0$ and $\sigma_r \in E_0 \cup E_1$. We can prove similarly to Case 1.1.

Case 6: $\sigma_t \in E_1$ and $\sigma_r \in E_1$. Then $\sigma_t \circ_{CG} \sigma_r = \sigma_t$ and $\sigma_r \circ_{CG} \sigma_t = \sigma_r$. So, $\sigma_r \circ_{CG} \sigma_t$, $\sigma_t \circ_{CG} \sigma_r \in E_1 \subset \mathcal{UR}(Cohyp_G(2))$.

Hence $\mathcal{UR}(Cohyp_G(2))$ is a submonoid of $\underline{Cohyp_G(2)}$.
Therefore, $\mathcal{UR}(Cohyp_G(2))$ is a unit-regular submonoid of $\underline{Cohyp_G(2)}$. ■

Theorem 4.7. *$\mathcal{UR}(Cohyp_G(2))$ is a maximal unit-regular submonoid of $\underline{Cohyp_G(2)}$.*

Proof. Let M be a proper unit-regular submonoid of $Cohyp_G(2)$ such that $\mathcal{UR}(Cohyp_G(2)) \subseteq M \subset \underline{Cohyp_G(2)}$. Let $\sigma_t \in M$. Then σ_t is a unit-regular element.

Case 1: If $\sigma_t \in E_2 \setminus E_2'$, then $t = f[e_0^2, s]$ where $E(t) \cap \{e_0^2, e_1^2\} = \{e_0^2\}, s \in CT_{(2)}$ and $rightmost_{inj}(s) = e_0^2$. We choose $\sigma_r \in E_3' \subseteq M$. Then $r = f[s', e_1^2]$ where $E(r) \cap \{e_0^2, e_1^2\} = \{e_1^2\}, s' \in CT_{(2)}$ and $leftmost_{inj}(s') \neq e_1^2$. Consider

$$\begin{aligned}(\sigma_r \circ_{CG} \sigma_t)(f) &= \hat{\sigma}_r(f[e_0^2, s]) \\ &= (\sigma_r(f))[e_0^2, \hat{\sigma}_r(s)] \\ &= (f[s', e_1^2])[e_0^2, \hat{\sigma}_r(s)] \\ &= f[s'[e_0^2, \hat{\sigma}_r(s)], e_1^2[e_0^2, \hat{\sigma}_r(s)]] \\ &= f[s'[e_0^2, \hat{\sigma}_r(s)], \hat{\sigma}_r(s)].\end{aligned}$$

Since $E(r) \cap \{e_0^2, e_1^2\} = \{e_1^2\}$ and $rightmost_{inj}(s) = e_0^2$, so $e_0^2 \in E(\hat{\sigma}_r(s))$. Since $e_0^2 \in E(\hat{\sigma}_r(s))$, this force that $\sigma_r \circ_{CG} \sigma_t$ is not unit-regular. It is a contradiction. So $\sigma_t \in E_2'$.

Case 2: If $\sigma_t \in E_3 \setminus E_3'$, then $t = f[s, e_1^2]$ where $E(t) \cap \{e_0^2, e_1^2\} = \{e_1^2\}, s \in CT_{(2)}$ and $leftmost_{inj}(s) = e_1^2$. We choose $\sigma_r \in E_2' \in M$. Then $r = f[e_0^2, s']$ where $E(r) \cap \{e_0^2, e_1^2\} = \{e_0^2\}, s' \in CT_{(2)}$ and $rightmost_{inj}(s') \neq e_0^2$. Consider

$$\begin{aligned}(\sigma_r \circ_{CG} \sigma_t)(f) &= \hat{\sigma}_r(f[s, e_1^2]) \\ &= (\sigma_r(f))[\hat{\sigma}_r(s), e_1^2] \\ &= (f[e_0^2, s'])[\hat{\sigma}_r(s), e_1^2] \\ &= f[e_0^2[\hat{\sigma}_r(s), e_1^2], s'[\hat{\sigma}_r(s), e_1^2]] \\ &= f[\hat{\sigma}_r(s), s'[\hat{\sigma}_r(s), e_1^2]].\end{aligned}$$

Since $E(r) \cap \{e_0^2, e_1^2\} = \{e_0^2\}$ and $leftmost_{inj}(s) = e_1^2$, so $e_1^2 \in E(\hat{\sigma}_r(s))$. Since $e_1^2 \in E(\hat{\sigma}_r(s))$, we have that $\sigma_r \circ_{CG} \sigma_t$ is not unit-regular. It is a contradiction. So $\sigma_t \in E_3'$.

Case 3: If $\sigma_t \in E_4 \setminus E_4'$, then $t = f[s, e_0^2]$ where $E(t) \cap \{e_0^2, e_1^2\} = \{e_0^2\}, s \in CT_{(2)}$ and $leftmost_{inj}(s) = e_0^2$. Since $e_0^2 \in E(s)$ and $leftmost_{inj}(s) = e_0^2, \hat{\sigma}_t(s) = t$. Consider

$$\begin{aligned}(\sigma_t \circ_{CG} \sigma_t)(f) &= \hat{\sigma}_t(f[s, e_0^2]) \\ &= (\sigma_t(f))[\hat{\sigma}_t(s), e_0^2] \\ &= (f[s, e_0^2])[\hat{\sigma}_t(s), e_0^2] \\ &= (f[s, e_0^2])[t, e_0^2] \\ &= f[s[t, e_0^2], e_0^2[t, e_0^2]] \\ &= f[s[t, e_0^2], t].\end{aligned}$$

Since $e_0^2 \in E(s)$ and t occurs in $s[t, e_0^2]$, so $e_0^2 \in E(s[t, e_0^2])$. Since$E(t) \cap \{e_0^2, e_1^2\} = \{e_0^2\}$ and $e_0^2 \in E(s[t, e_0^2])$, we have $\sigma_t \circ_{CG} \sigma_t$ is not unit-regular. It is a contradiction. So $\sigma_t \in E_4'$.

Case 4: If $\sigma_t \in E_5 \setminus E'_5$, then $t = f[e_1^2, s]$ where $E(t) \cap \{e_0^2, e_1^2\} = \{e_1^2\}, s \in CT_{(2)}$ and $rightmost_{inj}(s) = e_1^2$. Since $e_1^2 \in E(s)$ and $rightmost_{inj}(s) = e_1^2, \hat{\sigma}_t(s) = t$. Consider

$$\begin{aligned}(\sigma_r \circ_{CG} \sigma_t)(f) &= \hat{\sigma}_t(f[e_1^2, s]) \\ &= (\sigma_t(f))[e_1^2, \hat{\sigma}_t(s)] \\ &= (f[e_1^2, s])[e_1^2, t] \\ &= f[t, s[e_1^2, t]].\end{aligned}$$

Since $e_1^2 \in E(s)$ and t occurs in $s[e_1^2, t]$, so $e_1^2 \in E(s[e_1^2, t])$. Since $E(t) \cap \{e_0^2, e_1^2\} = \{e_1^2\}$ and $e_1^2 \in E(s[e_1^2, t])$, we have $\sigma_t \circ_{CG} \sigma_t$ is not unit-regular. It is a contradiction. So $\sigma_t \in E'_5$.

Thus $M \subseteq \mathcal{UR}(Cohyp_G(2))$. Hence $M = \mathcal{UR}(Cohyp_G(2))$. Therefore, $\mathcal{UR}(Cohyp_G(2))$ is a maximal unit-regular submonoid of $\underline{Cohyp_G(2)}$. ■

Definition 4.8. [10] Let S be a semigroup and $E(S)$ be the set of all idempotents in S. We say S is *left [right] factorisable* if $S = GE(S)$ $[S = E(S)H]$ for some subgroup G, H of S. Moreover, S is *factorisable* if S is both left and right factorisable.

Theorem 4.9. [10] *A monoid S is factorisable if and only if it is unit-regular.*

Corollary 4.10. *$\mathcal{UR}(Cohyp_G(2))$ is factorisable.*

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