Thai Journal of Mathematics
Volume 7 (2009) Number 1 : 115–132

www.math.science.cmu.ac.th/thaijournal
Online ISSN 1686-0209

Common Fixed Point of Modified Noor Iterations with Errors for Non-Lipschitzian Mappings in Banach Spaces

Nawitcha Onjai-uea and Suthep Suantai

Abstract : In this paper, we introduce and study the modified Noor iterative scheme for three asymptotically nonexpansive mappings in the intermediate sense. Weak and strong convergence theorems of such iterations to a common fixed point of three asymptotically nonexpansive mappings in a uniformly convex Banach space are established. The results obtained in this paper extend and improve the recent ones announced by Xu and Noor and many others.

Keywords : asymptotically nonexpansive mapping in the intermediate sense, completely continuous, uniformly convex, modified Noor iterations with errors.
2000 Mathematics Subject Classification : 47H09; 47H10 (2000 MSC)

1 Introduction

Fixed-point iterations process for asymptotically nonexpansive mapping in Banach spaces including Mann and Ishikawa iterations processes have been studied extensively by many authors. Many of them are used widely to study the approximate solutions of the certain problems. Recently, Xu and Noor [20] introduced and studied a three-step scheme to approximate fixed points of asymptotically nonexpansive mappings in Banach space. In 2004, Cho, Zhou and Guo [3] extended the work of Xu and Noor [20] to the three-step iterative scheme with errors and gave weak and strong convergence theorems for asymptotically nonexpansive mapping in Banach space. Suantai [14] defined a new three-step iterations which is an extension of the modified Noor iterations for asymptotically nonexpansive mappings in uniformly Banach space. Nammanee and Suantai [7] gave strong convergence theorem of the modified Noor iterations with errors for completely

[0]This work was completed with the support of the the Commission on Higher Education, Thailand.

continuous asymptotically nonexpansive mappings in the intermediate sense. Inspired and motivated by research going on this area, we consider and study the modified Noor iterations with errors for three asymptotically nonexpansive mappings in a uniformly convex Banach space. The scheme is defined as follows.

Let X be a normed space, C be a nonempty convex subset of X, and T_1, T_2 and $T_3 : C \to C$ be given mappings. Then for a given $x_1 \in C$, compute the sequences $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ by the iterative scheme

$$\begin{aligned} z_n &= a_nT_3^nx_n + (1-a_n-\gamma_n)x_n + \gamma_nu_n, \\ y_n &= b_nT_2^nz_n + c_nT_3^nx_n + (1-b_n-c_n-\mu_n)x_n + \mu_nv_n, \\ x_{n+1} &= \alpha_nT_1^ny_n + \beta_nT_2^nz_n + (1-\alpha_n-\beta_n-\lambda_n)x_n + \lambda_nw_n, \quad n \geq 1, \end{aligned} \tag{1.1}$$

where $\{a_n\}$, $\{b_n\}$, $\{c_n\}$, $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{\lambda_n\}$ are appropriate sequences in $[0,1]$ and $\{u_n\}$, $\{v_n\}$ and $\{w_n\}$ are bounded sequences in C.

The iterative schemes (1.1) are a generalization of the modified Noor iterations with errors. If $T_1 = T_2 = T_3 := T,$ then (1.1) reduces to modified Noor iterations with errors .

$$\begin{aligned} z_n &= a_nT^nx_n + (1-a_n-\gamma_n)x_n + \gamma_nu_n, \\ y_n &= b_nT^nz_n + c_nT^nx_n + (1-b_n-c_n-\mu_n)x_n + \mu_nv_n, \\ x_{n+1} &= \alpha_nT^ny_n + \beta_nT^nz_n + (1-\alpha_n-\beta_n-\lambda_n)x_n + \lambda_nw_n, \quad n \geq 1, \end{aligned} \tag{1.2}$$

Noor iterations include the Mann-Ishikawa iterations as spacial cases.
If $T_1 = T_2 = T_3 := T$, and $\gamma_n = \mu_n = \lambda_n \equiv 0$, then (1.1) reduces to the modified Noor iterations defined by Suantai [14]

$$\begin{aligned} z_n &= a_nT^nx_n + (1-a_n)x_n, \\ y_n &= b_nT^nz_n + c_nT^nx_n + (1-b_n-c_n)x_n, \\ x_{n+1} &= \alpha_nT^ny_n + \beta_nT^nz_n + (1-\alpha_n-\beta_n)x_n, \quad n \geq 1, \end{aligned} \tag{1.3}$$

where $\{a_n\}, \{b_n\}, \{c_n\}, \{\alpha_n\}, \{\beta_n\}$, are appropriate sequences in [0,1].
If $\gamma_n = \mu_n = \lambda_n = c_n = \beta_n \equiv 0$ and $T_1 = T_2 = T_3 := T$, then (1.1) reduces to Noor iterations defined by Xu and Noor [20].

$$\begin{aligned} z_n &= a_nT^nx_n + (1-a_n)x_n, \\ y_n &= b_nT^nz_n + (1-b_n)x_n, \\ x_{n+1} &= \alpha_nT^ny_n + (1-\alpha_n)x_n, \quad n \geq 1, \end{aligned} \tag{1.4}$$

where $\{a_n\}$, $\{b_n\}$, $\{\alpha_n\}$ are appropriate sequences in $[0,1]$.
For $\gamma_n = \mu_n = \lambda_n = c_n = \beta_n = a_n \equiv 0$ and $T_1 = T_2 = T_3 := T$, then (1.1) reduces to the usual Ishikawa iterative scheme

$$\begin{aligned} y_n &= b_nT^nx_n + (1-b_n)x_n, \\ x_{n+1} &= \alpha_nT^ny_n + (1-\alpha_n)x_n, \quad n \geq 1, \end{aligned} \tag{1.5}$$

where $\{b_n\}$, $\{\alpha_n\}$ are appropriate sequences in $[0, 1]$.

The purpose of this paper is to establish several strong convergence theorems for the modified Noor iterations with errors (1.1) for three asymptotically nonexpansive mappings in the intermediate sense, and weak convergence theorems for three asymptotically nonexpansive mappings in the intermediate sense in a uniformly convex Banach space with Opial's condition. Our results extend and improve the corresponding ones announced by Xu and Noor [20] and many others.

Now, we recall the well-known concepts and results.

Let X be normed space and C be a nonempty subset of X. A mapping $T : C \to C$ is said to be *asymptotically nonexpansive* on C if there exists a sequence $\{k_n\}$, $k_n \geq 1$ with $\lim_{n\to\infty} k_n = 1$, such that

$$\|T^n x - T^n y\| \leq k_n \|x - y\|,$$

for all $x, y \in C$ and each $n \geq 1$.

If $k_n \equiv 1$, then T is known as a *nonexpansive mapping.* The mapping T is called *uniformly L- Lipschitzian* if there exists a positive constant L such that

$$\|T^n x - T^n y\| \leq L\|x - y\|,$$

for all $x, y \in C$ and each $n \geq 1$.

T is called *asymptotically nonexpansive in the intermediate sense* [2] provided T is uniformly continuous and

$$\limsup_{n\to\infty} \sup_{x,y\in C} (\|T^n x - T^n y\| - \|x - y\|) \leq 0, \qquad \forall\ x, y \in C.$$

From the definitions, it follows that asymptotically nonexpansive mapping must be asymptotically nonexpansive in the intermediate sense and L-Lipschitzian mapping. But the convergence does not hold such as in the following example.

Example 1.(see [10]) Let $X = \mathbb{R}$, $C = [\frac{-1}{\P}, \frac{1}{\P}]$ and $|k| < 1$. For each $x \in C$, define

$$T(x) = \begin{cases} kx \sin\frac{1}{x} & if\ \ x \neq 0, \\ 0 & if\ \ x = 0. \end{cases}$$

Then T is asymptotically nonexpansive in the intermediate sense. It is well known [9] that $T^n x \to 0$ $\ uniformly$, but is not a Lipschitzian mapping so that it is not asymptotically nonexpansive mapping.

It is known [5] that if X is a uniformly convex Banach space and T is asymptotically nonexpansive in the intermediate sense, then $F(T) \neq \emptyset$.

Recall that a Banach space X is said to satisfy *Opial's condition* [10] if $x_n \to x$ weakly as $n \to \infty$ and $x \neq y$ imply that

$$\limsup_{n\to\infty} \|x_n - x\| < \limsup_{n\to\infty} \|x_n - y\|.$$

2 Preliminaries

Lemma 2.1. [16, Lemma 1]. *Let $\{a_n\}, \{b_n\}$ and $\{\delta_n\}$ be sequences of nonnegative real numbers satisfying the inequality*

$$a_{n+1} \leq (1+\delta_n)a_n + b_n, \ \ \forall n = 1, 2, ...$$

If $\sum_{n=1}^{\infty} \delta_n < \infty$ and $\sum_{n=1}^{\infty} b_n < \infty$, then
(i) $\lim_{n\to\infty} a_n$ exists.
(ii) $\lim_{n\to\infty} a_n = 0$ whenever $\liminf_{n\to\infty} a_n = 0$.

Lemma 2.2. [6, Lemma 1.4]. *Let X be a uniformly convex Banach space and $B_r = \{x \in X : \|x\| \leq r\}, r > 0$. Then there exists a continuous, strictly increasing, and convex function $g : [0,\infty) \to [0,\infty), g(0) = 0$ such that*

$$\|\alpha x + \beta y + \mu z + \lambda w\|^2 \leq \alpha\|x\|^2 + \beta\|y\|^2 + \mu\|z\|^2 + \lambda\|w\|^2 - \alpha\beta g(\|x-y\|),$$

for all x, y, z, $w \in B_r$ and all α, β, μ, $\lambda \in [0,1]$ with $\alpha+\beta+\mu+\lambda = 1$.

Lemma 2.3. [3, Lemma 1.6] *Let X be a uniformly convex Banach space,C a nonempty closed convex subset of X, and $T : C \to C$ be an asymptotically nonexpansive mapping. Then $I - T$ is demiclosed at 0, i.e., if $x_n \to x$ weakly and $x_n - Tx_n \to 0$ strongly, then $x \in F(T)$, where $F(T)$ is the set of fixed point of T.*

Lemma 2.4. [14, Lemma 2.7]. *Let X be a Banach space which satisfies Opial's condition and let $\{x_n\}$ be a sequence in X. Let $u, v \in X$ be such that $\lim_{n\to\infty} \|x_n - u\|$ and $\lim_{n\to\infty} \|x_n - v\|$ exist. If $\{x_{n_k}\}$ and $\{x_{m_k}\}$ are subsequences of $\{x_n\}$ which converge weakly to u and v, respectively, then $u = v$.*

3 Main Results

In this section, we prove strong convergence theorems for the modified Noor iterations with errors (1.1) for three asymptotically nonexpansive mappings in the intermediate sense in a uniformly convex Banach space. In order to prove our main results, the following lemmas are needed.

The next lemma is crucial for proving the main theorems.

Lemma 3.1. *Let X be a uniformly convex Banach space and let C be a nonempty closed and convex subset of X. Let T_1, T_2 and T_3 be an asymptotically nonexpansive in the intermediate sense self-maps of C*

Put

$$G_n^{(i)} = \sup_{x,y\in C} (\|T_i^n x - T_i^n y\| - \|x-y\|) \vee 0$$

for all $x, y \in C$, $n \geq 1$, $i = 1, 2, 3$, *so that* $\sum_{n=1}^{\infty} G_n^{(i)} < \infty, i = 1, 2, 3$. *Let* $\{a_n\}$, $\{b_n\}$, $\{c_n\}$, $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{\mu_n\}$ *and* $\{\lambda_n\}$ *be real sequences in* $[0, 1]$ *such that* $a_n + \gamma_n$, $b_n + c_n + \mu_n$ *and* $\alpha_n + \beta_n + \lambda_n$ *are in* $[0, 1]$ *for all* $n \geq 1$ *and* $\sum_{n=1}^{\infty} \gamma_n < \infty$, $\sum_{n=1}^{\infty} \mu_n < \infty$, $\sum_{n=1}^{\infty} \lambda_n < \infty$, *and let* $\{u_n\}$, $\{v_n\}$ *and* $\{w_n\}$ *be bounded sequences in* C. *For a given* $x_1 \in C$, *let* $\{x_n\}$, $\{y_n\}$ *and* $\{z_n\}$ *be the sequences defined as in (1.1). Then we have*

(i) $\lim_{n\to\infty} \|x_n - p\|$ *exists for all* $p \in F(T_1) \cap F(T_2) \cap F(T_3)$.

(ii) If $0 < \liminf_{n\to\infty} \alpha_n \leq \limsup_{n\to\infty}(\alpha_n + \beta_n + \lambda_n) < 1$, *then* $\lim_{n\to\infty} \|T_1^n y_n - x_n\| = 0$.

(iii) If $0 < \liminf_{n\to\infty} b_n \leq \limsup_{n\to\infty}(b_n + c_n + \mu_n) < 1$, *then* $\lim_{n\to\infty} \|T_2^n z_n - x_n\| = 0$.

(iv) If $0 < \liminf_{n\to\infty} a_n \leq \limsup_{n\to\infty}(a_n + \gamma_n) < 1$, *then* $\lim_{n\to\infty} \|T_3^n x_n - x_n\| = 0$.

Proof. (i) Let $p \in F(T_1) \cap F(T_2) \cap F(T_3)$. Since $\{G_n^{(i)}\}, \{u_n\}, \{v_n\}$ and $\{w_n\}$ are bounded sequences in C.

We can put $M = \sup_{n\geq 1} G_n^{(i)} \vee \sup_{n\geq 1} \|u_n - p\| \vee \sup_{n\geq 1} \|v_n - p\| \vee \sup_{n\geq 1} \|w_n - p\|$. Then

$$
\begin{aligned}
\|z_n - p\| &= \|a_n T_3^n x_n + (1 - a_n - \gamma_n)x_n + \gamma_n u_n - p\| \\
&= \|a_n(T_3^n x_n - p) + (1 - a_n - \gamma_n)(x_n - p) + \gamma_n(u_n - p)\| \\
&\leq (1 - a_n - \gamma_n)\|x_n - p\| + a_n\|T_3^n x_n - p\| + \gamma_n\|u_n - p\| \\
&\leq (1 - a_n - \gamma_n)\|x_n - p\| + a_n[\|x_n - p\| + G_n^{(3)}] + \gamma_n\|u_n - p\| \\
&= (1 - \gamma_n)\|x_n - p\| + a_n G_n^{(3)} + \gamma_n\|u_n - p\| \\
&\leq \|x_n - p\| + G_n^{(3)} + M\gamma_n, \qquad (3.1)
\end{aligned}
$$

and

$$
\begin{aligned}
\|y_n - p\| &= \|b_n T_2^n z_n + c_n T_3^n x_n + (1 - b_n - c_n - \mu_n)x_n + \mu_n v_n - p\| \\
&= \|b_n(T_2^n z_n - p) + c_n(T_3^n x_n - p) + (1 - b_n - c_n - \mu_n)(x_n - p) + \mu_n(v_n - p)\| \\
&\leq (1 - b_n - c_n - \mu_n)\|x_n - p\| + b_n\|T_2^n z_n - p\| + c_n\|T_3^n x_n - p\| + \mu_n\|v_n - p\| \\
&\leq (1 - b_n - c_n - \mu_n)\|x_n - p\| + b_n[\|z_n - p\| + G_n^{(2)}] \\
&\quad + c_n[\|x_n - p\| + G_n^{(3)}] + M\mu_n \\
&\leq (1 - b_n - c_n - \mu_n)\|x_n - p\| + b_n[\|x_n - p\| + G_n^{(3)} + M\gamma_n + G_n^{(2)}] \\
&\quad + c_n[\|x_n - p\| + G_n^{(3)}] + M\mu_n \\
&= (1 - \mu_n)\|x_n - p\| + b_n[G_n^{(3)} + M\gamma_n + G_n^{(2)}] + c_n G_n^{(3)} + M\mu_n \\
&\leq \|x_n - p\| + G_n^{(2)} + 2G_n^{(3)} + M(\gamma_n + \mu_n). \qquad (3.2)
\end{aligned}
$$

It follows from (3.1) and (3.2) that

$$
\begin{aligned}
\|x_{n+1} - p\| &= \|\alpha_n T_1^n y_n + \beta_n T_2^n z_n + (1 - \alpha_n - \beta_n - \lambda_n)x_n + \lambda_n w_n - p\| \\
&= \|\alpha_n(T_1^n y_n - p) + \beta_n(T_2^n z_n - p) + (1 - \alpha_n - \beta_n - \lambda_n)(x_n - p) + \lambda_n(w_n - p)\| \\
&\leq (1 - \alpha_n - \beta_n - \lambda_n)\|x_n - p\| + \alpha_n\|T_1^n y_n - p\| + \beta_n\|T_2^n z_n - p\| + \lambda_n\|w_n - p\|
\end{aligned}
$$

$$
\begin{aligned}
&\leq (1-\alpha_n-\beta_n-\lambda_n)\|x_n-p\| + \alpha_n[\|y_n-p\| + G_n^{(1)}] + \beta_n[\|z_n-p\| + G_n^{(2)}] + M\lambda_n \\
&\leq (1-\alpha_n-\beta_n-\lambda_n)\|x_n-p\| \\
&\quad + \alpha_n[\|x_n-p\| + G_n^{(2)} + 2G_n^{(3)} + M(\gamma_n+\mu_n) + G_n^{(1)}] \\
&\quad + \beta_n[\|x_n-p\| + G_n^{(3)} + M\gamma_n + G_n^{(2)}] + M\lambda_n \\
&= (1-\lambda_n)\|x_n-p\| + \alpha_n[G_n^{(1)} + G_n^{(2)} + 2G_n^{(3)} + M(\gamma_n+\mu_n)] \\
&\quad + \beta_n[G_n^{(3)} + M\gamma_n + G_n^{(2)}] + M\lambda_n \\
&\leq \|x_n-p\| + G_n^{(1)} + 2G_n^{(2)} + 3G_n^{(3)} + M(2\gamma_n+\mu_n+\lambda_n)
\end{aligned}
$$

Since $\sum_{n=1}^{\infty} G_n^{(i)}<\infty, i=1,2,3,$ $\sum_{n=1}^{\infty}\gamma_n<\infty,$ $\sum_{n=1}^{\infty}\mu_n<\infty$ and $\sum_{n=1}^{\infty}\lambda_n<\infty,$ it follows from Lemma 2.1 that $\lim_{n\to\infty}\|x_n-p\|$ exists.

(ii) By (i) we have that $\lim_{n\to\infty}\|x_n-p\|$ exists for any $p\in F(T_1)\cap F(T_2)\cap F(T_3)$. It follow that $\{x_n-p\}$, $\{y_n-p\}$, $\{z_n-p\}$, $\{T_1^n y_n-p\}$, $\{T_2^n z_n-p\}$, $\{T_3^n x_n-p\}$ are all bounded also both $\{u_n-p\}$, $\{v_n-p\}$ and $\{w_n-p\}$ by assumption. Next, we set

$$
\begin{aligned}
&r_1=\sup_n\|x_n-p\|,\ r_2=\sup_n\|T_3^n x_n-p\|,\ r_3=\sup_n\|T_2^n z_n-p\|,\ r_4=\sup_n\|T_1^n y_n-p\|,\\
&r_5=\sup_n\|y_n-p\|,\ r_6=\sup_n\|z_n-p\|,\ r_7=\sup_n\|u_n-p\|,\ r_8=\sup_n\|v_n-p\|\\
&\text{and } r_9=\sup_n\|w_n-p\|.
\end{aligned}
$$

Choose a number $r=\max\{r_1, r_2, r_3, r_4, r_5, r_6, r_7, r_8, r_9\}$. Then B_r contains the following sequences $\{x_n-p\}$, $\{y_n-p\}$,$\{z_n-p\}$, $\{T_1^n y_n-p\}$, $\{T_2^n z_n-p\}$,$\{T_3^n x_n-p\}$, $\{u_n-p\}$, $\{v_n-p\}$,$\{w_n-p\}$. By Lemma 2.2, there exists a continuous, strictly increasing, and convex function $g:[0,\infty)\to[0,\infty)$, $g(0)=0$ such that

$$
\|\alpha x+\beta y+\mu z+\lambda w\|^2 \leq \alpha\|x\|^2+\beta\|y\|^2+\mu\|z\|^2+\lambda\|w\|^2-\alpha\beta g(\|x-y\|), \tag{3.3}
$$

for all $x, y, z, w\in B_r$ and all $\alpha, \beta, \mu, \lambda\in[0,1]$ with $\alpha+\beta+\mu+\lambda=1$.
It follows from (3.3) that

$$
\begin{aligned}
\|z_n - p\|^2 &= \|a_nT_3^n x_n + (1-a_n-\gamma_n)x_n + \gamma_n u_n - p\|^2 \\
&= \|a_n(T_3^n x_n-p) + (1-a_n-\gamma_n)(x_n-p) + \gamma_n(u_n-p)\|^2 \\
&\leq (1-a_n-\gamma_n)\|x_n-p\|^2 + a_n\|T_3^n x_n-p\|^2 + \gamma_n\|u_n-p\|^2 \\
&\quad - a_n(1-a_n-\gamma_n)g(\|T_3^n x_n-x_n\|) \\
&\leq (1-a_n-\gamma_n)\|x_n-p\|^2 + a_n[\|x_n-p\| + G_n^{(3)}]^2 + M^2\gamma_n \\
&\quad - a_n(1-a_n-\gamma_n)g(\|T_3^n x_n-x_n\|) \\
&= (1-\gamma_n)\|x_n-p\|^2 + a_n[2G_n^{(3)}\|x_n-p\| + (G_n^{(3)})^2] \\
&\quad + M^2\gamma_n - a_n(1-a_n-\gamma_n)g(\|T_3^n x_n-x_n\|) \\
&\leq \|x_n-p\|^2 + 2G_n^{(3)}\|x_n-p\| + (G_n^{(3)})^2 + M^2\gamma_n \\
&\quad - a_n(1-a_n-\gamma_n)g(\|T_3^n x_n-x_n\|),
\end{aligned} \tag{3.4}
$$

$$
\begin{aligned}
\|y_n - p\|^2 &= \|b_n T_2^n z_n + c_n T_3^n x_n + (1 - b_n - c_n - \mu_n)x_n + \mu_n v_n - p\|^2 \\
&= \|b_n(T_2^n z_n - p) + (1 - b_n - c_n - \mu_n)(x_n - p) + c_n(T_3^n x_n - p) + \mu_n(v_n - p)\|^2 \\
&\leq (1 - b_n - c_n - \mu_n)\|x_n - p\|^2 + b_n\|T_2^n z_n - p\|^2 \\
&\quad + c_n\|T_3^n x_n - p\|^2 + \mu_n\|v_n - p\|^2 - b_n(1 - b_n - c_n - \mu_n)g(\|T_2^n z_n - x_n\|) \\
&\leq (1 - b_n - c_n - \mu_n)\|x_n - p\|^2 + b_n[\|z_n - p\| + G_n^{(2)}]^2 \\
&\quad + c_n[\|x_n - p\| + G_n^{(3)}]^2 + M^2\mu_n - b_n(1 - b_n - c_n - \mu_n)g(\|T_2^n z_n - x_n\|) \\
&= (1 - b_n - c_n - \mu_n)\|x_n - p\|^2 + b_n[\|z_n - p\|^2 + 2G_n^{(2)}\|z_n - p\| + (G_n^{(2)})^2] \\
&\quad + c_n[\|x_n - p\|^2 + 2G_n^{(3)}\|x_n - p\| + (G_n^{(3)})^2] \\
&\quad + M^2\mu_n - b_n(1 - b_n - c_n - \mu_n)g(\|T_2^n z_n - x_n\|) \\
&\leq (1 - b_n - c_n - \mu_n)\|x_n - p\|^2 + b_n[\|x_n - p\|^2 + 2G_n^{(3)}\|x_n - p\| + (G_n^{(3)})^2 \\
&\quad + M^2\gamma_n + 2G_n^{(2)}(\|x_n - p\| + G_n^{(3)} + M\gamma_n) + (G_n^{(2)})^2] \\
&\quad + c_n[\|x_n - p\|^2 + 2G_n^{(3)}\|x_n - p\| + (G_n^{(3)})^2] + M^2\mu_n \\
&\quad - b_n(1 - b_n - c_n - \mu_n)g(\|T_2^n z_n - x_n\|) \\
&= (1 - \mu_n)\|x_n - p\|^2 + b_n[(2G_n^{(2)} + 2G_n^{(3)})\|x_n - p\| + (G_n^{(2)})^2 + (G_n^{(3)})^2 \\
&\quad + 2G_n^{(2)}G_n^{(3)} + 2M\gamma_n G_n^{(2)} + M^2\gamma_n] + c_n[2G_n^{(3)}\|x_n - p\| + (G_n^{(3)})^2] \\
&\quad + M^2\mu_n - b_n(1 - b_n - c_n - \mu_n)g(\|T_2^n z_n - x_n\|) \\
&\leq \|x_n - p\|^2 + (2G_n^{(2)} + 4G_n^{(3)})\|x_n - p\| + (G_n^{(2)})^2 + 2(G_n^{(3)})^2 + 2G_n^{(2)}G_n^{(3)} \\
&\quad + 2MG_n^{(2)} + M^2(\gamma_n + \mu_n) - b_n(1 - b_n - c_n - \mu_n)g(\|T_2^n z_n - x_n\|). \qquad (3.5)
\end{aligned}
$$

And

$$
\begin{aligned}
\|x_{n+1} - p\|^2 &= \|\alpha_n T_1^n y_n + \beta_n T_2^n z_n + (1 - \alpha_n - \beta_n - \lambda_n)x_n + \lambda_n w_n - p\|^2 \\
&= \|\alpha_n(T_1^n y_n - p) + (1 - \alpha_n - \beta_n - \lambda_n)(x_n - p) + \beta_n(T_2^n z_n - p) + \lambda_n(w_n - p)\|^2 \\
&\leq (1 - \alpha_n - \beta_n - \lambda_n)\|x_n - p\|^2 + \alpha_n\|T_1^n y_n - p\|^2 + \beta_n\|T_2^n z_n - p\|^2 \\
&\quad + \lambda_n\|w_n - p\|^2 - \alpha_n(1 - \alpha_n - \beta_n - \lambda_n)g(\|T_1^n y_n - x_n\|) \\
&\leq (1 - \alpha_n - \beta_n - \lambda_n)\|x_n - p\|^2 + \alpha_n[\|y_n - p\| + G_n^{(1)}]^2 + \beta_n[\|z_n - p\| + G_n^{(2)}]^2 \\
&\quad + M^2\lambda_n - \alpha_n(1 - \alpha_n - \beta_n - \lambda_n)g(\|T_1^n y_n - x_n\|) \\
&\leq (1 - \alpha_n - \beta_n - \lambda_n)\|x_n - p\|^2 + \alpha_n[\|y_n - p\|^2 + 2G_n^{(1)}\|y_n - p\| + (G_n^{(1)})^2] \\
&\quad + \beta_n[\|z_n - p\|^2 + 2G_n^{(2)}\|z_n - p\| + (G_n^{(2)})^2] + M^2\lambda_n \\
&\quad - \alpha_n(1 - \alpha_n - \beta_n - \lambda_n)g(\|T_1^n y_n - x_n\|) \\
&\leq (1 - \alpha_n - \beta_n - \lambda_n)\|x_n - p\|^2 + \alpha_n[\|x_n - p\|^2 + (2G_n^{(2)} + 4G_n^{(3)})\|x_n - p\| \\
&\quad + (G_n^{(2)})^2 + 2(G_n^{(3)})^2 + 2G_n^{(2)}G_n^{(3)} + 2MG_n^{(2)} + M^2(\gamma_n + \mu_n) \\
&\quad - b_n(1 - b_n - c_n - \mu_n)g(\|T_2^n z_n - x_n\|) \\
&\quad + 2G_n^{(1)}(\|x_n - p\| + G_n^{(2)} + 2G_n^{(3)} + M(\gamma_n + \mu_n)) + (G_n^{(1)})^2] \\
&\quad + \beta_n[\|x_n - p\|^2 + 2G_n^{(3)}\|x_n - p\| + (G_n^{(3)})^2 + M^2\gamma_n \\
&\quad - a_n(1 - a_n - \gamma_n)g(\|T_3^n x_n - x_n\|) \\
&\quad + 2G_n^{(2)}(\|x_n - p\| + G_n^{(3)} + M\gamma_n) + (G_n^{(2)})^2] + M^2\lambda_n \\
&\quad - \alpha_n(1 - \alpha_n - \beta_n - \lambda_n)g(\|T_1^n y_n - x_n\|) \\
&= (1 - \lambda_n)\|x_n - p\|^2 + \alpha_n[(2G_n^{(1)} + 2G_n^{(2)} + 4G_n^{(3)})\|x_n - p\| + (G_n^{(1)})^2 + (G_n^{(2)})^2 \\
&\quad + 2(G_n^{(3)})^2 + 2G_n^{(1)}G_n^{(2)} + 4G_n^{(1)}G_n^{(3)} + 2G_n^{(2)}G_n^{(3)} + 2M(\gamma_n + \mu_n)G_n^{(1)} \\
&\quad + 2MG_n^{(2)} + M^2(\gamma_n + \mu_n) - b_n(1 - b_n - c_n - \mu_n)g(\|T_2^n z_n - x_n\|)] \\
&\quad + \beta_n[(2G_n^{(2)} + 2G_n^{(3)})\|x_n - p\| + (G_n^{(2)})^2 + (G_n^{(3)})^2 \\
&\quad + 2G_n^{(2)}G_n^{(3)} + 2M\gamma_n G_n^{(2)} + M^2 - \gamma_n a_n(1 - a_n - \gamma_n)g(\|T_3^n x_n - x_n\|)] \\
&\quad + M^2\lambda_n - \alpha_n(1 - \alpha_n - \beta_n - \lambda_n)g(\|T_1^n y_n - x_n\|)
\end{aligned}
$$

$$
\begin{aligned}
\leq\ & \|x_n - p\|^2 + (2G_n^{(1)} + 4G_n^{(2)} + 6G_n^{(3)})\|x_n - p\| + (G_n^{(1)})^2 + 2(G_n^{(2)})^2 + 3(G_n^{(3)})^2 \\
& + 2G_n^{(1)}G_n^{(2)} + 4G_n^{(1)}G_n^{(3)} + 4G_n^{(2)}G_n^{(3)} + 4MG_n^{(1)} + 4MG_n^{(2)} \\
& + M^2(2\gamma_n + \mu_n + \lambda_n) - \alpha_n(1 - \alpha_n - \beta_n - \lambda_n)g(\|T_1^n y_n - x_n\|) \\
& - b_n(1 - b_n - c_n - \mu_n)g(\|T_2^n z_n - x_n\|) - a_n(1 - a_n - \gamma_n)g(\|T_3^n x_n - x_n\|) \\
\leq\ & \|x_n - p\|^2 + (2G_n^{(1)} + 4G_n^{(2)} + 6G_n^{(3)})\|x_n - p\| + MG_n^{(1)} + 2MG_n^{(2)} + 3MG_n^{(3)} \\
& + 2MG_n^{(1)} + 4MG_n^{(1)} + 4MG_n^{(2)} + 4MG_n^{(1)} + 4MG_n^{(2)} \\
& + M^2(2\gamma_n + \mu_n + \lambda_n) - \alpha_n(1 - \alpha_n - \beta_n - \lambda_n)g(\|T_1^n y_n - x_n\|) \\
& - b_n(1 - b_n - c_n - \mu_n)g(\|T_2^n z_n - x_n\| - a_n(1 - a_n - \gamma_n)g(\|T_3^n x_n - x_n\|) \qquad (3.6)
\end{aligned}
$$

This implies that

$$
\begin{aligned}
\alpha_n(1 - \alpha_n - \beta_n - \lambda_n)g(\|T_1^n y_n - x_n\|) \leq\ & \|x_n - p\|^2 - \|x_{n+1} - p\|^2 + L(2G_n^{(1)} + 4G_n^{(2)} + 6G_n^{(3)}) \\
& + M(11G_n^{(1)} + 10G_n^{(2)} + 3G_n^{(3)}) + M^2(2\gamma_n + \mu_n + \lambda_n), \qquad (3.7)
\end{aligned}
$$

$$
\begin{aligned}
b_n(1 - b_n - c_n - \mu_n)g(\|T_2^n z_n - x_n\|) \leq\ & \|x_n - p\|^2 - \|x_{n+1} - p\|^2 + L(2G_n^{(1)} + 4G_n^{(2)} + 6G_n^{(3)}) \\
& + M(11G_n^{(1)} + 10G_n^{(2)} + 3G_n^{(3)}) + M^2(2\gamma_n + \mu_n + \lambda_n), \qquad (3.8)
\end{aligned}
$$

$$
\begin{aligned}
a_n(1 - a_n - \gamma_n)g(\|T_3^n x_n - x_n\|) \leq\ & \|x_n - p\|^2 - \|x_{n+1} - p\|^2 + L(2G_n^{(1)} + 4G_n^{(2)} + 6G_n^{(3)}) \\
& + M(11G_n^{(1)} + 10G_n^{(2)} + 3G_n^{(3)}) + M^2(2\gamma_n + \mu_n + \lambda_n), \qquad (3.9)
\end{aligned}
$$

where $L = \sup\{\|x_n - p\| : n \geq 1\}$.
If $0 < \liminf_{n\to\infty} \alpha_n \leq \limsup_{n\to\infty}(\alpha_n + \beta_n + \lambda_n) < 1$, then there exist a positive integer n_0 and $\eta_1,\ \eta_2 \in (0,1)$ such that

$0 < \eta_1 < \alpha_n,$ and $\alpha_n + \beta_n + \lambda_n < \eta_2 < 1$ for all $n \geq n_0$.

This implies by (3.7) that

$$
\begin{aligned}
\eta_1(1 - \eta_2)g(\|T_1^n y_n - x_n\|) \leq\ & \|x_n - p\|^2 - \|x_{n+1} - p\|^2 + L(2G_n^{(1)} + 4G_n^{(2)} + 6G_n^{(3)}) \\
& + M(11G_n^{(1)} + 10G_n^{(2)} + 3G_n^{(3)}) + M^2(2\gamma_n + \mu_n + \lambda_n) \\
\leq\ & \|x_n - p\|^2 - \|x_{n+1} - p\|^2 + K(2G_n^{(1)} + 4G_n^{(2)} + 6G_n^{(3)}) \\
& + K(11G_n^{(1)} + 10G_n^{(2)} + 3G_n^{(3)}) + M^2(2\gamma_n + \mu_n + \lambda_n) \\
=\ & \|x_n - p\|^2 - \|x_{n+1} - p\|^2 + K(13G_n^{(1)} + 14G_n^{(2)} + 9G_n^{(3)}) \\
& + M^2(2\gamma_n + \mu_n + \lambda_n), \qquad (3.10)
\end{aligned}
$$

where $K = \max\{M, L\}$, for all $n \geq n_0$. It follows from (3.10) that for $m \geq n_0$

$$\begin{aligned}\sum_{n=n_0}^{m} g(\|T_1^n y_n - x_n\|) &\leq \frac{1}{\eta_1(1-\eta_2)}\Bigg(\sum_{n=n_0}^{m}(\|x_n - p\|^2 - \|x_{n+1} - p\|^2) \\ &\quad + \sum_{n=n_0}^{m}(K(13G_n^{(1)} + 14G_n^{(2)} + 9G_n^{(3)}) + M^2(2\gamma_n + \mu_n + \lambda_n))\Bigg) \\ &\leq \frac{1}{\eta_1(1-\eta_2)}\Bigg(\|x_{n_0} - p\|^2 + K\sum_{n=n_0}^{m}(13G_n^{(1)} + 14G_n^{(2)} + 9G_n^{(3)}) \\ &\quad + M^2\sum_{n=n_0}^{m}(2\gamma_n + \mu_n + \lambda_n)\Bigg). \qquad (3.11)\end{aligned}$$

Since $\sum_{n=1}^{\infty}\gamma_n < \infty$, $\sum_{n=1}^{\infty}\mu_n < \infty$, $\sum_{n=1}^{\infty}\lambda_n < \infty$, and $\sum_{n=1}^{\infty}G_n^{(i)} < \infty$, $i = 1, 2, 3$,

by letting $m \to \infty$ in inequality (3.11), we get that $\sum_{n=n_0}^{\infty} g(\|T_1^n y_n - x_n\|) < \infty$, and therefore $\lim_{n\to\infty} g(\|T_1^n y_n - x_n\|) = 0$. Since g is strictly increasing and continuous at 0 with $g(0) = 0$, it follows that $\lim_{n\to\infty}\|T_1^n y_n - x_n\| = 0$.

(iii) If $\liminf_{n\to\infty} b_n \leq \limsup_{n\to\infty}(b_n + c_n + \mu_n) < 1$, then by using a similar method, together with inequality (3.8), it can be shown that $\lim_{n\to\infty}\|T_2^n z_n - x_n\| = 0$.

(iv) If $\liminf_{n\to\infty} a_n \leq \limsup_{n\to\infty}(a_n + \gamma_n) < 1$, then by using a similar method, together with inequality (3.9), it can be shown that $\lim_{n\to\infty}\|T_3^n x_n - x_n\| = 0$. □

Lemma 3.2. *Let X be a uniformly convex Banach space and let C be a nonempty closed and convex subset of X. Let T_1, T_2 and T_3 be an asymptotically nonexpansive in the intermediate sense self-maps of C.*

Put

$$G_n^{(i)} = \sup_{x,y\in C}(\|T_i^n x - T_i^n y\| - \|x - y\|) \vee 0$$

for all $x, y \in C$, $n \geq 1$, $i = 1, 2, 3$, so that $\sum_{n=1}^{\infty} G_n^{(i)} < \infty$, $i = 1, 2, 3$. Let $\{a_n\}$, $\{b_n\}$, $\{c_n\}$, $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{\mu_n\}$ and $\{\lambda_n\}$ be real sequences in $[0, 1]$ such that $a_n + \gamma_n$, $b_n + c_n + \mu_n$ and $\alpha_n + \beta_n + \lambda_n$ are in $[0, 1]$ for all $n \geq 1$ and $\sum_{n=1}^{\infty}\gamma_n < \infty$, $\sum_{n=1}^{\infty}\mu_n < \infty$, $\sum_{n=1}^{\infty}\lambda_n < \infty$, and let $\{u_n\}$, $\{v_n\}$ and $\{w_n\}$ be bounded sequences in C. For a given $x_1 \in C$, let $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ be the sequences defined as in (1.1) and

(i) $0 < \liminf_{n\to\infty}\alpha_n \leq \limsup_{n\to\infty}(\alpha_n + \beta_n + \lambda_n) < 1$,

(ii) $0 < \liminf_{n\to\infty} b_n \leq \limsup_{n\to\infty}(b_n + c_n + \mu_n) < 1$,

(iii) $0 < \liminf_{n\to\infty} a_n \le \limsup_{n\to\infty}(a_n + \gamma_n) < 1$,

then $\lim_{n\to\infty} \|T_1x_n - x_n\| = \lim_{n\to\infty} \|T_2x_n - x_n\| = \lim_{n\to\infty} \|T_3x_n - x_n\| = 0$.

Proof. By Lemma 3.1, we have

$$\lim_{n\to\infty} \|T_1^n y_n - x_n\| = 0,\ \lim_{n\to\infty} \|T_2^n z_n - x_n\| = 0,\ \lim_{n\to\infty} \|T_3^n x_n - x_n\| = 0. \tag{3.12}$$

From $x_{n+1} = \alpha_n T_1^n y_n + \beta_n T_2^n z_n + (1-\alpha_n-\beta_n-\lambda_n)x_n + \lambda_n w_n$, we have

$$\begin{aligned} \|x_{n+1} - x_n\| &= \|\alpha_n T_1^n y_n + \beta_n T_2^n z_n + (1-\alpha_n-\beta_n-\lambda_n)x_n + \lambda_n w_n - x_n\| \\ &\le \alpha_n\|T_1^n y_n - x_n\| + \beta_n\|T_2^n z_n - x_n\| + \lambda_n\|w_n - x_n\| \to 0, \end{aligned}$$

and

$$\begin{aligned} \|y_n - x_n\| &= \|b_n T_2^n z_n + c_n T_3^n x_n + (1-b_n-c_n-\mu_n)x_n + \mu_n v_n - x_n\| \\ &\le b_n\|T_2^n z_n - x_n\| + c_n\|T_3^n x_n - x_n\| + \mu_n\|v_n - x_n\| \to 0. \end{aligned} \tag{3.13}$$

By (3.12) and (3.13), we have

$$\begin{aligned} \|T_1^n x_n - x_n\| &\le \|T_1^n x_n - T_1^n y_n\| + \|T_1^n y_n - x_n\| \\ &\le \|x_n - y_n\| + G_n^{(1)} + \|T_1^n y_n - x_n\| \to 0. \end{aligned}$$

Hence

$$\begin{aligned} \|x_{n+1} - T_1^n x_{n+1}\| &\le \|x_{n+1} - x_n\| + \|T_1^n x_{n+1} - T_1^n x_n\| + \|T_1^n x_n - x_n\| \\ &\le 2\|x_{n+1} - x_n\| + G_n^{(1)} + \|T_1^n x_n - x_n\| \to 0. \end{aligned}$$

Since

$$\begin{aligned} \|x_{n+1} - T_1 x_{n+1}\| &\le \|x_{n+1} - T_1^{n+1} x_{n+1}\| + \|T_1 x_{n+1} - T_1^{n+1} x_{n+1}\| \to 0 \quad \text{and} \\ \|x_n - T_1 x_n\| &\le \|x_n - x_{n+1}\| + \|x_{n+1} - T_1 x_{n+1}\| + \|T_1 x_{n+1} - T_1 x_n\|, \end{aligned}$$

it follows from uniform continuity of T_1 that $\lim_{n\to\infty} \|T_1x_n - x_n\| = 0$.

From $z_n = a_n T_3^n x_n + (1-a_n-\gamma_n)x_n + \gamma_n u_n$, we have

$$\begin{aligned} \|z_n - x_n\| &= \|a_n T_3^n x_n + (1-a_n-\gamma_n)x_n + \gamma_n u_n - x_n\| \\ &\le a_n\|T_3^n x_n - x_n\| + \gamma_n\|u_n - x_n\| \to 0. \end{aligned} \tag{3.14}$$

By (3.12) and (3.14), we have

$$\begin{aligned} \|T_2^n x_n - x_n\| &\le \|T_2^n x_n - T_2^n z_n\| + \|T_2^n z_n - x_n\| \\ &\le \|x_n - z_n\| + G_n^{(2)} + \|T_2^n z_n - x_n\| \to 0, \end{aligned}$$

and

$$\begin{aligned} \|x_{n+1} - T_2^n x_{n+1}\| &\le \|x_{n+1} - x_n\| + \|T_2^n x_{n+1} - T_2^n x_n\| + \|T_2^n x_n - x_n\| \\ &\le 2\|x_{n+1} - x_n\| + G_n^{(2)} + \|T_2^n x_n - x_n\| \to 0, \end{aligned}$$

Since

$$\begin{aligned}\|x_{n+1}-T_2x_{n+1}\| &\le \|x_{n+1}-T_2^{n+1}x_{n+1}\|+\|T_2x_{n+1}-T_2^{n+1}x_{n+1}\|\to 0 \quad \text{and}\\ \|x_n-T_2x_n\| &\le \|x_n-x_{n+1}\|+\|x_{n+1}-T_2x_{n+1}\|+\|T_2x_{n+1}-T_2x_n\|,\end{aligned}$$

it follows from uniform continuity of T_2 that $\lim_{n\to\infty}\|T_2x_n-x_n\|=0$.
From $\lim_{n\to\infty}\|T_3^nx_n-x_n\|=0$, we have

$$\begin{aligned}\|x_{n+1}-T_3^nx_{n+1}\| &\le \|x_{n+1}-x_n\|+\|T_3^nx_{n+1}-T_3^nx_n\|+\|T_3^nx_n-x_n\|\\ &\le 2\|x_{n+1}-x_n\|+G_n^{(3)}+\|T_3^nx_n-x_n\|\to 0.\end{aligned}$$

Since

$$\begin{aligned}\|x_{n+1}-T_3x_{n+1}\| &\le \|x_{n+1}-T_3^{n+1}x_{n+1}\|+\|T_3x_{n+1}-T_3^{n+1}x_{n+1}\|\to 0 \quad \text{and}\\ \|x_n-T_3x_n\| &\le \|x_n-x_{n+1}\|+\|x_{n+1}-T_3x_{n+1}\|+\|T_3x_{n+1}-T_3x_n\|,\end{aligned}$$

it follows from uniform continuity of T_3 that $\lim_{n\to\infty}\|T_3x_n-x_n\|=0$. □

Theorem 3.3. *Let X be a uniformly convex Banach space and let C be a nonempty closed and convex subset of X. Suppose that T_1 be a completely continuous asymptotically nonexpansive in the intermediate sense self-maps of C and T_2 and T_3 are continuous asymptotically nonexpansive in the intermediate sense self-maps of C*
Put

$$G_n^{(i)}=\sup_{x,y\in C}(\|T_i^nx-T_i^ny\|-\|x-y\|)\vee 0$$

for all $x,\, y\in C$, $n\ge 1$ and $i=1,2,3$, so that $\sum_{n=1}^{\infty}G_n^{(i)}<\infty, i=1,2,3$. Let $\{a_n\}$, $\{b_n\}$, $\{c_n\}$, $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{\mu_n\}$ and $\{\lambda_n\}$ be real sequences in $[0,1]$ such that $a_n+\gamma_n$, $b_n+c_n+\mu_n$ and $\alpha_n+\beta_n+\lambda_n$ are in $[0,1]$ for all $n\ge 1$ and $\sum_{n=1}^{\infty}\gamma_n<\infty$, $\sum_{n=1}^{\infty}\mu_n<\infty$, $\sum_{n=1}^{\infty}\lambda_n<\infty$ and let $\{u_n\}$, $\{v_n\}$ and $\{w_n\}$ be bounded sequences in C For a given $x_1\in C$, let $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ be the sequences defined as in (1.1) and
(i) $0<\liminf_{n\to\infty}\alpha_n\le\limsup_{n\to\infty}(\alpha_n+\beta_n+\lambda_n)<1$,
(ii) $0<\liminf_{n\to\infty}b_n\le\limsup_{n\to\infty}(b_n+c_n+\mu_n)<1$,
(iii) $0<\liminf_{n\to\infty}a_n\le\limsup_{n\to\infty}(a_n+\gamma_n)<1$,
then the sequences $\{x_n\},\{y_n\}$ and $\{z_n\}$ converge strongly to a common fixed point of T_1,T_2 and T_3.

Proof. By Lemma 3.2, we have
$\lim_{n\to\infty}\|T_1x_n-x_n\|=\lim_{n\to\infty}\|T_2x_n-x_n\|=\lim_{n\to\infty}\|T_3x_n-x_n\|=0$.
Since T_1 is completely continuous and $\{x_n\}$ is bounded, there exists a subsequence

$\{x_{n_k}\}$ of $\{x_n\}$ such that $\{Tx_{n_k}\}$ converges. Therefore from $\lim_{n\to\infty} \|T_1x_n - x_n\| = 0$, $\{x_{n_k}\}$ converges. Let $\lim_{k\to\infty} x_{n_k} = p$. By continuity of T_1 and $\lim_{n\to\infty} \|T_1x_n - x_n\| = 0$, we have that $T_1p = p$, so p is a fixed point of T_1. Since T_2, T_3 are continuous, $T_2x_{n_k} \to T_2p$ and $T_3x_{n_k} \to T_3p$. Since $\|T_2x_{n_k} - x_{n_k}\| \to 0$ and $\|T_3x_{n_k} - x_{n_k}\| \to 0$, it follows that $x_{n_k} = T_2x_{n_k} - (T_2x_{n_k} - x_{n_k}) \to T_2p$ and $x_{n_k} = T_3x_{n_k} - (T_3x_{n_k} - x_{n_k}) \to T_3p$. Hence $T_2p = p$ and $T_3p = p$. So $p \in F(T_1) \cap F(T_2) \cap F(T_3)$. By Lemma 3.1, $\lim_{n\to\infty} \|x_n - p\| = 0$ exists. But $\lim_{n\to\infty} \|x_{n_k} - p\| = 0$. Thus $\lim_{n\to\infty} \|x_n - p\| = 0$. Since $\|y_n - x_n\| \to 0$ and $\|z_n - x_n\| \to 0$ as $n \to \infty$, it follows that $\lim_{n\to\infty} y_n = p$ and $\lim_{n\to\infty} z_n = p$. □

For $T_1 = T_2 = T_3 := T$ in Theorem 3.3, the following results are obtained.

Corollary 3.4. [7, Theorem 2.2]. *Let X be a uniformly convex Banach space, and let C be a nonempty bounded closed and convex subset of X. Let T be a completely continuous asymptotically nonexpansive in the intermediate sense. Put*

$$G_n = \sup_{x,y\in C} (\|T^nx - T^ny\| - \|x - y\|) \vee 0 \quad \forall n \geq 1,$$

so that $\sum_{n=1}^{\infty} G_n < \infty$. Let $\{a_n\}$, $\{b_n\}$, $\{c_n\}$, $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{\mu_n\}$ and $\{\lambda_n\}$ be real sequences in $[0,1]$ such that $a_n + \gamma_n$, $b_n + c_n + \mu_n$ and $\alpha_n + \beta_n + \lambda_n$ are in $[0,1]$ for all $n \geq 1$, and $\sum_{n=1}^{\infty} \gamma_n < \infty$, $\sum_{n=1}^{\infty} \mu_n < \infty$, $\sum_{n=1}^{\infty} \lambda_n < \infty$, and let $\{u_n\}$, $\{v_n\}$ and $\{w_n\}$ be bounded sequences in C. For a given $x_1 \in C$,let $\{x_n\}, \{y_n\}$ and $\{z_n\}$ be the sequences defined as in (1.2) and
(i) $0 < \liminf_{n\to\infty} \alpha_n \leq \limsup_{n\to\infty} (\alpha_n + \beta_n + \lambda_n) < 1$ and
(ii) $0 < \liminf_{n\to\infty} b_n \leq \limsup_{n\to\infty} (b_n + c_n + \mu_n) < 1$.
Then $\{x_n\}, \{y_n\}$ and $\{z_n\}$ converge strongly to a fixed point of T.

Corollary 3.5. [6, Theorem 2.3]. *Let X be a uniformly convex Banach space, and C a nonempty closed, bounded and convex subset of X. Let T be a completely continuous asymptotically nonexpansive self-map of C with $\{k_n\}$ satisfying $k_n \geq 1$ and $\sum_{n=1}^{\infty} (k_n - 1) < \infty$. Let $\{a_n\}$, $\{b_n\}$, $\{c_n\}$, $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}, \{\mu_n\}$ and $\{\lambda_n\}$ be sequences of real numbers in $[0,1]$ with $b_n + c_n + \mu_n \in [0,1]$ and $\alpha_n + \beta_n + \lambda_n \in [0,1]$ for all $n \geq 1$, and $\sum_{n=1}^{\infty} \gamma_n < \infty$, $\sum_{n=1}^{\infty} \mu_n < \infty$, $\sum_{n=1}^{\infty} \lambda_n < \infty$ and*
(i) $0 < \liminf_{n\to\infty} b_n \leq \limsup_{n\to\infty} (b_n + c_n + \mu_n) < 1$, and
(ii) $0 < \liminf_{n\to\infty} \alpha_n \leq \limsup_{n\to\infty} (\alpha_n + \beta_n + \lambda_n) < 1$.
Let $\{x_n\}, \{y_n\}$ and $\{z_n\}$ be sequences defined by the modified Noor iterations with errors (1.2). Then $\{x_n\}, \{y_n\}$ and $\{z_n\}$ converge strongly to a fixed point of T.

Corollary 3.6. [14, Theorem 2.3]. *Let X be a uniformly convex Banach space, and C a nonempty closed , bounded and convex subset of X. Let T be a completely continuous asymptotically nonexpansive self-map of C with $\{k_n\}$ satisfying $k_n \geq 1$ and $\sum_{n=1}^{\infty}(k_n - 1) < \infty$. Let $\{a_n\}$, $\{b_n\}$, $\{c_n\}$, $\{\alpha_n\}$, $\{\beta_n\}$ be sequences of real numbers in $[0,1]$ with $b_n + c_n \in [0,1]$ and $\alpha_n + \beta_n \in [0,1]$ for all $n \geq 1$, and*
(i) $0 < \liminf_{n\to\infty} b_n \leq \limsup_{n\to\infty}(b_n + c_n) < 1$, *and*
(ii) $0 < \liminf_{n\to\infty} \alpha_n \leq \limsup_{n\to\infty}(\alpha_n + \beta_n) < 1$.
Let $\{x_n\}, \{y_n\}$ and $\{z_n\}$ be the sequences defined by the modified Noor iterations (1.3). Then $\{x_n\}, \{y_n\}$ and $\{z_n\}$ converge strongly to a fixed point of T.

For $T_1 = T_2 = T_3 := T$ and $c_n = \beta_n = \gamma_n = \mu_n = \lambda_n \equiv 0$ in Theorem 3.3, we obtain the following result.

Corollary 3.7. [20, Theorem 2.1]. *Let X be a uniformly convex Banach space, and let C be a bounded, closed and convex subset of X. Let T be a completely continuous asymptotically nonexpansive self-map of C with $\{k_n\}$ satisfying $k_n \geq 1$ and $\sum_{n=1}^{\infty}(k_n - 1) < \infty$. Let $\{a_n\}$, $\{b_n\}$, $\{\alpha_n\}$ be real sequences in $[0,1]$ satisfying*
(i) $0 < \liminf_{n\to\infty} b_n \leq \limsup_{n\to\infty} b_n < 1$, *and*
(ii) $0 < \liminf_{n\to\infty} \alpha_n \leq \limsup_{n\to\infty} \alpha_n < 1$.
For a given $x_1 \in C$, define

$$\begin{aligned} z_n &= a_n T^n x_n + (1 - a_n)x_n, \\ y_n &= b_n T^n z_n + (1 - b_n)x_n, \quad n \geq 1, \\ x_{n+1} &= \alpha_n T^n y_n + (1 - \alpha_n)x_n. \end{aligned}$$

Then $\{x_n\}, \{y_n\}$ and $\{z_n\}$ converge strongly to a fixed point of T.

When $T_1 = T_2 = T_3 := T$ and $a_n = c_n = \beta_n = \gamma_n = \mu_n = \lambda_n \equiv 0$ in Theorem 3.3, we can obtain Ishikawa-type convergence result.

Corollary 3.8. *Let X be a uniformly convex Banach space, and let C be a bounded, closed and convex subset of X. Let T be a completely continuous asymptotically nonexpansive self-map of C with $\{k_n\}$ satisfying $k_n \geq 1$ and $\sum_{n=1}^{\infty}(k_n - 1) < \infty$. Let $\{b_n\}$, $\{\alpha_n\}$ be a real sequences in $[0,1]$ satisfying*
(i) $0 < \liminf_{n\to\infty} b_n \leq \limsup_{n\to\infty} b_n < 1$, *and*
(ii) $0 < \liminf_{n\to\infty} \alpha_n \leq \limsup_{n\to\infty} \alpha_n < 1$.
For a given $x_1 \in C$, define

$$\begin{aligned} y_n &= b_n T^n z_n + (1 - b_n)x_n, \\ x_{n+1} &= \alpha_n T^n y_n + (1 - \alpha_n)x_n, \quad n \geq 1. \end{aligned}$$

Then $\{x_n\}$ and $\{y_n\}$ converge strongly to a fixed point of T.

In the next result, we prove weak convergence for the iterative scheme (1.1) for asymptotically nonexpansive mapping in the intermediate sense in a uniformly convex Banach space satisfying Opial's condition.

Theorem 3.9. *Let X be a uniformly convex Banach space which satisfies Opial 's condition, and let C be a nonempty closed and convex subset of X. Let T_1, T_2 and T_3 be asymptotically nonexpansive in the intermediate sense self-maps of C*

Put

$$G_n^{(i)} = \sup_{x,y\in C} (\|T_i^n x - T_i^n y\| - \|x-y\|) \vee 0$$

for all $x, y \in C$, $n \geq 1$, $i = 1,2,3$, so that $\sum_{n=1}^{\infty} G_n^{(i)} < \infty, i = 1,2,3$. Let $\{a_n\}$, $\{b_n\}$, $\{c_n\}$, $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{\mu_n\}$ and $\{\lambda_n\}$ be real sequences in $[0,1]$ such that $a_n + \gamma_n$, $b_n + c_n + \mu_n$ and $\alpha_n + \beta_n + \lambda_n$ are in $[0,1]$ for all $n \geq 1$ and $\sum_{n=1}^{\infty} \gamma_n < \infty$, $\sum_{n=1}^{\infty} \mu_n < \infty$, $\sum_{n=1}^{\infty} \lambda_n < \infty$ and let $\{u_n\}$, $\{v_n\}$ and $\{w_n\}$ be bounded sequences in C. If $F = F(T_1) \cap F(T_2) \cap F(T_3) \neq \emptyset$ and

(i) $0 < \liminf_{n\to\infty} \alpha_n \leq \limsup_{n\to\infty} (\alpha_n + \beta_n + \lambda_n) < 1$,

(ii) $0 < \liminf_{n\to\infty} b_n \leq \limsup_{n\to\infty} (b_n + c_n + \mu_n) < 1$,

(iii) $0 < \liminf_{n\to\infty} a_n \leq \limsup_{n\to\infty} (a_n + \gamma_n) < 1$,

then the sequence $\{x_n\}, \{y_n\}$ and $\{z_n\}$ defined by (1.1) converges weakly to a common fixed point of T_1, T_2 and T_3.

Proof. It follows from Lemma 3.2 that $\lim_{n\to\infty} \|T_1 x_n - x_n\| = 0$, $\lim_{n\to\infty} \|T_2 x_n - x_n\| = 0$, $\lim_{n\to\infty} \|T_3 x_n - x_n\| = 0$. Since X is uniformly convex and $\{x_n\}$ is bounded, we may assume that $x_n \to u$ weakly as $n \to \infty$, without loss of generality. By Lemma 2.3, we have $u \in F$. Suppose that subsequences $\{x_{n_k}\}$ and $\{x_{m_k}\}$ of $\{x_n\}$ converge weakly to u and v, respectively. From Lemma 2.3, u, $v \in F$. By Lemma 3.1 (i), $\lim_{n\to\infty} \|x_n - u\|$ and $\lim_{n\to\infty} \|x_n - v\|$ exist. It follows from Lemma 2.4 that $u = v$. Therefore $\{x_n\}$ converges weakly to a common fixed point of T_1, T_2 and T_3.Since $\|y_n - x_n\| \to 0$ (as $n \to \infty$) and $\|z_n - x_n\| \to 0$ (as $n \to \infty$) and $x_n \to u$ weakly as $n \to \infty$, it follows that $y_n \to u$ and $z_n \to u$ weakly as $n \to \infty$. □

For $T_1 = T_2 = T_3 := T$, in Theorem 3.9, we follow results.

Corollary 3.10. [7, Theorem 2.7]. *Let X be a uniformly convex Banach space which satisfies Opial 's condition, and let C be a nonempty bounded, closed and convex subset of X. Let T be an asymptotically nonexpansive in the intermediate sense. Put*

$$G_n = \sup_{x,y\in C} (\|T^n x - T^n y\| - \|x-y\|) \vee 0, \quad \forall n \geq 1,$$

so that $\sum_{n=1}^{\infty} G_n < \infty$. *For a given* $x_1 \in C$,*let* $\{x_n\}, \{y_n\}$ *and* $\{z_n\}$ *be the sequences defined as in (1.2) and*
(i) $0 < \liminf_{n\to\infty} \alpha_n \le \limsup_{n\to\infty}(\alpha_n + \beta_n + \lambda_n) < 1$ *and*
(ii) $0 < \liminf_{n\to\infty} b_n \le \limsup_{n\to\infty}(b_n + c_n + \mu_n) < 1$.
Then $\{x_n\}$ *converges weakly to a fixed point of* T.

Corollary 3.11. [6, Theorem 2.8]. *Let* X *be a uniformly convex Banach space which satisfies Opial 's condition, and* C *a nonempty closed,bounded and convex subset of* X. *Let* T *be an asymptotically nonexpansive self-map of* C *with* $\{k_n\}$ *satisfying* $k_n \ge 1$ *and* $\sum_{n=1}^{\infty}(k_n - 1) < \infty$. *Let* $\{a_n\}$, $\{b_n\}$, $\{c_n\}$, $\{\alpha_n\}$ $\{\beta_n\}$, $\{\mu_n\}$, $\{\lambda_n\}$ *be sequences of real numbers in* $[0, 1]$ *with* $a_n + \gamma_n$, $b_n + c_n + \mu_n$ *and* $\alpha_n + \beta_n + \lambda_n$ *are in* $[0, 1]$ *for all* $n \ge 1$, *and* $\sum_{n=1}^{\infty} \gamma_n < \infty, \sum_{n=1}^{\infty} \mu_n < \infty, \sum_{n=1}^{\infty} \lambda_n < \infty$ *and*
(i) $0 < \liminf_{n\to\infty} b_n \le \limsup_{n\to\infty}(b_n + c_n + \mu_n) < 1$, *and*
(ii) $0 < \liminf_{n\to\infty} \alpha_n \le \limsup_{n\to\infty}(\alpha_n + \beta_n + \lambda_n) < 1$.
Let $\{x_n\}$ *be the sequence defined by the modified Noor iterations with errors (1.2). Then* $\{x_n\}$ *converges weakly to a fixed point of* T.

Corollary 3.12. [14, Theorem 2.8]. *Let* X *be a uniformly convex Banach space which satisfies Opial 's condition, and* C *a nonempty closed, bounded and convex subset of* X. *Let* T *be an asymptotically nonexpansive self-map of* C *with* $\{k_n\}$ *satisfying* $k_n \ge 1$ *and* $\sum_{n=1}^{\infty}(k_n - 1) < \infty$. *Let* $\{a_n\}, \{b_n\}, \{c_n\}, \{\alpha_n\}, \{\beta_n\}$ *be sequences of real numbers in* $[0, 1]$ *with* $b_n + c_n \in [0, 1]$ *and* $\alpha_n + \beta_n \in [0, 1]$ *for all* $n \ge 1$, *and*
(i) $0 < \liminf_{n\to\infty} b_n \le \limsup_{n\to\infty}(b_n + c_n) < 1$, *and*
(ii) $0 < \liminf_{n\to\infty} \alpha_n \le \limsup_{n\to\infty}(\alpha_n + \beta_n) < 1$.
Let $\{x_n\}$ *be the sequence defined by the modified Noor iterations (1.3). Then* $\{x_n\}$ *converges weakly to a fixed point of* T.

When $T_1 = T_2 = T_3 := T$ and $\gamma_n = \mu_n = \lambda_n = c_n = \beta_n \equiv 0$ in Theorem 3.9, we obtain the following result.

Corollary 3.13. *Let* X *be a uniformly convex Banach space which satisfies Opial 's condition, and* C *a nonempty bounded, closed and convex subset of* X. *Let* T *be an asymptotically nonexpansive self-map of* C *with* $\{k_n\}$ *satisfying* $k_n \ge 1$ *and* $\sum_{n=1}^{\infty}(k_n - 1) < \infty$. *Let* $\{a_n\}$, $\{b_n\}$, $\{\alpha_n\}$ *be sequences of real numbers in* $[0, 1]$ *and*
(i) $0 < \liminf_{n\to\infty} b_n \le \limsup_{n\to\infty} b_n < 1$, *and*

(ii) $0 < \liminf_{n\to\infty} \alpha_n \leq \limsup_{n\to\infty} \alpha_n < 1$.
Let $\{x_n\}, \{y_n\}$ *and* $\{z_n\}$ *be the sequences defined by*

$$\begin{aligned} z_n &= a_n T^n x_n + (1 - a_n) x_n, \\ y_n &= b_n T^n z_n + (1 - b_n) x_n, \quad n \geq 1, \\ x_{n+1} &= \alpha_n T^n y_n + (1 - \alpha_n) x_n. \end{aligned}$$

Then $\{x_n\}$ *converges weakly to a fixed point of* T.

When $T_1 = T_2 = T_3 := T$ and $\gamma_n = \mu_n = \lambda_n = c_n = \beta_n = a_n \equiv 0$ in Theorem 3.9, we obtain Ishikawa-type weak convergence theorem as follows:

Corollary 3.14. *Let* X *be a uniformly convex Banach space which satisfies Opial 's condition, and* C *a nonempty bounded, closed and convex subset of* X*. Let* T *be an asymptotically nonexpansive self-map of* C *with* $\{k_n\}$ *satisfying* $k_n \geq 1$ *and* $\sum_{n=1}^{\infty}(k_n - 1) < \infty$*. Let* $\{b_n\}$, $\{\alpha_n\}$ *be sequences of real numbers in* $[0, 1]$ *such that*
(i) $0 < \liminf_{n\to\infty} b_n \leq \limsup_{n\to\infty} b_n < 1$, *and*
(ii) $0 < \liminf_{n\to\infty} \alpha_n \leq \limsup_{n\to\infty} \alpha_n < 1$.
Let $\{x_n\}$ *and* $\{y_n\}$ *be the sequences defined by*

$$\begin{aligned} y_n &= b_n T^n z_n + (1 - b_n) x_n, \\ x_{n+1} &= \alpha_n T^n y_n + (1 - \alpha_n) x_n, \quad n \geq 1. \end{aligned}$$

Then $\{x_n\}$ *converges weakly to a fixed point of* T.

Acknowledgement(s) : The author would like to thank the Graduate School, Chiang Mai University, Thailand for their financial support.

References

[1] S.C. Bose, *Weak convergence to the fixed point of an asymptotically nonexpansive map,* Proc. Amer. Math. Soc. **68** (1978) 305 – 308.

[2] R. E. Bruck, T. Kuczumow, S. Reich, *Convergence of iterates of asymptotically nonexpansive mappings in Banach spaces with the uniform Opial property,* Colloq. Math. **65** (1993) 169 – 179.

[3] Y. J. Cho, H. Y. Zhou and G. Guo, *Weak and strong convergence theorems for three-step iterations with errors for asymptotically nonexpansive mappings,* Computers and Mathematics with Applications **47** (2004) 707 – 717.

[4] K. Goebel and W.A. Kirk, *A fixed point theorem for asymptotically nonexpansive mappings,* Proc. Amer. Math. Soc. **35** (1972) 171 – 174.

[5] W. A. Kirk, *Fixed point theorems for non-Lipschitzian mappings of asymptotically nonexpansive type,* Israel J. Math **17** (1974) 339 – 346.

[6] K. Nammanee, M. Aslam Noor, S. Suantai, *Convergence Criteria of modified Noor Iterations with errors for asymptotically nonexpansive mappings,* J. Math. Anal. Appl. **314** (2006) 320 – 334.

[7] K. Nammanee, S. Suantai, *The modified Noor iterations with errors for non-Lipschitzian mappings in Banach spaces,* Applied Mathematics and Computation.**187** (2007) 669 – 679.

[8] M. A. Noor, *New approximation schemes for general variational inequalities,* J. Math. Anal. Appl. **251** (2000) 217 – 229.

[9] M. A. Noor, *Three-step iterative algorithms for multivalued quasi variational inclusions,* J. Math. Anal. Appl. **255** (2001) 589 – 604.

[10] Z. Opial, *Weak convergence of successive approximations for nonexpansive mappings,* Bull. Amer. Math. Soc. **73** (1967) 591 – 567.

[11] S. Plubtieng, R. Wangkeeree *Strong convergence theorems for multi-step Noor iterations with errors in Banach spaces,* J. Math. Anal. Appl. **321** (2006) 10 – 23.

[12] J. Schu, *Iterative construction of fixed points of asymptotically nonexpansive mappings,* J. Math. Anal. Appl.**158** (1991) 407 – 413.

[13] J. Schu, *Weak and strong convergence to fixed points of asymptotically nonexpansive mappings,* Bull. Austral. Math. Soc. **43** (1991) 153 – 159.

[14] S. Suantai, *Weak and strong convergence Criteria of Noor Iterations for asymptotically nonexpansive mappings,* J. Math. Anal.Appl. **311** (2005) 506 – 517.

[15] W. Takahashi, Fixed point theory and its applications, Yokohama Publishers, 2000.

[16] K.K. Tan and H. K. Xu, *Approximating fixed points of nonexpansive mapping by the Ishikawa iteration process,* J. Math. Anal.Appl. **178** (1993) 301 – 308.

[17] K.K. Tan and H. K. Xu, *Fixed point iteration processes for asymptotically nonexpansive mapping,* Proc. Amer. Math. Soc. **122** (1994) 733 – 739.

[18] H. K. Xu, *Inequalities in Banach spaces with applications,* Nonlinear Anal. **16** (1991) 1127 – 1138.

[19] H.K. Xu, *Existence and convergence for fixed points of asymptotically nonexpansive type,* Nonlinear Anal. **16** (1991) 1139 – 1146.

[20] B. L. Xu and M. Aslam Noor, *Fixed point iterations for asymptotically nonexpansive mappings in Banach spaces,* J. Math. Anal. Appl. **267** (2002) 444 – 453.

(Received)

Nawitcha Onjai-uea
Program of Mathematics ,
Faculty of Science and Technology,
Nakhon Pathom Rajabhat University,
Nakhon Pathom, 73000, THAILAND
e-mail : nawitcha9@hotmail.com

Suthep Suantai
Department of Mathematics,
Faculty of Science,
Chiang Mai University,
Chiang Mai, 50200, THAILAND
e-mail : scmti005@chiangmai.ac.th.