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On Property (k-NUC) in Cesaro-Musielak-Orlicz Sequence Spaces

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Abstract: In this paper we define a generalized Cesaro sequence space Ces_M , where $M = (M_k)$ is a Musielak-Orlicz function , and consider it equipped with the Luxemburg norm. We call this space the Cesaro Musielak-Orlicz sequence space. The main purpose of the paper is to show that if $M \in \delta_2 \cap \delta_2^*$ and satisfies condition $(*)$ then Ces_M is $k-$ nearly uniform convex $(k - NUC)$ for $k \geq 2$.

Keywords: k-nearly uniform convex, Cesaro Musielak-Orlicz sequence space, Luxemburg norm.
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1 Introduction

The Cesaro sequence space ces_p $(1 < p < \infty)$ were introduced by J.S.Shue [19].They are useful for theory of matrix operators. Y. A. Cui [3] showed that ces_p $(1 < p < \infty)$ is $k - NUC$ for any $k \geq 2$.

In this paper, we define a new sequence space, Ces_M, which is a generalization of the space ces_p, by using a Musielak-Orlicz function, and we call the space Ces_M, Cesaro-Musielak-Orlicz sequence space. We show that if $M \in \delta_2 \cap \delta^*$ and M satisfies the condition $(*)$, then Ces_M is $k - NUC$, so it has Banach-Sake property.

We now introduce the basic notations and defitions. In the following, let $\mathbb{R}$ be the real line and $\mathbb{N}$ the set of natural numbers. Let $(X, ||.||)$ be a real Banach space, and let $B(X)$ (resp. $S(X)$) be the closed unit ball (resp. the unit sphere) of X. Clarkson [2] introduced the concept of uniform convexity. The norm $||.||$ is called *uniformly convex* (write **(UC)**) if for each $\epsilon > 0$ there is $\delta > 0$ such that for $x, y \in S(X)$ inequality $||x - y|| > \epsilon$ implies

$$||\frac{1}{2}(x + y)|| < 1 - \delta$$

For any $x \notin B(X)$, the *drop* determined by x is the set

$$D(x, B(X)) = \text{conv}(\{x\} \cup B(X)).$$

Rolewicz in [17], basing on Danes drop theorem [5], introduced the notion of drop property for Banach spaces.

A Banach space X has the *drop property* (write (**D**)) if for every closed set C disjoint with $B(X)$ there exists an element $x \in C$ such that

$$D(x, B(X)) \cap C = \{x\}.$$

A Banach space X is said to have the *Kadac-Klee property* (or *property (***H***))* if every weakly convergent sequence on the unit sphere is convergent in norm.

In [18] Rolewicz proved that if the Banach space X has the drop property, then X is reflexive. Montesinos [14] extended this result by showing that X has the drop property if and only if X is reflexive and X has the property (**H**).

Recall that a sequence $\{x_n\} \subset X$ is said to be $\epsilon-$*separated sequence* for some $\epsilon > 0$ if

$$sep(x_n) = \inf\{\|x_n - x_m\| : n \neq m\} > \epsilon.$$

A Banach space X is said to be *nearly uniformly convex* (write (**NUC)**) if for every $\epsilon > 0$ there exists $\delta \in (0,1)$ such that for every sequence $(x_n) \subseteq B(X)$ with $sep(x_n) > \epsilon$, we have

$$conv(x_n) \cap ((1-\delta)B(X)) \neq \emptyset.$$

Huff [7] proved that every (NUC) Banach space is reflexive and it has property (H)

Kutzarova [9] has defined k-nearly uniformly convex Banach spaces. Let $k \geq 2$ be an integer . A Banach space X is said to be $k-$ *nearly uniformly convex* (write $k - NUC$) if for any $\epsilon > 0$ there exists $\delta > 0$ such that for any sequence $(x_n) \subset B(X)$ with $sep(x_n) \geq \epsilon$ there are $n_1, n_2, ..., n_k \in \mathbb{N}$ such that

$$\|\frac{x_{n_1} + x_{n_2} + x_{n_3} + ... + x_{n_k}}{k}\| < 1 - \delta.$$

Clearly $k - NUC$ Banach spaces are NUC but the opposite implication does not hold in general (see [9]).

Fan and Glicksberg [6] have introduced fully k-convex Banach spaces. A Banach spaces X is said to be $fully\ k - rotund$ (write kR) if for every sequence $(x_n) \subset B(X)$, $\|x_{n_1} + x_{n_2} + ... + x_{n_k}\| \to k$ as $n_1, n_2, ..., n_k \to \infty$ implies that (x_n) is convergent.

It is well known that $UC \Rightarrow kR \Rightarrow (k+1)R$, and kR spaces are reflexive and rotund, and it is easy to see that $k - NUC \Rightarrow kR$.

Let X be a real vector space. A functional $\varrho : X \to [0, \infty]$ is called a *modular* if it satisfies the conditions

(i) $\varrho(x) = 0$ if and only if $x = 0$;

(ii) $\varrho(\alpha x) = \varrho(x)$ for all scalar α with $|\alpha| = 1$;

(iii) $\varrho(\alpha x + \beta y) \leq \varrho(x) + \varrho(y)$, for all $x, y \in X$ and all $\alpha, \beta \geq 0$ with $\alpha + \beta = 1$. The modular ϱ is called *convex* if

(iv) $\varrho(\alpha x+\beta y) \leq \alpha\varrho(x)+\beta\varrho(y)$, for all $x, y \in X$ and all $\alpha, \beta \geq 0$ with $\alpha+\beta = 1$.

If ϱ is a modular in X, we define

$$X_\varrho = \{x \in X : \lim_{\lambda \to 0^+} \varrho(\lambda x) = 0 \}$$

$$\text{and} \quad X_\varrho^* = \{x \in X : \varrho(\lambda x) < \infty \text{ for some } \lambda > 0 \}.$$

It is clear that $X_\varrho \subseteq X_\varrho^*$. If ϱ is a convex modular, for $x \in X_\varrho$ we define

$$\|x\| = \inf\{\lambda > 0 : \varrho\left(\frac{x}{\lambda}\right) \leq 1 \} \tag{1.1}$$

It is known that if ϱ is a convex modular on X, then $X_\varrho = X_\varrho^*$ and $\|.\|$ is a norm on X_ϱ for which it is a Banach space. The norm $\|.\|$ defined as in (1.1) is called the Luxemburg norm.

The following known results gave some relationships between the modular ϱ and the Luxemburg norm $\|.\|$ on X_ϱ.

Theorem 1.1 *Let ϱ be a convex modular on X and let $x \in X_\varrho$ and (x_n) a sequence in X_ϱ. Then $\|x_n - x\| \to 0$ as $n \to \infty$ if and only if $\varrho(\lambda(x_n - x)) \to 0$ as $n \to \infty$ for every $\lambda > 0$.*

Proof. See [13, Theorem 1.3]. □

Let l^0 be the space of all real sequences. For $1 \leq p < \infty$, the Cesaro sequence space (ces_p , for short) is defined by

$$ces_p = \{x \in l^0 : \textstyle\sum_{n=1}^{\infty}(\frac{1}{n}\sum_{i=1}^{n}|x(i)|)^p < \infty\}$$

equipped with the norm

$$\|x\| = (\sum_{n=1}^{\infty}(\frac{1}{n}\sum_{i=1}^{n}|x(i)|)^p)^{\frac{1}{p}}$$

This space was introduced by Shue [19]. It is useful in the theory of Matrix operator and others (see [10] and [11]). Some geometric properties of the Cesaro sequence space ces_p were studied by many authors.

A map $\phi : \mathbb{R} \to [0, \infty]$ is said to be an *Orlicz function* if ϕ vanishes only at 0, and ϕ is even and convex.

A sequence $M = (M_k)$ of Orlicz functions is called a *Musielak-Orlicz function.* In addition, a Musielak-Orlicz function $N = (N_k)$ is called a *complementary function* of a Musielak-Orlicz function M if

$$N_k(v) = \sup\{|v|u - M_k(u) : u \geq 0\} \ , \quad k = 1, 2, ...$$

For a given Musielak-Orlicz function M, the *Musielak-Orlicz sequence space* l_M and its subspace h_M are defined as follows:

$$l_M := \{x \in l^0 : I_M(cx) < \infty \text{ for some } c > 0\},$$

$$h_M := \{x \in \ell^0 : I_M(cx) < \infty \text{ for all } c > 0\},$$

where I_M is a convex modular defined by

$$I_M(x) = \sum_{k=1}^{\infty} M_k(x(k)), \quad x = (x(k)) \in l_M.$$

We consider ℓ_M equipped with the *Luxemburg norm*

$$\|x\| = \inf\{k > 0 : I_M(\frac{x}{k}) \leq 1\}$$

or equipped with the Orlicz norm

$$\|x\|^0 = \inf\{\frac{1}{k}(1 + I_M(kx)) : k > 0\}.$$

To simplify notation, we put $l_M := (l_M, \|.\|)$ and $l_M^0 := (l_M, \|.\|^0)$. Both of them are Banach spaces (see [1] and [16]).

Let $M = (M_k)$ be the Musielak-Orlicz function. The *Cesàro-Musielak-Orlicz sequence space* is defined by

$$Ces_M := \{x \in l^0 : \rho_M(cx) < \infty \text{ for some } c > 0\},$$

where ρ_M is a convex modular defined by $\rho_M(x) = \sum_{k=1}^{\infty} M_k(\frac{1}{k}\sum_{i=1}^{k}|x(i)|)$.

We consider Ces_M equipped with the Luxemburg norm

$$\|x\| = \inf\{\lambda > 0 : \rho_M(\frac{x}{\lambda}) \leq 1\}$$

under with it is a Banach space. We define the subspace $SCes_M$ of Ces_M by

$$SCes_M := \{x \in l^0 : \rho_M(cx) < \infty \text{ for all } c > 0\}.$$

We say that a Musielak-Orlicz function M *satisfies the δ_2-condition* (we will write $M \in \delta_2$ for short) if there exist constants $K \geq 2, u_0 > 0$ and a sequence (c_k) of positive numbers such that $\sum_{k=1}^{\infty} c_k < \infty$ and the inequality

$$M_k(2u) \leq KM_k(u) + c_k$$

hold for every $k \in \mathbb{N}$ and $u \in \mathbb{R}$ satisfying $M_k(u) \leq u_0$.

If $M \in \delta_2$ and $N \in \delta_2$, then we write $M \in \delta_2 \cap \delta_2^*$. It is known that $\ell_M = h_M$ if and only if $M \in \delta_2$ (see [16]). Moreover, we say that a Musielak-Orlicz function M *satisfies the* $(*)$*-condition* if for any $\epsilon \in (0, 1)$, there exists $\delta > 0$ such that $M_k((1 + \delta)u) \leq 1$ whenever $M_k(u) \leq 1 - \epsilon$ for all $k \in \mathbb{N}$ and $u \in \mathbb{R}$.

2 Main Results

We start with a basic property of ρ_M.

Proposition 2.1 *For any $x \in Ces_M$, we have*

(1) if $\|x\| \leq 1$, then $\rho_M(x) \leq \|x\|$, and

(2) if $\|x\| > 1$, then $\rho_M(x) \geq \|x\|$.

Proof. (1) If $x = 0$, then the inequality holds. Let $x \neq 0$. By the definition of $\|.\|$, there is a sequence (ϵ_n) such that $\epsilon_n \downarrow \|x\|$ such that $\rho_M(\frac{x}{\epsilon_n}) \leq 1$. This implies $\rho_M(\frac{x}{\|x\|}) \leq 1$. Since ρ_M is convex, we have $\rho_M(x) \leq \|x\|\rho_M(\frac{x}{\|x\|}) \leq \|x\|$.

(2) Let $\|x\| > 1$. Then for $\epsilon \in (0, \frac{\|x\|-1}{\|x\|})$, we have $(1-\epsilon)\|x\| > 1$. By convexity of ρ_M, we have $1 < \rho_M(\frac{x}{(1-\epsilon)\|x\|}) \leq \frac{\rho_M(x)}{(1-\epsilon)\|x\|}$, so that $(1-\epsilon)\|x\| < \rho_M(x)$. By taking $\epsilon \to 0$, we have $\rho_M(x) \geq \|x\|$. □

The following result is directly obtained from Proposition 2.1(1).

Corollary 2.2 *If (x_n) is a sequence in Ces_M such that $x_n \to 0$ as $n \to \infty$, then $\rho_M(x_n) \to 0$ as $n \to \infty$.*

Proposition 2.3 *If a Musielak-Orlicz function $M = (M_k) \in \delta_2$, then $SCes_M = Ces_M$.*

Proof . If $x \in Ces_M$, then the sequence $a = (a(k))$, defined by $a(k) = \frac{1}{k}\sum_{i=1}^{k}|x(i)|$ for all $k \in \mathbb{N}$, is in ℓ_M. By $M \in \delta_2$ we have that $\ell_M = h_M$. This implies that $\rho_M(\lambda x) = I_M(\lambda a) < \infty \; \forall \lambda > 0$, hence $x \in SCes_M$. □

Proposition 2.4 *If the Musielak-Orlicz function $M = (M_k) \in \delta_2$, then*

(1) $\|x\| = 1 \Leftrightarrow \rho_M(x) = 1$,

(2) for every $\epsilon > 0$, there exists a $\delta > 0$ such that $\|x\| < 1 - \delta$ whenever $\rho_M(x) < 1 - \epsilon$.

Proof. (1) Assume that $\rho_M(x) = 1$. By definition of $\|.\|$, we have that $\|x\| \leq 1$. If $\|x\| < 1$, then we have by Proposition 2.1(1) that $\rho_M(x) \leq \|x\| < 1$, which contradicts our assumption. Therefore $\|x\| = 1$.

Conversely, assume that $\|x\| = 1$. By Proposition 2.1(1), $\rho_M(x) \leq 1$. Suppose that $\rho_M(x) < 1$. By Proposition 2.3, we have $\rho_M(cx) < \infty$ for all $c > 1$. Since the function $t \mapsto \rho_M(tx)$ is continuous, there exists $c > 1$ such that $\rho_M(cx) = 1$. By using the same proof as in the first part, we have that $\|cx\| = 1$, so $c = 1$, which is a contradiction.

(2) Suppose that (2) is not true. Then there exists a $\epsilon_0 > 0$ and $x_n \in Ces_M$ such that $\rho_M(x_n) < 1-\epsilon_0$ and $\frac{1}{2} \leq \|x_n\|$ and $\|x_n\| \to 1$. Let $L = \sup_n\{\rho_M(2x_n)\}$. Then $L < \infty$ since $M \in \delta_2$. Let $a_n = \frac{1}{\|x_n\|} - 1$, we have $a_n \leq 1$ and $a_n \to 0$. Then

$$\begin{aligned}1 &= \rho_M(\frac{x_n}{\|x_n\|})\\ &= \rho_M(2a_nx_n + (1-a_n)x_n)\\ &\leq a_n\rho_M(2x_n) + (1-a_n)\rho_M(x_n)\\ &\leq a_nL + (1-\epsilon_0).\end{aligned}$$

This implies $1 \leq \lim_{n\to\infty}(a_nL + (1-\epsilon_0)) = 1-\epsilon_0$, which is a contradiction. Hence (2) is satisfied. □

Proposition 2.5 *If the Musielak-Orlicz function $M = (M_k)$ satisfies condition $(*)$ and $M \in \delta_2$, then*

(1) for every $\epsilon > 0$ and $c > 0$, there exists a $\delta > 0$ such that for any $x, y \in Ces_M$, we have

$$|\rho_M(x+y) - \rho_M(x)| < \epsilon$$

whenever $\rho_M(x) \leq c$ and $\rho_M(y) \leq \delta$ and

(2) for every $\epsilon > 0$, there exists $\delta > 0$ such that $\|x\| > 1+\delta$ whenever $\rho_M(x) > 1+\epsilon$.

Proof. (1) Let $\epsilon > 0$ and $c > 0$. By [4, Lemma 8], there exists a $\delta > 0$ such that for any $a, b \in l_M$, we have

$$|I_M(a+b) - I_M(a)| < \epsilon \tag{2.1}$$

whenever $I_M(a) \leq c$ and $I_M(b) \leq \delta$. For each $i \in \mathbb{N}$, let

$$s(i) = \begin{cases}\operatorname{sgn}(x(i)+y(i)) & \text{if } x(i)+y(i) \neq 0,\\ 1 & \text{if } x(i)+y(i) \neq 0.\end{cases}$$

Then we have

$$\begin{aligned}\rho_M(x+y) &= \sum_{k=1}^{\infty} M_k(\frac{1}{k}\sum_{i=1}^{k}|x(i)+y(i)|)\\ &= \sum_{k=1}^{\infty} M_k(\frac{1}{k}\sum_{i=1}^{k}s(i)x(i) + \frac{1}{k}\sum_{i=1}^{k}s(i)y(i)).\end{aligned} \tag{2.2}$$

Let $a(k) = \frac{1}{k}\sum_{i=1}^{k} s(i)x(i)$ and $b(k) = \frac{1}{k}\sum_{i=1}^{k} s(i)y(i)$ for all $k \in \mathbb{N}$. Then $a = (a(k)) \in l_M$ and $b = (b(k)) \in l_M$, and from (2.2) , we have

$$\rho_M(x+y) = I_M(a+b), I_M(a) \le \rho_M(x) \text{ and } I_M(b) \le \rho_M(y).$$

Let $x, y \in Ces_M$ be such that $\rho_M(x) \le c$ and $\rho_M(y) \le \delta$. Then $I_M(a) \le c$ and $I_M(b) \le \delta$. By (2.1), we have $\rho_M(x+y) - \rho_M(x) \le I_M(a+b) - I_M(a) < \epsilon$, that is

$$\rho_M(x+y) < \rho_M(x) + \epsilon. \tag{2.3}$$

Next, we shall show that

$$\rho_M(x) < \rho_M(x+y) + \epsilon. \tag{2.4}$$

For each $i \in \mathbb{N}$, let

$$s(i) = \begin{cases} sgn(x(i)) & \text{if} \quad x(i) \neq 0, \\ 1 & \text{if} \quad x(i) = 0. \end{cases}$$

Then

$$\begin{aligned} \rho_M(x) &= \rho_M((x+y) + (-y)) \\ &= \sum_{k=1}^{\infty} M_k \left(\frac{1}{k} \sum_{i=1}^{k} |(x(i) + y(i)) + (-y(i))| \right) \\ &= \sum_{k=1}^{\infty} M_k \left(\frac{1}{k} \sum_{i=1}^{k} s(i)(x(i) + y(i)) + \frac{1}{k} \sum_{i=1}^{k} s(i)(-y(i)) \right). \end{aligned}$$

Let $a(k) = \frac{1}{k}\sum_{i=1}^{k} s(i)(x(i)+y(i))$ and $b(k) = \frac{1}{k}\sum_{i=1}^{k} s(i)(-y(i))$ for all $k \in \mathbb{N}$. It is clear that $a = (a(k)) \in l_M$ and $b = (b(k)) \in l_M$, and $\rho_M(x) = I_M(a+b)$, $I_M(a) \le \rho_M(x+y)$ and $I_M(b) \le \rho_M(y)$. Hence we have $I_M(a+b) \le c$ and $I_M(-b) \le \delta$. By (2.3), we have

$$\begin{aligned} |I_M(a+b) - I_M(a)| &= |I_M(a) - I_M(a+b)| && (2.5) \\ &= |I_M((a+b) + (-b)) - I_M(a+b)| < \epsilon. && (2.6) \end{aligned}$$

This implies that $\rho_M(x) - \rho_M(x+y) \le I_M(a+b) - I_M(a) < \epsilon$, hence $\rho_M(x) < \rho_M(x+y) + \epsilon$, so (2.4) holds. Therefore (1) is obtained by (2.3) and (2.4).

(2) Let $\epsilon > 0$ be given. By (1), there exists $\delta \in (0,1)$ such that

$$\rho_M(u) \le 1, \rho_M(v) \le \delta \Rightarrow \rho_M(u+v) \le \rho_M(u) + \epsilon. \tag{2.7}$$

If $\|x\| \le 1+\delta$, then $\rho_M(\frac{x}{1+\delta}) \le 1$ and $\rho_M(\frac{\delta x}{1+\delta}) \le \delta\rho_M(\frac{x}{1+\delta}) \le \delta$. By (2.5), we have $\rho_M(x) = \rho_M(\frac{x}{1+\delta} + \frac{\delta x}{1+\delta}) \le \rho_M(\frac{x}{1+\delta}) + \epsilon \le 1 + \epsilon$. Hence (2) is satisfied. □

Proposition 2.6 *If the Musielak-Orlicz function $M = (M_k)$ satisfies condition $(*)$ and $M \in \delta_2$, then for any sequence $(x_n) \subset Ces_M$, $\|x_n\| \to 1$ implies $\rho_M(x_n) \to 1$.*

Proof. Suppose that $\rho_M(x_n) \nrightarrow 1$ as $n \to \infty$. We may assume that there exits $\epsilon_0 > 0$ such that $|\rho_M(x_n) - 1| > \epsilon_0$ for all $n \in \mathbb{N}$. If $\rho_M(x_n) - 1 > \epsilon_0$, then $\rho_M(x_n) > 1 + \epsilon_0$, by Proposition 2.5(2), there exists $\delta > 0$ such that $\|x_n\| > 1 + \delta$. If $\rho_M(x_n) - 1 < -\epsilon_0$, then $\rho_M(x_n) < 1 - \epsilon_0$, by Proposition 2.4(2), there exists a $\delta' > 0$ such that $\|x_n\| < 1 - \delta'$. These imply that $\|x_n\| \nrightarrow 1$ as $n \to \infty$. □

Proposition 2.7*In Cesàro-Musielak-Orlicz sequence space. If a Musielak-Orlicz function $M = (M_k)$ satisfies condition $(*)$ and $M \in \delta_2$, then the norm convergence and modular convergence coincide.*

Proof. From Corollary 2.2 and because $M \in \delta_2$, it suffices to prove that modular convergence implies norm convergence. To do this, let $\epsilon \in (0, \frac{1}{2})$, choose a positive integer K such that $\frac{1}{2^{K+1}} < \epsilon \leq \frac{1}{2^K}$. By Proposition 2.5(1), there exists a $\delta \in (0, \frac{1}{2^{K+1}})$ such that

$$\rho_M(u) \leq 1, \rho_M(v) \leq \delta \Rightarrow \rho_M(u+v) < \rho_M(u) + \epsilon. \tag{2.8}$$

If $\rho_M(x) < \delta$, by application of (2.6), we have $\rho_M(nx) < \rho_M(x) + n\epsilon$, for $n = 1, ..., 2^{K-1}$. In particular, $\rho_M(\frac{x}{4\epsilon}) \leq \rho_M(2^{K-1}x) < \rho_M(x) + 2^{K-1}\epsilon < \frac{1}{2} + \frac{1}{2} = 1$. This implies $\|x\| < 4\epsilon$. □

Theorem 2.8 *If the Musielak-Orlicz function $M = (M_i)$ satisfies $(*)$ -condition and $M \in \delta_2 \cap \delta_2^*$, then Ces_M is $k - NUC$.*

Proof. Let $\epsilon > 0$ be given and sequence $(x_n) \subseteq B(Ces_M)$ with $sep\{(x_n)\} > \epsilon.$, By $M \in \delta_2$ there exists a $\delta > 0$ such that

$$\inf\{\rho_M(\frac{x_n - x_m}{2}) : n \neq m\} \geq \delta$$

.Next, we will show that for any $j \in \mathbb{N}$ there exists $n_j \in \mathbb{N}$ such that

$$\sum_{i=j}^{\infty} M_i(\frac{1}{i}\sum_{l=1}^{i} |x_{n_j}(l)|) \geq \frac{\delta}{3} \tag{2.9}$$

otherwise, there exists a $j_0 \in \mathbb{N}$ such that

$$\sum_{i=j_0}^{\infty} M_i(\frac{1}{i}\sum_{l=1}^{i} |x_{n_j}(l)|) < \frac{\delta}{3}$$

for any $j \in \mathbb{N}$. Put $y_n = (x_n(1), x_n(2), ..., x_n(j_0), 0, 0, ...)$ for $n \in \mathbb{N}$. Then there exists a subsequence $(y_{n_l}) \subset (y_n)$ such that

$$\rho_M(\frac{y_{n_l} - y_{n_j}}{2}) < \frac{\delta}{3}$$

for any $l \neq j$. Hence

$$\begin{aligned}
\rho_M(\frac{x_{n_l} - x_{n_j}}{2}) &= \rho_M(\frac{\sum_{m=1}^{j_0}(x_{n_l}(m) - x_{n_j}(m))e_m}{2}) \\
&+ \rho_M(\frac{\sum_{m=j_0+1}^{j_0}(x_{n_l}(m) - x_{n_j}(m))e_m}{2}) \\
&\leq \rho_M(\frac{\sum_{m=1}^{j_0}(x_{n_l}(m) - x_{n_j}(m))e_m}{2}) \\
&+ \frac{1}{2}\sum_{i=j_0+1}^{\infty} M_i(\frac{1}{i}\sum_{m=1}^{i}|x_{n_l}(m)|) \\
&+ \frac{1}{2}\sum_{i=j_0+1}^{\infty} M_i(\frac{1}{i}\sum_{m=1}^{i}|x_{n_j}(m)|) \\
&= \rho_M(\frac{y_{n_l} - y_{n_j}}{2}) + \frac{1}{2}\sum_{i=j_0+1}^{\infty} M_i(\frac{1}{i}\sum_{m=1}^{i}|x_{n_l}(m)|) \\
&+ \frac{1}{2}\sum_{i=j_0+1}^{\infty} M_i(\frac{1}{i}\sum_{m=1}^{k}|x_{n_j}(m)|) \\
&< \frac{\delta}{3} + \frac{\delta}{6} + \frac{\delta}{6} = \frac{2\delta}{3} < \delta.
\end{aligned}$$

This contradiction show that (2.7) is correct. Since $N \in \delta_2$, there exists $\theta \in (0, 1)$ and a sequence (h_i) in $\mathbb{R}^+$ with $\sum_{i=1}^{\infty} M_i(h_i) < \infty$ such that

$$M_i(\frac{u}{k}) \leq \frac{1-\theta}{k} M_i(u)$$

hold for every $i \in \mathbb{N}$ and u satisfying $M_i(h_i) \leq M_i(u) \leq 1$. Using $M \in \delta_2$ again, there exists $\delta_1 > 0$ such that

$$|\rho_M(x + y) - \rho_M(x)| < \frac{\theta\delta}{12k}$$

whenever $\rho_M(x) \leq 1$ and $\rho_M(y) < \delta_1$. Take $n_1 < n_2 < ... < n_{k-1} \in \mathbb{N}$. Notice that

$$\rho_M(\frac{x_{n_1} + x_{n_2} + ... + x_{n_{k-1}}}{k}) < \infty$$

and $\rho_M(x_{n_l}) < \infty$ for $l = 1, 2, .., k - 1$. This exists a $j_0 \in \mathbb{N}$ such that

$$\sum_{i=j_0+1}^{\infty} M_i(\frac{1}{i}\sum_{l=1}^{i}|\frac{x_{n_1}(l) + x_{n_2}(l) + ... + x_{n_{k-1}}(l)}{k}|) < \delta_1, \tag{2.10}$$

$$\sum_{i=j_0+1}^{\infty} M_i(\frac{1}{i}\sum_{l=1}^{i}|x_{n_j}(l)|) < \frac{\delta}{3}, (j = 1, 2, .., k-1) \tag{2.11}$$

$$\sum_{i=j_0+1}^{\infty} M_i(h_i) < \frac{\delta\theta}{12k}. \tag{2.12}$$

By (1) there exists a $n_k \in \mathbb{N}$ such that

$$\sum_{i=j_0+1}^{\infty} M_i(\frac{1}{i}\sum_{l=1}^{i}|x_{n_k}(l)|) \geq \frac{\delta}{3}. \tag{2.13}$$

Hence, by (2.8),(2.9),(2.10) and (2.11), we have

$$\begin{aligned}
\rho_M(\frac{x_{n_1} + x_{n_2} + ... + x_{n_k}}{k}) &= \sum_{i=1}^{j_0} M_i(\frac{1}{i}\sum_{l=1}^{i}|\frac{x_{n_1}(l) + x_{n_2}(l) + ... + x_{n_k}(l)}{k}|) \\
&+ \sum_{i=j_0+1}^{\infty} M_i(\frac{1}{i}\sum_{l=1}^{i}|\frac{x_{n_1}(l) + x_{n_2}(l) + ... + x_{n_k}(l)}{k}|) \\
&\leq \frac{1}{k}\sum_{j=1}^{k}\sum_{i=1}^{j_0} M_i(\frac{1}{i}\sum_{l=1}^{i}|x_{n_j}(l)|) \\
&+ \sum_{i=j_0+1}^{\infty} M_i(\frac{1}{i}\sum_{l=1}^{i}|\frac{x_{n_k}(l)}{k}|) + \frac{\delta\theta}{12k} \\
&\leq \frac{1}{k}\sum_{j=1}^{k}\sum_{i=1}^{j_0} M_i(\frac{1}{i}\sum_{l=1}^{i}|x_{n_j}(l)|) \\
&+ \frac{1-\theta}{k}\sum_{i=j_0+1}^{\infty} M_i(\frac{1}{i}\sum_{l=1}^{i}|x_{n_k}(l)|) \\
&+ \sum_{i=j_0+1}^{\infty} M_i(h_i) + \frac{\delta\theta}{12k} \\
&\leq \frac{1}{k}\sum_{j=1}^{k}\sum_{i=1}^{\infty} M_i(\frac{1}{i}\sum_{l=1}^{i}|x_{n_j}(l)|) \\
&- \frac{\theta}{k}\sum_{i=j_0+1}^{\infty} M_i(\frac{1}{i}\sum_{l=1}^{i}|x_{n_k}(l)|) + \frac{\delta\theta}{6k} \\
&\leq 1 - \frac{\delta\theta}{3k} + \frac{\delta\theta}{6k} = 1 - \frac{\delta\theta}{6k}.
\end{aligned}$$

Since $M \in \delta_2$ and satisfies $(*)$-condition, by Proposition 2.4(2) there is $\gamma \in (0,1)$ such that

$$||\frac{x_{n_1} + x_{n_2} + ... + x_{n_k}}{k}|| < 1 - \gamma.$$

Thus Ces_M is $(k - NUC)$.

Corollary 2.9 *If the Musielak-Orlicz function $M = (M_k)$ satisfies $(*)$ -condition and $M \in \delta_2 \cap \delta_2^*$, then*

(1) Ces_M has the Banach-saks property i.e. Ces_M is reflexive and it has weak Banach-saks property.

(2) Ces_M is NUC.

Since $k - NUC \Rightarrow kR \Rightarrow R$ & Rfx and $k - NUC \Rightarrow (NUC) \Rightarrow$ property (H) & Rfx, where Rfx denotes for reflexive, by Theorem 2.8, the following results are obtained.

Corollary 2.10 [3] *For $1 < p < \infty$, the space ces_p is $k - NUC$.*

Corollary 2.11 *For $1 < p < \infty$, the space ces_p is kR and (NUC).*

Corollary 2.12 *For $1 < p < \infty$, the space ces_p has the drop property.*

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