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Note on Chain Sequences of Laguerre and Romanovski-Laguerre Type Polynomials

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Abstract In the last few years, the recurrence relations of classical orthogonal polynomials are useful in the studying of chain sequences and some properties of Mass-Spring system. In this work, we discuss chain sequences of Laguerre and Romanovski-Laguerre type finite class of classical orthogonal polynomials (COPs). We also introduce the application of chain sequences for discussing some properties of Mass-Spring system.

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1. Introduction

The theory of classical orthogonal polynomials plays important role in the discussing of continued fractions associated with moments problems and chain sequences using the three term recurrence relations. These continued fractions and chain sequences are very useful in further study of many important properties of classical orthogonal polynomials (COPs). The classification of the birth-death process is also discussed with the help of chain sequences [1–11]. The orthogonal polynomials are also useful in observing of asymptotical behaviour and some operators which are given in [12, 13].

We consider the three-terms recurrence relations for classical orthogonal polynomials

$$\begin{cases} \phi_{n+1}(x) = (A_n x + B_n)\phi_n(x) - C_n\phi_{n-1}(x), & n \geq 1, \quad \text{and} \\ \phi_0(x) = 1, \phi_1(x) = A_0 x + B_0, \end{cases} \tag{1.1}$$

where $A_0 \neq 0$ and $A_n C_n \neq 0$ for $n \geq 1$. Define $P_n(x) = \phi_n(x)/(A_0 A_1 A_2 \ldots A_{n-1})$. Hence (1.1) becomes a sequences of monic polynomials $\{P_n(x)\}_{n=0}^{\infty}$ satisfying three-terms

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recurrence relations,

$$\begin{cases} P_n(x) = (x - c_n)P_{n-1}(x) - \lambda_n P_{n-2}(x), & n > 1, \quad \text{and} \\ P_0(x) = 1, P_1(x) = x - c_1 \end{cases} \tag{1.2}$$

where the coefficients are real [3, 4, 14–16].
It is well known that when $\lambda_n > 0$ for all $n > 1$, the zeros of $P_n(x)$ are real, distinct and between each pair of consecutive zeros of $P_{n+1}(x)$ there is precisely one zeros of $P_n(x)$. Moreover, there exits a positive Borel measure ψ on $\mathbb{R}$ such that,

$$\int_a^b P_n(x)P_m(x)\psi(dx) = k_n\delta_{nm} \tag{1.3}$$

with $k_n > 0$. When $\lambda_n > 0$ for all $n > 1$ we shell refer to $\{P_n(x)\}$ as an orthogonal polynomials sequence (OPS).
In general, $\lambda_n \neq 0$ for all $n > 1$, we shell refer to $P_n(x)$ as a generalized orthogonal polynomials sequence (GOPS). We can be said in general about the polynomials of a GOPS but for the existence of a finite signed Borel measure ψ on $\mathbb{R}$ such that (1.3) holds with $k_n \neq 0$. However, as shown in [7, 17], there exits a class of GOPSs which, in general, are not OPSs but have properties resembling those of OPSs as far as zeros are concerned. This class is denoted by $\wp$ and defined as follows.

Definition 1.1. Let $\{P_n(x)\}$ be a GOPS satisfying (1.2). Then $\{P_n(x)\} \in \wp$ if $c_n \neq 0$ for all $n > 1$ and the sequences $\{\alpha_n\}_{n=1}^{\infty}$ defined by

$$\alpha_n \equiv \frac{\lambda_{n+1}}{c_n c_{n+1}} \tag{1.4}$$

constitutes a chain sequences. That is, there exits a parameter sequence $\{g_n\}_{n=0}^{\infty}$ of $\{\alpha_n\}_{n=1}^{\infty}$ satisfying $0 \leq g_0 < 1$ and $0 < g_n < 1$, $n \geq 1$, such that $\alpha_n = g_n(1 - g_{n-1})$, $n \geq 1$ and the continued fraction expansion

$$\frac{1}{1} - \frac{(1-g_0)g_1 x}{1} - \frac{(1-g_1)g_2 x}{1} - \frac{(1-g_2)g_3 x}{1} - \dots, \tag{1.5}$$

is also called g-fraction or g-sequence [4, 11, 18–20].

The elements of a GOPS in $\wp$ will be called chain sequences polynomials. Of course, if $c_n > 0$ for all $n \geq 1$ or $c_n < 0$ for all $n \geq 1$, hence $\lambda_n > 0$ for all $n > 1$, then $\{P_n(x)\} \in \wp$ constitute an OPS and we are on familiar grounds. And hence the λ_n, differ in sign. The following was proved in [7, 17].
Chihara [4] has shown that their true interval of orthogonality lies in $(0, \infty)$ if, and only if, there exits a real sequences $\{\delta_n\}$ such that,

$$\begin{cases} \delta_1 = 0, \quad \delta_n > 0 \quad \text{for} \quad n > 1 \\ \delta_{2n}\delta_{2n+1} = \lambda_{n+1} \quad \text{and} \quad \delta_{2n} + \delta_{2n-1} = c_n \quad \text{for} \quad n \geq 1. \end{cases} \tag{1.6}$$

According to Law [21], some recurrence polynomials families emerge naturally in the analysis of certain physical systems, such as the chain of harmonic oscillators represented in Figure 1.
In the absence of externally applied forces, the equations of motion are as follows,

$$\begin{cases} m_0\ddot{x_0} = -(k_0 + k_1)x_0 + k_1 x_1, \\ m_n\ddot{x_n} = k_n x_{n-1} - (k_n + k_{n+1})x_n + k_{n+1}x_{n+1}, \quad n \geq 1, \end{cases} \tag{1.7}$$

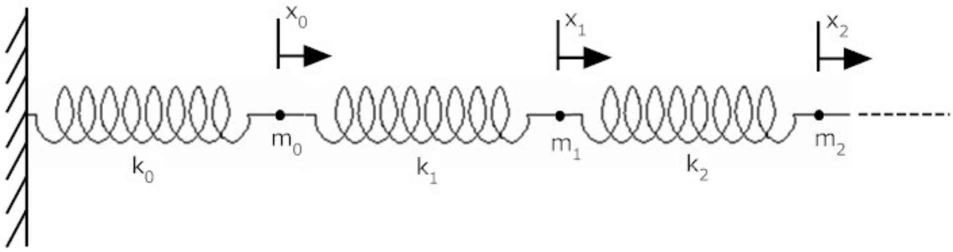

FIGURE 1. A half-infinite frictionless chain of spring and mass with nearest neighbor coupling.

where the masses m_n and spring constants k_n satisfy

$$\begin{aligned} &k_0 > 0, \\ &k_n > 0 \quad \text{for} \quad n \geq 1, \\ &m_n > 0 \quad \text{for} \quad n \geq 0. \end{aligned} \tag{1.8}$$

Solutions of the equations (1.7) can be expressed in a closed form (1.8) using the secular polynomials associated with the system.

Further, if any polynomials system (1.1) are changed into the form (1.2), then they are secular polynomials associated with some spring-mass system (1.7) if, and only if, there exist sequences $\{k_n\}$ and $\{m_n\}$, satisfying (1.8), such that,

$$\begin{aligned} c_{n+1} &= -\frac{(k_n + k_{n+1})}{m_n}, \quad n \geq 0, \\ \lambda_{n+1} &= \frac{k_n^2}{m_n m_{n-1}}, \quad n \geq 1. \end{aligned} \tag{1.9}$$

Note that here orthogonality is understood that orthogonality with respect to a distribution having an infinite spectrum [4] and the true interval of orthogonality of an orthogonal polynomial family $\{P_n(x)\}$ means, as usual [4], the smallest interval which contains all zeros of all $P_n(x)$.

To find the relation between (1.6) and (1.8) with (1.9), the following procedure is devised

(1) Let $k_0 = 0$ and let m_0 be an arbitrary positive constant.
(2) For $i \geq 1$, let $k_i = \delta_{2i} m_{i-1}$, then $m_i = k_i / k_{\delta_{2i+1}}$. Then (1.8) is surely satisfied. Furthermore,
(3) if $n \geq 1$, then $\dfrac{k_{n-1} + k_n}{m_{n-1}} = \delta_{2n-1} + \delta_{2n} = -c_n$ and $\dfrac{k_n^2}{m_n m_{n-1}} = \lambda_{n+1}$ and hence (1.9) are satisfied by $\{k_n\}$ and $\{m_n\}$.

For the converse, assume that $\{k_n\}$ and $\{m_n\}$ satisfy (1.8) and assume the conditions (1.9) holds for the coefficients c_{n+1} and λ_{n+1} of (1.2). Then for k_0 equal to zero or not, simply define the $\{\delta_n\}$ by $\delta_1 = 0$ and for $i \geq 1$, let $\delta_{2i} = k_i / m_{i-1}$ and $\delta_{2i+1} = k_i / m_i$. Hence, for $n \geq 1$, $\delta_{2n}\delta_{2n+1} = \lambda_{n+1}$ and this leads to the theorem.

Theorem 1.2. [21] *Let the polynomials $p_n(x)$ satisfy a recurrence relation (1.2) in which λ_{n+1} for $n \geq 1$. Then their true interval of orthogonality lies in $(0, \infty)$ if, and only if, there exits a spring-mass system $\{k_n\}$ and $\{m_n\}$, satisfying (1.8), for which (1.9) holds.*

Lemma 1.3. [22] *The continued fraction*

$$l(x) = \cfrac{1}{1 + \cfrac{(1-r_1)r_2 x}{1 + \cfrac{(1-r_2)r_3 x}{1 + \dots}}} \tag{1.10}$$

where $r_n \in [0,1]$, $n = 1, 2, \dots$ *with* $r_i = \alpha_{i+1}$, $i \geq 1$, *converges uniformly for* $x \in [0, \infty)$ *and satisfies*

$$0 \leq l(x) < 1. \tag{1.11}$$

In next two sections, we find the chain sequences of Laguerre and Romanovski-Laguerre finite class of classical orthogonal polynomials (COPs) and introduce the application of chain sequences for Laguerre and Romanovski-Laguerre finite class of COPs which are shown in theorem [2.3] and [3.3] respectively. We also investigate some sequences for both Laguerre and Romanovski-Laguerre finite class of COPs which are satisfying the conditions of the mass-spring system. In last theorem [3.5], the bound of zeros for Romanovski-Laguerre finite class of COPs are also introduced.

2. Laguerre Polynomials

First, we consider well-known generalized Laguerre polynomials related to probability density function $w_1(x) = x^\alpha e^{-x}; \alpha > -1$ of Gamma distribution [23] and their orthogonality interval is infinite $[0, \infty)$ [3, 4, 9, 15], now the three-terms recurrence relations for monic generalized Laguerre polynomials [4, 9], that is (1.2) becomes in this form,

$$\begin{aligned} &P_n(x) = (x - (2n + \alpha - 1))P_{n-1} - (n-1)(n-1+\alpha)P_{n-2}(x), \quad n > 1, \\ &P_0(x) = 1 \quad \text{and} \quad P_1(x) = x - (1 + \alpha). \end{aligned} \tag{2.1}$$

Theorem 2.1. *A sequences* $\alpha_n = \left\{ \frac{n}{(\alpha + 2n + 1)} \left(1 - \frac{n-1}{(\alpha + 2n - 1)} \right) \right\}_{n=1}^{\infty}$ *is called a chain sequence if there exits a sequence* $g_n = \left\{ \frac{n}{(\alpha + 2n + 1)} \right\}_{n=0}^{\infty}$ *such that,* $0 \leq g_0 < 1$, *and* $0 < g_n < 1$, $n \geq 1$ *for* $\alpha > -1$.

Proof. From (1.2) and (2.1), then we get

$$c_n = 2n + \alpha - 1 \quad \text{and} \quad \lambda_n = (n-1)(n-1+\alpha).$$

Now using the definition of chain sequence and (1.4),

$$\alpha_n = \frac{n(n+\alpha)}{(2n+\alpha-1)(2n+\alpha+1)} \quad \text{for} \quad n = 1, 2, \dots. \tag{2.2}$$

we know that $\alpha_1 = g_1$, so we take $n = 1$ then we can get, $\alpha_1 = g_1 = \frac{1}{3+\alpha}$. For $n = 2$, $\alpha_2 = \frac{2(\alpha+2)}{(\alpha+3)(\alpha+3)}$ and $g_2 = \frac{2}{(\alpha+5)}$. By applying induction method, $\alpha_{n-1} = g_{n-1}(1 - g_{n-2})$, we can get $g_n = \frac{n}{(\alpha + 2n + 1)}$ and here $g_0 = 0$.
Now we have to prove $0 < g_n < 1$. It is obvious that,

$$0 < g_n \quad \text{for} \quad n \geq 1 \quad \text{and} \quad \alpha > -1. \tag{2.3}$$

Again we know that $\alpha > -1$, that is,

$$\alpha + 2n + 1 > 2n \Rightarrow \frac{1}{\alpha + 2n + 1} < \frac{1}{2n} \Rightarrow \frac{n}{\alpha + 2n + 1} < \frac{1}{2} \Rightarrow g_n < \frac{1}{2}. \tag{2.4}$$

Hence we can prove from (2.3) and (2.4) that, $0 < g_n < 1$. ∎

Lemma 2.2. *The sequence $\{\delta_n\}$ for the generalized Laguerre polynomials are given by*

$$\delta_{2n} = n + \alpha, \qquad \delta_{2n+1} = n, \quad n = 1, 2, \ldots.$$

Proof. From the three-terms recurrence relation for generalized Laguerre polynomials (2.1), and the procedure given in (1.6), then we can get the result follows from direct computation. ∎

Theorem 2.3. *Let $f(x) \in \mathbb{W}$ with corresponding parameters $\{\alpha_n\}_{n=1}^{\infty}$. Then we have,*

$$A_1(x) \leq f(x) \leq B_1(x) \quad 0 < x < +\infty.$$

$$A_1(x) = \frac{(\alpha+3)(1+x)}{(1+x)\alpha + (3x+2)}, \quad \textit{and} \quad B_1(x) = \frac{(\alpha+3+x)}{(\alpha+3)}$$

$$f(x) = \frac{(\alpha+3)^2(\alpha+5) + 2(\alpha+2)^2 x l(x)}{(\alpha+3)^2(\alpha+5) + (\alpha+3)(\alpha+5)x + 2(\alpha+2)^2 x l(x)}.$$

$l(x)$ is defined in Lemma 1.3.

Proof. Let $\{\alpha_n\}_{n=1}^{\infty}$ be a chain sequence for generalized Laguerre polynomials. which is given by (2.2) and then the inequality (1.11) implies

$$1 + \frac{2(\alpha+2)^2}{(\alpha+3)^2(\alpha+5)} x l(x) \in \left[1, 1 + \frac{(\alpha+2)}{(\alpha+3)x}\right]$$

$$\frac{(\alpha+3)^2(\alpha+5) + 2(\alpha+2)^2 x l(x)}{(\alpha+3)^2(\alpha+5)} \in \left[1, \frac{(1+x)\alpha + (3x+2)}{(\alpha+3)}\right].$$

where we have used the fact that $\dfrac{2(\alpha+2)}{(\alpha+3)(\alpha+5)} \in [0,1]$. one can verify that,

$$1 + \frac{(\alpha+3)(\alpha+5)x}{(\alpha+3)^2(\alpha+5) + 2(\alpha+2)^2 x l(x)} \in \left[1 + \frac{x}{(1+x)\alpha + (3x+2)}, 1 + \frac{x}{(\alpha+3)}\right]$$

$$\frac{(\alpha+3)(\alpha+5)(\alpha+3+x) + 2(\alpha+2)^2 x l(x)}{(\alpha+3)^2(\alpha+5) + 2(\alpha+2)^2 x l(x)} \in \left[\frac{(1+x)(\alpha+3)}{(1+x)\alpha + (3x+2)}, \frac{(\alpha+3)+x}{(\alpha+3)}\right].$$

and further

$$\frac{1}{\dfrac{(\alpha+3)^2(\alpha+5) + 2(\alpha+2)^2 x l(x)}{(\alpha+3)(\alpha+5)(\alpha+3+x) + 2(\alpha+2)^2 x l(x)}} \in \left[\frac{(1+x)(\alpha+3)}{(1+x)\alpha + (3x+2)}, \frac{(\alpha+3)+x}{(\alpha+3)}\right].$$

$$\frac{(\alpha+3)^2(\alpha+5) + 2(\alpha+2)^2 x l(x)}{(\alpha+3)(\alpha+5)(\alpha+3+x) + 2(\alpha+2)^2 x l(x)} \in \left[\frac{(1+x)(\alpha+3)}{(1+x)\alpha + (3x+2)}, \frac{(\alpha+3)+x}{(\alpha+3)}\right].$$

The proof is done. ∎

Theorem 2.4. *The mass m_n and spring constants k_n satisfy the mass-spring system (1.7) and (1.8). where $m_n = \frac{(1+\alpha)_n}{n!}m_0$ and $k_n = \frac{(1+\alpha)_n}{(n-1)!}$ for $m_0 = 1$ is a positive constant.*

Proof. According to Theorem 1.2 and lemma 2.2, then we get,

$$\frac{k_n}{m_{n-1}} = (n+\alpha) \tag{2.5}$$

$$\frac{k_n}{m_n} = n. \tag{2.6}$$

From (2.5) and (2.6),

$$m_n = \frac{(n+\alpha)}{n}m_{n-1}. \tag{2.7}$$

For $n = 0, 1, 2, \ldots$. where $m_{-1} = 1$. we can get m_1, m_2, m_3 in terms of m_0. where m_0 is an arbitrary positive constant. $m_1 = (1+\alpha)m_0$, $m_2 = \frac{(1+\alpha)(2+\alpha)}{2!}m_0$, $m_3 = \frac{(1+\alpha)(2+\alpha)(3+\alpha)}{3!}m_0$. By applying the induction method and take $m_0 = 1$ is positive constant for $\alpha > -1$. Then we can get m_n in the form,

$$m_n = \frac{(1+\alpha)(2+\alpha)\ldots(n+\alpha)}{n!}.$$

In close form,

$$m_n = \frac{(1+\alpha)_n}{n!}. \tag{2.8}$$

From (2.6) and (2.8), we can easily get k_n,

$$k_n = \frac{(1+\alpha)_n}{(n-1)!}. \tag{2.9}$$

Here mass m_n and spring constants k_n are satisfying the mass-spring system (1.7) and (1.8) for $m_0 = 1$ is positive constants. ∎

3. Romanovski-Laguerre type Polynoials

Now we consider the less-known Romanovski-Laguerre type finite class of classical orthogonal polynomials related to probability density function $w_2(x,p) = x^{-p}e^{-\frac{1}{x}}; p > 2n+1$ of inverse Gamma distribution [1, 2, 24, 25]. We further note that, even though we call as finite, due to the fact that this is finitely orthogonal for every $n \in \mathbb{Z}^+$, and their orthogonality interval is quite infinite $[0,\infty)$ and changing the linear variable does not change the main interval of orthogonality [24, 26, 27]. Now the three-terms recurrence relations for Romanovski-Laguerre finite class of classical orthogonal polynomials is given by,

$$N_{n+1}(x) = \left(\frac{(p-2n-2)(p-2n-1)}{(p-n-1)}x \right. \tag{3.1}$$

$$\left. - \frac{p(p-2n-1)}{(p-n-1)(p-2n)}\right)N_n(x) - \frac{n(p-2n-2)}{(p-n-1)(p-2n)}N_{n-1}(x). \tag{3.2}$$

Now we can easily change (3.1) into the form (1.2). That is, the three-terms recurrence relations for monic Romanovski-Laguerre finite class of classical orthogonal polynomials is given as follows.

$$N_n(x) = (x - c_n)N_{n-1}(x) - \lambda_n N_{n-2}(x), \quad n > 1 \quad \text{and}$$
$$N_1(x) = 1 \quad \text{and} \quad N_1(x) = x - c_1. \tag{3.3}$$

Where

$$c_n = \frac{p}{(p-2n)(p-2n+2)} \quad \text{and} \quad \lambda_n = \frac{(n-1)(p-n+1)}{(p-2n+1)(p-2n+2)^2(p-2n+3)}. \tag{3.4}$$

Theorem 3.1. *A sequences* $\alpha_n = \left\{\frac{n(p-2n-2)}{p(p-2n-1)}\left(1 - \frac{(n-1)(p-2n)}{p(p-2n+1)}\right)\right\}_{n=1}^{\infty}$ *is called a chain sequence if there exits a sequence* $g_n = \left\{\frac{n(p-2n-2)}{p(p-2n-1)}\right\}_{n=0}^{\infty}$ *such that,* $0 \le g_0 < 1$*, and* $0 < g_n < 1$*,* $n \ge 1$ *for* $p > 2n+2$*.*

Proof. From (3.3) and (3.4)

$$c_n = \frac{p}{(p-2n)(p-2n+2)} \quad \text{and} \quad \lambda_n = \frac{(n-1)(p-n+1)}{(p-2n+1)(p-2n+2)^2(p-2n+3)}.$$

Now using (1.4) then we get,

$$\alpha_n = \frac{\dfrac{n(p-n)}{(p-2n)^2(p-2n-1)(p-2n+1)}}{\dfrac{p}{(p-2n+2)(p-2n)}\dfrac{p}{(p-2n-2)(p-2n)}}$$

$$\alpha_n = \frac{n(p-n)(p-2n+2)(p-2n-2)}{p^2(p-2n-1)(p-2n+1)} \quad \text{for} \quad n = 1, 2, \ldots. \tag{3.5}$$

we know that $\alpha_1 = g_1$, so we take $n = 1$ then we can get, $\alpha_1 = g_1 = \frac{(p-4)}{p(p-3)}$. For $n = 2$, $\alpha_2 = \frac{2(p-2)^2(p-6)}{p^2(p-3)(p-5)}$. By applying induction method, $\alpha_{n-1} = g_{n-1}(1 - g_{n-2})$, then we can get $g_n = \frac{n(p-2n-1)}{p(p-2n-1)}$ and here $g_0 = 0$.
Now we have to prove $0 < g_n < 1$. we can write g_n in this form,

$$\frac{n(p-2n-1)}{p(p-2n-1)} = \frac{n}{p}\left(1 - \frac{1}{(p-2n-1)}\right)$$

According to our condition $p > 2n+2$, we can easily prove

$$\frac{n}{p}\left(1 - \frac{1}{(p-2n-1)}\right) > 0 \Rightarrow g_n > 0. \tag{3.6}$$

Again use the condition $p > 2n+2$,

$$\frac{n}{p} < \frac{1}{2}. \tag{3.7}$$

Now,

$$\left(\frac{p-2n-2}{p-2n-1}\right) = \left(1+\frac{1}{p-2n-2}\right)^{-1}.$$

we know that $p > 2n+2$, then we can say,

$$\left(1+\frac{1}{p-2n-2}\right)^{-1} < 1 \Rightarrow \left(1+\frac{1}{p-2n-2}\right) > 1. \tag{3.8}$$

After using (3.7), (3.8)

$$g_n = \frac{n(p-2n-1)}{p(p-2n-1)} = \frac{n}{p}\left(1+\frac{1}{p-2n-2}\right)^{-1} < \frac{1}{2}. \tag{3.9}$$

From (3.6) and (3.9), hence $0 < g_n < 1$. ■

Lemma 3.2. *If $p > 2n+2$, then the sequence $\{\delta_n\}$ for Romanovski-Laguerre finite class of classical orthogonal polynomials are given by*

$$\delta_{2n} = \frac{(p-n)}{(p-2n)(p-2n+1)}, \qquad \delta_{2n+1} = \frac{n}{(p-2n-1)(p-2n)}, \quad n = 1, 2, \ldots.$$

Proof. From the three-terms recurrence relation for Romanovski-Laguerre finite class of classical orthogonal polynomials (3.3) and (3.4),

$$c_n = \frac{p}{(p-2n)(p-2n+2)} \quad \text{and} \quad \lambda_n = \frac{(n-1)(p-n+1)}{(p-2n+1)(p-2n+2)^2(p-2n+3)}.$$

We take $n = 1$ and get $c_1 = \delta_2 = \dfrac{1}{(p-2)}$ and after that $\delta_3 = \dfrac{1}{(p-2)(p-3)}$.
For $n = 2$, $c_2 = \delta_4 + \delta_3$,

$$\frac{p}{(p-2)(p-4)} = \delta_4 + \frac{1}{(p-2)(p-3)} \Rightarrow \delta_4 = \frac{(p-2)}{(p-3)(p-4)}.$$

and

$$\delta_4\delta_5 = \frac{2(p-2)}{(p-3)(p-4)^2(p-5)} \Rightarrow \delta_5 = \frac{2}{(p-4)(p-5)}.$$

For $n = 3$ and using the relation $\dfrac{p}{(p-4)(p-6)} = \delta_6 + \delta_5$ and $\dfrac{3(p-3)}{(p-5)(p-6)^2(p-7)} = \delta_6\delta_7$, we can get

$$\delta_6 = \frac{(p-3)}{(p-5)(p-6)} \quad \text{and} \quad \delta_7 = \frac{3}{(p-6)(p-7)}.$$

For $n = 4$,

$$\delta_8 = \frac{(p-4)}{(p-7)(p-8)} \quad \text{and} \quad \delta_9 = \frac{4}{(p-7)(p-8)}.$$

After applying the induction method, finally we get the result follows from direct computation. ■

Theorem 3.3. *Let $f(x) \in \mathbb{W}$ with corresponding parameters $\{\alpha_n\}_{n=1}^{\infty}$. Then we have,*

$$A_1(x) \leq f(x) \leq B_1(x) \quad 0 < x < +\infty.$$

Where

$$A_1(x) = \frac{p(p-3)(1+x)}{p(p-3)+(p-2)^2 x}, \quad \text{and} \quad B_1(x) = \frac{p(p-3)+(p-4)x}{p(p-3)}$$

$$f(x) = \frac{p^3(p-3)^2(p-5)+2(p-2)^2(p-6)xl(x)}{p^2(p-3)(p-5)(p^2-2p-4)+2(p-2)^4(p-6)xl(x)}.$$

$l(x)$ is defined in Lemma 1.3 and $\{\alpha_n\}_{n=1}^{\infty}$ is given by (3.5) for $n = 1, 2, \ldots$.

Proof. We can apply same procedure whatever we used in proof of Theorem 2.3. ∎

Theorem 3.4. *If $p > 2n+1$, the mass m_n and spring constants k_n satisfy the mass-spring system (1.7) and (1.8). where $m_n = \dfrac{(-1)^n(1-p)_n(p-2n-1)}{n!}$ and $k_n = \dfrac{(-1)^n(1-p)_n}{(n-1)!(p-2n)}$.*

Proof. According to Theorem 1.2 and lemma 3.2, then we get,

$$\frac{k_n}{m_{n-1}} = \frac{(p-n)}{(p-2n)(p-2n+1)} \tag{3.10}$$

$$\frac{k_n}{m_n} = \frac{n}{(p-2n-1)(p-2n)}. \tag{3.11}$$

From (3.10) and (3.11),

$$m_n = \frac{(p-n)(p-2n-1)}{n(p-2n+1)} m_{n-1}. \tag{3.12}$$

For $n = 0, 1, 2, \ldots$. where $m_{-1} = 1$. we can get m_1, m_2, m_3 in terms of m_0. $m_1 = (p-3)m_0$, $m_2 = \dfrac{(p-2)(p-5)}{2!} m_0$, $m_3 = \dfrac{(p-2)(p-3)(p-7)}{3!} m_0$. After using the induction method,

$$m_n = \frac{(p-1)(p-2)(p-3)\ldots(p-n)(p-2n-1)}{n!} m_0.$$

We take $m_0 = (p-1)$ is an arbitrary positive constant for $p > 2n+1$. Then we can get m_n in the close form,

$$m_n = \frac{(-1)^n(1-p)_n(p-2n-1)}{n!}. \tag{3.13}$$

From (3.11) and (3.13), we can easily get k_n,

$$k_n = \frac{(-1)^n(1-p)_n}{(n-1)!(p-2n)}. \tag{3.14}$$

Here mass m_n and spring constants k_n are satisfying the mass-spring system (1.7) and (1.8) for $m_0 = (p-1)$ is positive constants. ∎

Theorem 3.5. *Let $\{\alpha_n\}_{n=1}^{N-1}$ be a chain sequence, and set*

$$B = \max\{x_n : 0 < n < N\} \quad \text{and} \quad A = \min\{y_n : 0 < n < N\}, \tag{3.15}$$

where x_n, y_n and $x_n > y_n$ are the roots of the equation,

$$\left(x-\frac{p}{(p-2n)(p-2n-2)}\right)\left(x-\frac{p}{(p-2n)(p-2n+2)}\right)\alpha_n \\ =\frac{n(p-n)}{p(p-2n)^2(p-2n-1)(p-2n-2)}. \tag{3.16}$$

that is,

$$x_n=\frac{p}{(p-2n)^2-4}+\sqrt{\frac{p^2}{(p-2n)^2-4}+\frac{na(p-n)}{(p-2n)^2((p-2n)^2-1)}} \tag{3.17}$$

And

$$y_n=\frac{p}{(p-2n)^2-4}-\sqrt{\frac{p^2}{(p-2n)^2-4}+\frac{na(p-n)}{(p-2n)^2((p-2n)^2-1)}} \tag{3.18}$$

Then the zeros of $N_N(x)$ lie in (A,B).

Proof. We can change (3.1) into an another from,

$$xN_n^p(x)=\frac{(p-n-1)}{(p-2n-2)(p-2n)}N_{n+1}^{(p)}(x)+\frac{p}{(p-2n-2)(p-2n)}N_n^{(p)}(x) \\ +\frac{n}{p(p-2n-1)(p-2n)}N_{n-1}^{(p)}(x). \tag{3.19}$$

From equation (3.19), $\mu_n=\dfrac{(p-n-1)}{(p-2n-2)(p-2n)}$, $\nu_n=\dfrac{p}{(p-2n-2)(p-2n)}$ and $\xi_n=\dfrac{n}{p(p-2n-1)(p-2n)}$. From [[20], Theorem 2], we can get a quadratic equation,

$$(x-\nu_n)(x-\nu_{n-1})\alpha_n=\xi_n\mu_{n-1}. \tag{3.20}$$

Choose $\alpha_n=\dfrac{1}{a}$ is a constant.

$$a=4\cos^2\left(\frac{\pi}{N+1}\right)+\epsilon,\quad \text{for some}\quad \epsilon>0. \tag{3.21}$$

Now (3.20) becomes as follows,

$$\left(x-\frac{p}{(p-2n)(p-2n-2)}\right)\left(x-\frac{p}{(p-2n)(p-2n+2)}\right)\frac{1}{a} \\ =\frac{n(p-n)}{p(p-2n)^2(p-2n-1)(p-2n-2)}.$$

$$x^2-\frac{2p}{(p-2n+2)(p-2n-2)}x \\ +\frac{p^3(p-2n-1)-na(p-n)(p-2n+2)}{p(p-2n)^2(p-2n-1)(p-2n-2)(p-2n+2)}=0. \tag{3.22}$$

Let x_n and y_n are the roots of (3.22), thus

$$x_n = \frac{p}{(p-2n)^2-4} + \sqrt{\frac{p^2}{((p-2n)^2-4)^2} - \left(\frac{p^3(p-2n-1)-na(p-n)(p-2n+2)}{p(p-2n)^2(p-2n-1)((p-2n)^2-4)}\right)} \tag{3.23}$$

$$y_n = \frac{p}{(p-2n)^2-4} - \sqrt{\frac{p^2}{((p-2n)^2-4)^2} - \left(\frac{p^3(p-2n-1)-na(p-n)(p-2n+2)}{p(p-2n)^2(p-2n-1)((p-2n)^2-4)}\right)} \tag{3.24}$$

Is is clear that x_n increase with n, hence

$$B = \sqrt{\frac{p^2}{((p-2N+2)^2-4)^2} - \left(\frac{p^3(p-2N+1)-a(N-1)(p-N+1)(p-2N)}{p(p-2N+2)^2(p-2N+1)((p-2N+2)^2-4)}\right)} + \frac{p}{(p-2N+2)^2-4} \tag{3.25}$$

■

4. Conclusion

In this work, we considered the Laguerre and Romanovski-Laguerre finite class of classical orthogonal polynomials. The chain sequences are obtained for both polynomials and discussed the application of chain sequences in real life problems through mass-spring system. We also investigated the bound of zeros using chain sequences for Romanovski-Laguerre finite class of classical orthogonal polynomials.

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