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Convergence Theorem for Solving Split Equality Fixed Point Problem of Asymtotically Quasi-nonexpansive Semigroups in Hilbert Spaces

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Abstract In this article, we introduce new iteration method for solving a split equality common fixed point problem of asymptotically quasi-nonexpansive semigroups in Hilbert space. The weak and strong convergence theorems of the iteration are proved. Moreover, for applications, we utilize our results to study the split equality variational inequality problem, and the split equality equilibrium problem. the results presented in the article are new and generalize of some recent corresponding results.

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1. Introduction

Let H_1 and H_2 be two real Hilbert spaces, C and Q be nonempty closed convex subsets of H_1 and H_2, respectively. The split feasibility problem (SFP) is finding a point $q \in H_1$ satisfies:

$$q \in C \text{ and } Aq \in Q \tag{1.1}$$

where $A : H_1 \to H_2$ is a bounded linear operator. In 1994, Censor and Elfving [1] introduce the split feasibilty problem in finite dimensional Hilbert spaces for modelling inverse problems in medical image reconstruction.

If C and Q are the sets of fixed points of two nonlinear mappings, the split common fixed point problem (SCFP) for mapping S and T is to find a point $q \in H_1$ satisfies:

$$q \in C := F(S) \text{ and } Aq \in Q := F(T) \tag{1.2}$$

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where $F(S)$ and $F(T)$ are the sets of fixed point of S and T, respectively.

In 2013, Moudafi [2] introduced the following split equality feasibility problem (SEFP). Let H_1, H_2 and H_3 be real Hilbert spaces, C and Q be nonempty closed convex subsets of H_1 and H_2, $A : H_1 \to H_3$ and $B : H_2 \to H_3$ be two bounded linear operators respectively. The split equality feasibility problem (SEFP) is to find

$$x \in C, y \in Q \text{ such that } Ax = By. \tag{1.3}$$

After that he introduced new split equality fixed point problem (SEFPP). Let H_1, H_2 and H_3 be real Hilbert spaces, $S : H_1 \to H_1$ and $T : H_2 \to H_2$ be two nonlinear mappings with $C := F(S)$ and $Q := F(T)$ are nonempty, $A : H_1 \to H_3$ and $B : H_2 \to H_3$ be two bounded linear operators. The split equality fixed point problem for S and T is to find

$$\text{a point } x \in C \text{ and } y \in Q \text{ such that } Ax = By, \tag{1.4}$$

which allows asymmetric and partial relation between the variable x and y. It is used for instance in decomposition method for PDE's, application in game theory and in intensity modulated radiation therapy (IMRT) [3–5]. Recently, many authors study about convergence theorem of the split equality problem and other in [6–11].

For solving (1.4), Moudafi[12] introduced an algorithm:

$$\begin{aligned} x_{n+1} &= U(x_n - \gamma_n A^*(Ax_n - By_n)), \\ y_{n_1} &= T(y_n + \gamma_n B^*(Ax_{n+1} - By_n)) \end{aligned} \tag{1.5}$$

and proved the convergence of the sequence which generated by this algorithm for firmly quasi-nonexpansive operators U and T, where $\gamma_n \in (\epsilon, \min\{\frac{1}{\lambda_A}, \frac{1}{\lambda_B}\})$ is a non-decreasing sequence and λ_A, λ_B are spectral radii of A^*A and B^*B, respectively.

Recently, Shehu et al. [13] proved strong convergence theorem for solving split equality fixed point problem which does not involve the prior knowledge of operator norms by using algorithm: Let $u, x_1 \in H_1$ and $v, y_1 \in H_2$,

$$\begin{aligned} u_n &= x_n - \gamma A^*(Ax_n - By_n) \\ v_n &= y_n + \gamma B^*(Ax_n - By_n) \\ x_{n+1} &= (1 - \alpha_n - \beta_n)u_n + \alpha_n S u_n + \beta_n u \\ y_{n+1} &= (1 - \alpha_n - \beta_n)v_n + \alpha_n T v_n + \beta_n v \end{aligned} \tag{1.6}$$

where S and T are quasi-nonexpansive mappings on H_1 and H_2, respectively. In 2016, Ma and Wang [8] proved the weak and strong convergence theorem for finding a solution of a spilt equality common fixed point of asymptotically nonexpansive semigroup in Banach space by following algorithm:

$$\begin{aligned} z_n &\in J_3(Ax_n - By_n) \\ u_n &= S(t_n)(x_n - \gamma J_1^{-1} A^* z_n) \\ v_n &= T(t_n)(y_n + \gamma J_2^{-1} B^* z_n) \\ x_{n+1} &= \beta_n u_n + (1 - \beta_n)(x_n - \gamma J_1^{-1} A^* z_n) \\ y_{n+1} &= \beta_n v_n + (1 - \beta_n)(y_n + \gamma J_2^{-1} B^* z_n) \end{aligned} \tag{1.7}$$

In 2019, Saelee et al. [14] proved the weak and strong convergence theorem of iterative scheme (1.7) for spilt equality common fixed point of asymptotically quasi-nonexpansive semigroup in Banach space. In this paper, we introduce new iterative method to approximate a solution of the split equality common fixed point problems of asymptotically quasi-nonexpansive semigroups in Hilbert, motivated by [14] and [13] and establish weak convergence theorem and strong converge theorem with suitable condition and some application in split equality variational inequality problem and split equality equilibrium problem.

2. Preliminaries

Throughout of this section, Let H be a real Hilbert spaces.

A mapping $T : C \to C$ is said to be *quasi-nonexpansive* if $F(T) \neq 0$ such that

$$\|Tx - q\| = \|x - q\| \quad \forall (x, q) \in H \times F(T)$$

A mapping $T : C \to C$ is said to be *asymptotically quasi-nonexpansive* if $F(T) \neq 0$ and there exists a nonnegative real sequence $\{l_k\}$ with $l_k \to 0$ such that for each $k \geq 1$,

$$\left\|T^k x - q\right\|^2 = \|x - q\|^2 + l_k \|x - q\|^2 \quad \forall (x, q) \in H \times F(T)$$

Definition 2.1. [15] A one-parameter family $\mathcal{F} = \{T(t) : t \geq 0\}$ of H into itself is called *a Lipschitzian semigroup* on H if it satisfies the following conditions:

(i) $T(0)x = x$ for all $x \in H$,
(ii) $T(s + t) = T(s)T(t)$ for all $s, t \geq 0$,
(iii) for each $x \in H$, the mapping $t \mapsto T(t)x$ is continuous,
(iv) for each $t > 0$, there exists a bounded measurable function $L(t) : [0, \infty) \to [0, \infty)$ such that

$$||T(t)x - T(t)y|| \leq L(t)||x - y|| \text{ for all } x, y \in H.$$

A semigroup is called *quasi-nonexpansive semigroup* if $F(\mathcal{F}) \neq \emptyset$ and it satisfies (i)-(iii) above and $\|T(t)x - q\| \geq \|x - q\| \quad \forall x \in C,\ q \in F(\mathcal{F}),\ t \geq 0$

A semigroup is called *asymptotically quasi-nonexpansive semigroup* if $F(\mathcal{F}) \neq \emptyset$ and there exists a sequence $\{L(t)\}_{t>0}$ with $L(t) \geq 1$, $L(t)$ is nonincreasing in t and $\lim_{t\to\infty} L(t) = 1$ satisfies (i)-(iii) above and $||T^n(t)x - T^n(t)q|| \leq L(t)||x - q||\ \forall x, y \in C, q \in F(\mathcal{F}), n \geq 1$.

If $\mathcal{F}$ satisfies (i)-(iii) and

$$\lim_{t\to\infty} \sup_{x\in D} ||T(t)x - T(s)T(t)x|| = 0 \text{ for all } s > 0 \text{ and bounded } D \subseteq C.$$

then $\mathcal{F}$ is called *uniformly asymptotically regular* on C.

We denote the set of all common fixed point of $\mathcal{F}$ that is,

$$F(\mathcal{F}) := \{x \in H : T(t)x = x, 0 \leq t < \infty\} = \bigcap_{t\geq 0} F(T(t)).$$

Let $T : C \to C$ be a mapping with $F(T) \neq \emptyset$. Then T is said to be *demiclosed at zero* if for any $\{x_n\} \subset C$ with x_n converges weakly to x and $\|x_n - Tx_n\| \to 0, x = Tx$. A mapping $T : C \to C$ is said to be *semi-compact*, if for any sequence $\{x_n\} \in C$ such that $\|x_n - Tx_n\| \to 0$ as $n \to \infty$, there exists subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $\{x_{n_j}\}$ converges strongly to $x^* \in C$. A Banach space E is said to *satisfy Opial's property* if for any sequence $\{x_n\} \in E$, $x_n \rightharpoonup x$, for any $y \in E$ with $y \neq x$, we have

$$\liminf_{n\to\infty} \|x_n - x\| < \liminf_{n\to\infty} \|x_n - y\|$$

Lemma 2.2. *Let H be a real Hilbert space. Then the following identities are obtained:*

$$2\langle x, y\rangle = \|x\|^2 + \|y\|^2 - \|x - y\|^2 = \|x + y\|^2 - \|y\|^2 - \|x\|^2 .$$

Lemma 2.3. [16] *Let $\{a_n\}$, $\{b_n\}$ and $\{\delta_n\}$ be nonnegative real number sequences such that*

$$a_{n+1} \le (1+b_n)a_n + \delta_n, \quad \forall n \ge 1,$$

where $\sum_{n=1}^{\infty} b_n < \infty$ and $\sum_{n=1}^{\infty} \delta_n < \infty$. Then $\lim_{n\to\infty} a_n$ exists.

3. Main Results

Theorem 3.1. *Let H_1, H_2 and H_3 be real Hilbert spaces, $\{S(t) : t \ge 0\}$ and $\{T(t) : t \ge 0\}$ be two uniformly asymptotically regular family of self-mapping quasi-nonexpansive semigroup of H_1 and H_2 with $C := \bigcap_{t\ge 0} F(S(t)) \neq \emptyset$ and $Q := \bigcap_{t\ge 0} F(T(t)) \neq \emptyset$, $A : H_1 \to H_3$ and $B : H_2 \to H_3$ be two bounded linear operators. Assume that $S - I, T - I$ are demiclosed at 0 and S, T are uniformly L-Lipschitzian. For any $x_0, u \in H_1$ and $y_0, v \in H_2$, the sequence $\{(x_n, y_n)\}$ is generated by*

$$\begin{aligned} u_n &= S(t_n)(x_n - \gamma_n A^*(Ax_n - By_n)) \\ v_n &= T(t_n)(y_n + \gamma_n B^*(Ax_n - By_n)) \\ x_{n+1} &= \beta_n u_n + (1-\beta_n-\alpha_n)(x_n - \gamma_n A^*(Ax_n - By_n)) + \alpha_n u \\ y_{n+1} &= \beta_n v_n + (1-\beta_n-\alpha_n)(y_n + \gamma_n B^*(Ax_n - By_n)) + \alpha_n v \end{aligned} \tag{3.1}$$

for all $n \in \mathbb{N} \cup \{0\}$ where $\{t_n\}$ and $\{\gamma_n\}$ are two sequence of positive real numbers, $\{\beta_n\}$ is a sequence in $(0,1)$ and $\{\alpha_n\}$ is a sequence in $(0, 1-a)$ for some $a > 0$ satisfying

(1) $\lim_{n\to\infty} t_n = \infty$,

(2) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n < \infty$,

(3) $L(t) = \max\{L^{(1)}(t), L^{(2)}(t)\}$ *and* $\sum_{n=1}^{\infty}(L(t_n) - 1) < \infty$,

(4) $0 < \liminf_{n\to\infty} \beta_n \le \limsup_{n\to\infty} \beta_n < 1$,

(5) $\beta_n + \alpha_n < 1$,

(6) for small enough $\epsilon > 0$, $\gamma_n \in \left(\epsilon, \frac{2\|Ax_n - By_n\|^2}{\|A^*(Ax_n - By_n)\|^2 + \|B^*(Ax_n - By_n)\|^2} - \epsilon\right)$.

If $\Omega = \{(x^*, y^*) \in H_1 \times H_2 : Ax^* = By^*, x^* \in C, y^* \in Q\} \neq \emptyset$, then

(I) the sequence $\{(x_n, y_n)\}$ converges weakly to a solution $(x^*, y^*) \in \Omega$ of (1.4).

(II) In addition, if there exists at least one t such that $S(t) \in \{S(t) : t \ge 0\}$ and $T(t) \in \{T(t) : t \ge 0\}$ are semi-compact, then the sequence $\{(x_n, y_n)\}$ converges strongly to a solution $(x^*, y^*) \in \Omega$ of (1.4).

Proof. For the proof, we divide into four steps.

Step 1 We first show that for $(x, y) \in \Omega$, $\lim_{n\to\infty} \Gamma_{n+1}(x, y)$ exists

Setting $e_n = x_n - \gamma_n A^*(Ax_n - By_n)$ and $w_n = y_n + \gamma_n B^*(Ax_n - By_n)$. Let $(x, y) \in \Omega$, by convexity of $\|\cdot\|$ we have

$$\begin{aligned}\|x_{n+1}-x\|^2 &= \|(1-\beta_n-\alpha_n)e_n+\beta_n u_n+\alpha_n u-x\|^2\\ &= \|(1-\beta_n-\alpha_n)(e_n-x)+\beta_n(u_n-x)+\alpha_n(u-x)\|^2\\ &\le (1-\beta_n-\alpha_n)\|e_n-x\|^2+\beta_n\|u_n-x\|^2+\alpha_n\|u-x\|^2\\ &\le (1-\beta_n-\alpha_n)\|e_n-x\|^2+\beta_n L^2(t_n)\|e_n-x\|^2+\alpha_n\|u-x\|^2\\ &= (1-\alpha_n-\beta_n(1-L^2(t_n))\|e_n-x\|^2+\alpha_n\|u-x\|^2 \qquad (3.2)\end{aligned}$$

Further, from Lemma 2.2, we have

$$\begin{aligned}\|e_n-x\|^2 &= \|x_n-\gamma_n A^*(Ax_n-By_n)-x\|^2\\ &= \|(x_n-x)-\gamma_n A^*(Ax_n-By_n)\|^2\\ &= \|x_n-x\|^2+\gamma_n^2\|A^*(Ax_n-By_n)\|^2-2\gamma_n\langle x_n-x, A^*(Ax_n-By_n)\rangle\\ &= \|x_n-x\|^2+\gamma_n^2\|A^*(Ax_n-By_n)\|^2-2\gamma_n\langle Ax_n-Ax, Ax_n-By_n\rangle\\ &= \|x_n-x\|^2+\gamma_n^2\|A^*(Ax_n-By_n)\|^2-\gamma_n\|Ax_n-Ax\|^2 \qquad (3.3)\\ &\quad -\gamma_n\|Ax_n-By_n\|^2+\gamma_n\|By_n-Ax\|^2.\end{aligned}$$

Similarly, we have

$$\|y_{n+1}-y\|^2 \le (1-\alpha_n-\beta_n(1-L^2(t_n))\|w_n-y\|^2+\alpha_n\|v-y\|^2 \qquad (3.4)$$

and

$$\begin{aligned}\|w_n-y\|^2 &= \|y_n-y\|^2+\gamma_n^2\|B^*(Ax_n-By_n)\|^2-\gamma_n\|By_n-By\|^2\\ &\quad -\gamma_n\|Ax_n-By_n\|^2+\gamma_n\|Ax_n-By\|^2. \qquad (3.5)\end{aligned}$$

By (3.3), (3.5), assumption of γ_n and $Ax = By$, we have

$$\begin{aligned}\|e_n-x\|^2+\|w_n-y\|^2 &= \|x_n-x\|^2+\|y_n-y\|^2-\gamma_n[2\|Ax_n-By_n\|^2\\ &\quad -\gamma_n(\|A^*(Ax_n-By_n)\|^2+\|B^*(Ax_n-By_n)\|^2]\\ &\le \|x_n-x\|^2+\|y_n-y\|^2. \qquad (3.6)\end{aligned}$$

Adding (3.2), (3.4) and (3.6) we have

$$\begin{aligned}\|x_{n+1}-x\|^2+\|y_{n+1}-y\|^2 &\le (1-\alpha_n-\beta_n(1-L^2(t_n))(\|e_n-x\|^2+\|w_n-y\|^2)\\ &\quad +\alpha_n(\|u-x\|^2+\|v-y\|^2)\\ &\le (1-\alpha_n-\beta_n(1-L^2(t_n))(\|x_n-x\|^2+\|y_n-y\|^2)\\ &\quad +\alpha_n(\|u-x\|^2+\|v-y\|^2). \qquad (3.7)\end{aligned}$$

Let $\Gamma_n(x,y)=\|x_n-x\|^2+\|y_n-y\|^2$ and $\Gamma(x,y)=\|u-x\|^2+\|v-y\|^2$, we can rewrite (3.7) into

$$\begin{aligned}\Gamma_{n+1}(x,y) &\le (1-\alpha_n-\beta_n(1-L^2(t_n)))\Gamma_n(x,y)+\alpha_n\Gamma(x,y)\\ &\le (1+\beta_n(L^2(t_n)-1))\Gamma_n(x,y)+\alpha_n\Gamma(x,y). \qquad (3.8)\end{aligned}$$

By Lemma 2.3 and $\sum_{n=1}^{\infty}(L^2(t_n)-1)<\infty$ and $L(t_n)\to 1$, $\lim_{n\to\infty}\Gamma_n(x,y)$ exists.

Step 2 We prove that $\lim_{n\to\infty}\|Ax_n - By_n\| = 0$, $\lim_{n\to\infty}\|x_n - u_n\| = 0$ and $\lim_{n\to\infty}\|y_n - v_n\| = 0$. Adding (3.2), (3.3), (3.4) and (3.5) and $Ax = By$, we obtain

$$\begin{aligned}\Gamma_{n+1}(x,y) \leq (1-\alpha_n+\beta_n(L^2(t_n)-1))\{\Gamma_n(x,y) - \gamma_n[2\|Ax_n - By_n\|^2 \\ - \gamma_n(\|A^*(Ax_n-By_n)\|^2 + \|B^*(Ax_n-By_n)\|^2)]\} + \alpha_n\Gamma(x,y)\end{aligned} \tag{3.9}$$

From (3.9) we have

$$\begin{aligned}&(1-\alpha_n+\beta_n(L^2(t_n)-1))\gamma_n[2\|Ax_n-By_n\|^2 - \gamma_n(\|A^*(Ax_n-By_n)\|^2 \\ &+\|B^*(Ax_n-By_n)\|^2)] \\ &\leq (1-\alpha_n+\beta_n(L^2(t_n)-1))\Gamma_n(x,y) - \Gamma_{n+1}(x,y) + \alpha_n\Gamma(x,y).\end{aligned} \tag{3.10}$$

It follows from assumption of γ_n and α_n, we get

$$\lim_{n\to\infty}(\|A^*(Ax_n-By_n)\|^2 + \|B^*(Ax_n-By_n)\|^2) = 0. \tag{3.11}$$

So we have

$$\lim_{n\to\infty}\|A^*(Ax_n-By_n)\| = \lim_{n\to\infty}\|B^*(Ax_n-By_n)\| = 0. \tag{3.12}$$

Then we obtain

$$\lim_{n\to\infty}\|x_n - e_n\| = \lim_{n\to\infty}\|\gamma_n A^*(Ax_n-By_n)\| = 0 \quad \text{and} \tag{3.13}$$
$$\lim_{n\to\infty}\|y_n - w_n\| = \lim_{n\to\infty}\|-\gamma_n B^*(Ax_n-By_n)\| = 0 \tag{3.14}$$

Again by Lemma 2.2, we get

$$\begin{aligned}\|x_{n+1}-x\|^2 &= \|(1-\beta_n)(e_n-x)+\beta_n(u_n-x)+\alpha_n(u-e_n)\|^2 \\ &= \|(1-\beta_n)(e_n-x)+\beta_n(u_n-x)\|^2 + \alpha_n^2\|u-e_n\|^2 \\ &\quad + 2\alpha_n\langle u-e_n, (1-\beta_n)(e_n-x)+\beta_n(u_n-x)\rangle \\ &\leq (1-\beta_n)\|e_n-x\|^2 - \beta_n\|u_n-x\|^2 + \alpha_n^2\|u-e_n\|^2 \\ &\quad + 2\alpha_n\langle u-e_n, (1-\beta_n)(e_n-x)+\beta_n(u_n-x)\rangle \\ &= \|e_n-x\|^2 - \beta_n(\|e_n-x\|^2 - \|u_n-x\|^2) + \alpha_n^2\|u-e_n\|^2 \\ &\quad + 2\alpha_n\langle u-e_n, (1-\beta_n)(e_n-x)+\beta_n(u_n-x)\rangle \\ &\leq \|e_n-x\|^2 - \beta_n\|u_n-e_n\|^2 + \alpha_n^2\|u-e_n\|^2 \\ &\quad + 2\alpha_n\langle u-e_n, (1-\beta_n)(e_n-x)+\beta_n(u_n-x)\rangle\end{aligned} \tag{3.15}$$

Similarly,

$$\begin{aligned}\|y_{n+1}-y\|^2 &= \|(1-\beta_n)(w_n-y)+\beta_n(v_n-y)+\alpha_n(v-w_n)\|^2 \\ &\leq \|w_n-y\|^2 - \beta_n\|v_n-w_n\|^2 + \alpha_n^2\|v-w_n\|^2 \\ &\quad + 2\alpha_n\langle v-w_n, (1-\beta_n)(w_n-y)+\beta_n(v_n-y)\rangle\end{aligned} \tag{3.16}$$

From (3.15), (3.16) and (3.6), we get

$$\begin{aligned}\Gamma_{n+1}(x,y) \leq \Gamma_n(x,y) &- \beta_n(\|u_n - e_n\|^2 + \|v_n - w_n\|^2) \\ &+ \alpha_n^2(\|u - e_n\|^2 + \|v - w_n\|^2) \\ &+ 2\alpha_n[\langle u - e_n, (1-\beta_n)(e_n - x) + \beta_n(u_n - x)\rangle \\ &+ \langle v - w_n, (1-\beta_n)(w_n - y) + \beta_n(v_n - y)\rangle] \end{aligned} \tag{3.17}$$

Since $\{e_n\}, \{w_n\}, \{u_n\}$ and $\{v_n\}$ are all bounded, there exists $M > 0$ such that

$$\begin{aligned}&\alpha_n(\|u - e_n\|^2 + \|v - w_n\|^2) + 2\alpha_n[\langle u - e_n, (1-\beta_n)(e_n - x) \\ &\quad + \beta_n(u_n - x)\rangle + \langle v - w_n, (1-\beta_n)(w_n - y) + \beta_n(v_n - y)\rangle] \leq M\end{aligned}$$

Thus, from (3.17), we obtain

$$\beta_n(\|u_n - e_n\|^2 + \|v_n - w_n\|^2) \leq \Gamma_n(x,y) - \Gamma_{n+1}(x,y) - \alpha_n M \to 0.$$

by condition (3), we get

$$\|u_n - e_n\|^2 + \|v_n - w_n\|^2 \to 0.$$

That is $\|u_n - e_n\| \to 0$ and $\|v_n - w_n\| \to 0$.
Hence

$$\begin{aligned}\|x_n - u_n\| &\leq \|x_n - e_n\| + \|e_n - u_n\| \to 0 \\ \|y_n - v_n\| &\leq \|y_n - w_n\| + \|w_n - v_n\| \to 0.\end{aligned}$$

That is $\lim_{n\to\infty} \|x_n - u_n\| = 0$ and $\lim_{n\to\infty} \|y_n - v_n\| = 0$

Step 3 We prove that $\lim_{n\to\infty} \|x_n - S(t)x_n\| = 0$ and $\lim_{n\to\infty} \|y_n - T(t)x_n\| = 0$ for all $t \in [0,\infty)$.

$$\begin{aligned}\|x_{n+1} - x_n\|^2 &= \|(1-\beta_n-\alpha_n)e_n + \beta_n u_n + \alpha_n u - x_n\|^2 \\ &= \|(1-\beta_n-\alpha_n)(e_n - x_n) + \beta_n(u_n - x_n) + \alpha_n(u - x_n)\|^2 \\ &\leq (1-\beta_n-\alpha_n)\|e_n - x_n\|^2 + \beta_n\|u_n - x_n\|^2 + \alpha_n\|u - x_n\|^2\end{aligned}$$

So we obtain $\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0$. Similarly, we get $\lim_{n\to\infty} \|y_{n+1} - y_n\| = 0$. Since $S(t_n)$ and $T(t_n)$ are L-Lipschitzian, we can get

$$\begin{aligned}\|u_n - S(t_n)x_n\|^2 &= \|S(t_n)e_n - S(t_n)x_n\|^2 \\ &\leq L^2(t_n)\|e_n - x_n\|^2 \\ &= L^2(t_n)\gamma_n^2\|A^*(Ax_n - By_n)\|^2\end{aligned}$$

and $\|v_n - T(t_n)y_n\|^2 \leq L^2(t_n)\gamma_n^2\|B^*(Ax_n - By_n)\|^2$. By 3.12, we obtain

$$\lim_{n\to\infty} \|u_n - S(t_n)x_n\| = 0, \quad \lim_{n\to\infty} \|v_n - T(t_n)y_n\| = 0 \tag{3.18}$$

So we can get

$$\lim_{n\to\infty} \|x_n - S(t_n)x_n\| = 0, \quad \lim_{n\to\infty} \|y_n - T(t_n)y_n\| = 0 \tag{3.19}$$

Since $\|x_n - x\|^2 \leq \Gamma_n(x,y)$, $\|y_n - y\|^2 \leq \Gamma_n(x,y)$ and $\lim_{n\to\infty} \Gamma_n(x,y)$ exists, we know that $\{x_n\}$ and $\{y_n\}$ are bounded. Therefore, there exist bounded subsets $C_1 \subseteq H_1$ and

$Q_1 \subseteq H_2$ such that $\{x_n\} \subseteq C_1$ and $\{y_n\} \subseteq Q_1$. Since $\{S(t) : t \geq 0\}$ and $\{T(t) : t \geq 0\}$ are uniformly asymptotically regular and $\lim_{n\to\infty} t_n = \infty$, for all $t \geq 0$,

$$\lim_{n\to\infty} \|S(t)S(t_n)x_n - S(t_n)x_n\| \leq \lim_{n\to\infty} \sup_{x\in C_1} \|S(t)S(t_n)x - S(t_n)x\| = 0,$$

and

$$\lim_{n\to\infty} \|T(t)T(t_n)y_n - T(t_n)y_n\| \leq \lim_{n\to\infty} \sup_{x\in Q_1} \|T(t)T(t_n)y - T(t_n)y\| = 0$$

Since $\{S(t)x\}$ is continuous on t for all $x \in H_1$, $\{T(t)x\}$ is continuous on t for all $x \in H_2$ and

$$\begin{aligned}\|x_n - S(t)x_n\| &\leq \|x_n - S(t_n)x_n\| + \|S(t_n)x_n - S(t)S(t_n)x_n\| \\ &\quad + \|S(t)S(t_n)x_n - S(t)x_n\|, \\ \|y_n - T(t)y_n\| &\leq \|y_n - T(t_n)y_n\| + \|T(t_n)y_n - T(t)T(t_n)y_n\| \\ &\quad + \|T(t)T(t_n)x_n - T(t)x_n\|, \end{aligned} \tag{3.20}$$

we obtain

$$\lim_{n\to\infty} \|x_n - S(t)x_n\| = 0, \quad \lim_{n\to\infty} \|y_n - T(t)y_n\| = 0.$$

Step 4 We prove that (x^*, y^*) is the unique weak cluster point of $\{(x_n, y_n)\}$. Since $\{(x_n, y_n)\} \subseteq C_1 \times Q_1$, we may assume that (x^*, y^*) is a weak cluster point of $\{(x_n, y_n)\}$. By assumption of demiclosedness of $S - I$ and $T - I$, we have $x^* \in C = \bigcap_{t\geq 0} F(S(t)), y^* \in Q = \bigcap_{t\geq 0} F(T(t))$. Since A and B are bounded linear operators, we know that $Ax^* - By^*$ is a weak cluster point of $\{Ax_n - By_n\}$. By the weakly lower semi-continuous property of the norm and (3.12), we have

$$\|Ax^* - By^*\| \leq \liminf_{n\to\infty} \|Ax_n - By_n\| = 0.$$

So $Ax^* = By^*$. This implies $(x^*, y^*) \in \Omega$. Now we prove that (x^*, y^*) is a unique weak cluster point of $\{(x_n, y_n)\}$. Assume that there exists another subsequence $\{(x_{n_k}, y_{n_k})\}$ of $\{(x_n, y_n)\}$ such that $\{(x_{n_k}, y_{n_k})\}$ converges weakly to a point (p, q) with $(p, q) \neq (x^*, y^*)$. Similarly, we get $(p, q) \in \Gamma$. By Opial's property of H_1 ,

$$\begin{aligned}\liminf_{i\to\infty} \|x_{n_i} - p\| &< \liminf_{i\to\infty} \|x_{n_i} - x^*\| \\ &= \lim_{n\to\infty} \|x_n - x^*\| \\ &= \liminf_{k\to\infty} \|x_{n_k} - x^*\| \\ &< \liminf_{k\to\infty} \|x_{n_k} - p\| \\ &= \lim_{n\to\infty} \|x_n - p\| \\ &= \liminf_{i\to\infty} \|x_{n_i} - x^*\| \end{aligned}$$

which is a contradiction. Similarly by Opial's property of H_2, we get

$$\liminf_{i\to\infty} \|y_{n_i} - q\| < \liminf_{i\to\infty} \|y_{n_i} - q\|$$

This implies that $(p, q) = (x^*, y^*)$. Then (I) holds.

Next, assume there exist $S(t) \in \{S(t) : t \geq 0\}$ and $T(t) \in \{T(t) : t \geq 0\}$ are semi-compact. Then $\{(x_n, y_n)\}$ is bounded, $\lim_{n\to\infty} \|x_n - S(t)x_n\| = 0$ and $\lim_{n\to\infty} \|y_n - T(t)y_n\| = 0$ for all

$t \geq 0$. So there exists a subsequence $\{(x_{n_j}, y_{n_j})\}$ of $\{(x_n, y_n)\}$ such that $\{(x_{n_j}, y_{n_j})\}$ converges strongly to (u^*, v^*). Since $\{(x_n, y_n)\}$ converge weakly to (x^*, y^*), we get $(u^*, v^*) = (x^*, y^*)$. ■

Corollary 3.2. *Let H_1, H_2 and H_3 be real Hilbert spaces, $\{S(t) : t \geq 0\}$ and $\{T(t) : t \geq 0\}$ be two self-mapping quasi-nonexpansive semigroup of H_1 and H_2 with $C := \bigcap_{t\geq 0} F(S(t)) \neq \emptyset$ and $Q := \bigcap_{t\geq 0} F(T(t)) \neq \emptyset$, $A : H_1 \to H_3$ and $B : H_2 \to H_3$ be two bounded linear operators. Assume that $S - I, T - I$ are demiclosed at 0 and S, T are uniformly L-Lipschitzian. For any $x_0, u \in H_1$ and $y_0, v \in H_2$, the sequence $\{(x_n, y_n)\}$ is generated by*

$$\begin{aligned} u_n &= S(t_n)(x_n - \gamma_n A^*(Ax_n - By_n)) \\ v_n &= T(t_n)(y_n + \gamma_n B^*(Ax_n - By_n)) \\ x_{n+1} &= \beta_n u_n + (1 - \beta_n - \alpha_n)(x_n - \gamma_n A^*(Ax_n - By_n)) + \alpha_n u \\ y_{n+1} &= \beta_n v_n + (1 - \beta_n - \alpha_n)(y_n + \gamma_n B^*(Ax_n - By_n)) + \alpha_n v \end{aligned} \tag{3.21}$$

for all $n \in \mathbb{N} \cup \{0\}$ where $\{t_n\}$ and $\{\gamma_n\}$ are two sequence of positive real numbers, $\{\beta_n\}$ is a sequence in $(0, 1)$ and $\{\alpha_n\}$ is a sequence in $(0, 1 - a)$ for some $a > 0$ satisfying

(1) $\lim_{n\to\infty} t_n = \infty$,

(2) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$,

(3) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$,

(4) $\beta_n + \alpha_n < 1$,

(5) $\gamma_n \in \left(\epsilon, \frac{2\|Ax_n - By_n\|^2}{\|A^*(Ax_n - By_n)\|^2 + \|B^*(Ax_n - By_n)\|^2} - \epsilon\right)$ *for some* $\epsilon > 0$.

If $\Omega = \{(x^*, y^*) \in H_1 \times H_2 : Ax^* = By^*, x^* \in C, y^* \in Q\} \neq \emptyset$, then

(I) the sequence $\{(x_n, y_n)\}$ converges weakly to a solution $(x^*, y^*) \in \Omega$ of (1.4).

(II) In addition, if there exists at least one t such that $S(t) \in \{S(t) : t \geq 0\}$ and $T(t) \in \{T(t) : t \geq 0\}$ are semi-compact, then the sequence $\{(x_n, y_n)\}$ converges strongly to a solution $(x^*, y^*) \in \Omega$ of (1.4).

In Theorem 3.1, taking $B = I$ and $H_2 = H_3$, we obtain the following result for split common fixed point problem (1.2).

Corollary 3.3. *Let H_1 and H_2 be real Hilbert spaces, $\{S(t) : t \geq 0\}$ and $\{T(t) : t \geq 0\}$ be two self-mapping quasi-nonexpansive semigroup of H_1 and H_2 with $C := \bigcap_{t\geq 0} F(S(t)) \neq \emptyset$ and $Q := \bigcap_{t\geq 0} F(T(t)) \neq \emptyset$ and $A : H_1 \to H_2$ be a bounded linear operators. Assume that $S - I, T - I$ are demiclosed at 0 and S, T are uniformly L-Lipschitzian. For any*

$x_0, u \in H_1$ and $y_0, v \in H_2$, the sequence $\{(x_n, y_n)\}$ is generated by

$$\begin{aligned} u_n &= S(t_n)(x_n - \gamma_n A^*(Ax_n - y_n)) \\ v_n &= T(t_n)(y_n + \gamma_n(Ax_n - y_n)) \\ x_{n+1} &= \beta_n u_n + (1 - \beta_n - \alpha_n)(x_n - \gamma_n A^*(Ax_n - y_n)) + \alpha_n u \\ y_{n+1} &= \beta_n v_n + (1 - \beta_n - \alpha_n)(y_n + \gamma_n(Ax_n - y_n)) + \alpha_n v \end{aligned} \tag{3.22}$$

for all $n \in \mathbb{N} \cup \{0\}$ where $\{t_n\}$ and $\{\gamma_n\}$ are two sequence of positive real numbers, $\{\beta_n\}$ is a sequence in $(0,1)$ and $\{\alpha_n\}$ is a sequence in $(0, 1-a)$ for some $a > 0$ satisfying

(1) $\lim_{n\to\infty} t_n = \infty$,

(2) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$,

(3) $L(t) = \max\{L^{(1)}(t), L^{(2)}(t)\}$ *and* $\sum_{n=1}^{\infty}(L(t_n) - 1) < \infty$,

(4) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$,

(5) $\beta_n + \alpha_n < 1$,

(6) $\gamma_n \in \left(\epsilon, \frac{2\|Ax_n - y_n\|^2}{\|A^*(Ax_n - y_n)\|^2 + 1} - \epsilon\right)$ *for some* $\epsilon > 0$.

If $\Omega = \{q \in C, Aq \in Q\} \neq \emptyset$, then

(I) *the sequence $\{(x_n, y_n)\}$ converges weakly to a solution $(x^*, y^*) \in \Omega$ of (1.2).*

(II) *In addition, if there exists at least one t such that $S(t) \in \{S(t) : t \geq 0\}$ and $T(t) \in \{T(t) : t \geq 0\}$ are semi-compact, then the sequence $\{(x_n, y_n)\}$ converges strongly to a solution $(x^*, y^*) \in \Omega$ of (1.2).*

Corollary 3.4. *Let H_1, H_2 and H_3 be real Hilbert spaces, $A : H_1 \to H_3$ and $B : H_2 \to H_3$ be two bounded linear operators. Let $S : H_1 \to H_1$ and $T : H_2 \to H_2$ be quasi-nonexpansive mappings such that $S - I, T - I$ are demiclosed at 0 and $F(S) \neq \emptyset$ and $F(T) \neq \emptyset$. For any $x_0, u \in H_1$ and $y_0, v \in H_2$, the sequence $\{(x_n, y_n)\}$ is generated by*

$$\begin{aligned} u_n &= S(x_n - \gamma_n A^*(Ax_n - By_n)) \\ v_n &= T(y_n + \gamma_n B^*(Ax_n - By_n)) \\ x_{n+1} &= \beta_n u_n + (1 - \beta_n - \alpha_n)(x_n - \gamma_n A^*(Ax_n - By_n)) + \alpha_n u \\ y_{n+1} &= \beta_n v_n + (1 - \beta_n - \alpha_n)(y_n + \gamma_n B^*(Ax_n - By_n)) + \alpha_n v \end{aligned} \tag{3.23}$$

for all $n \in \mathbb{N} \cup \{0\}$ where $\{t_n\}$ and $\{\gamma_n\}$ are two sequence of positive real numbers, $\{\beta_n\}$ is a sequence in $(0,1)$ and $\{\alpha_n\}$ is a sequence in $(0, 1-a)$ for some $a > 0$ satisfying

(1) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$,

(2) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$,

(3) $\beta_n + \alpha_n < 1$,

(4) $\gamma_n \in \left(\epsilon, \frac{2\|Ax_n - By_n\|^2}{\|A^*(Ax_n - By_n)\|^2 + \|B^*(Ax_n - By_n)\|^2} - \epsilon\right)$ *for some* $\epsilon > 0$.

If $\Omega = \{(x^*, y^*) \in H_1 \times H_2 : Ax^* = By^*, x^* \in C, y^* \in Q\} \neq \emptyset$, then

(I) *the sequence $\{(x_n, y_n)\}$ converges weakly to a solution $(x^*, y^*) \in \Omega$ of (1.4).*

(II) In addition, S and T are semi-compact, then the sequence $\{(x_n, y_n)\}$ converges strongly to a solution $(x^*, y^*) \in \Omega$ of (1.4).

4. Applications

4.1 Application to the split equality variational inequality problem

Assume that C and Q are nonempty and closed convex subsets of H_1 and H_2, respectively. Let $M : C \to E_1$ be a mapping. Variational inequality problem (VIP) in a Hilbert space is formulated as the problem of finding a point x^* with property $x^* \in C$ such that

$$\langle Mx^*, z - x^* \rangle \geq 0, \quad \forall z \in C.$$

We will denote the solution set of VIP by $VI(M, C)$.

A mapping $M : C \to E_1$ is said to be α-strongly monotone if for all $x, y \in C$,

$$\langle Mx - My, x - y \rangle \geq \alpha ||x - y||^2 \text{ for } \alpha > 0.$$

A mapping $M : C \to E_1$ is said to be β-inverse strongly monotone if for all $x, y \in C$

$$\langle Mx - My, x - y \rangle \geq \beta ||Mx - My||^2 \text{ for } \beta > 0.$$

The equilibrium problem (for short, EP) is to find $x^* \in C$ such that

$$F(x^*, y) \geq 0, \quad \forall y \in C.$$

The set of solutions of EP is denoted by $EP(F)$. Given a mapping $T : C \to C$, let $F(x, y) = \langle Tx, y - x \rangle$ for all $x, y \in C$. Then $x^* \in EP(F)$ if and only if $x^* \in C$ is a solution of the variational inequality $\langle Tx, y - x \rangle \geq 0$ for all $y \in C$, that is, x^* is a solution of the variational inequality. Setting $F(x, y) = \langle Mx, y - x \rangle$, it is easy to show that F satisfies the following conditions (A1)–(A4) as M is a β-inverse strongly monotone mapping

(A1) $F(x, x) = 0, \forall x \in C$,
(A2) $F(x, y) + F(y, x) \leq 0, \forall x, y \in C$,
(A3) For all $x, y, z \in C, \lim_{t \downarrow 0} F(tz + (1 - t)x, y) \leq F(x, y)$,
(A4) For each $x \in C$, the function $y \mapsto F(x, y)$ is convex and lower semi-continuous.

Define the resolvent mapping $T_r(x), r > 0$ as

$$T_r(x) = \{z \in C : F(z, y) + \frac{1}{r}\langle y - z, z - x \rangle \geq 0, \ \forall y \in C\}.$$

It is well known that the resolvent mapping $T_r(x)$ is firmly nonexpansive mapping and hence quasi-nonexpansive mapping.

Let $B_1 : C \to H_1$ and $B_2 : Q \to H_2$ be two β-inverse-strongly monotone mappings, where C and Q are nonempty and closed convex subsets of H_1 and H_2, respectively. The split equality variational inequality problem is equivalent to find $x^* \in C, y^* \in Q$ such that

$$\langle B_1(x^*), x - x^* \rangle \geq 0, \quad \forall x \in C,$$

and

$$\langle B_2(y^*), y - y^* \rangle \geq 0, \quad \forall y \in Q,$$

and such that

$$Ax^* = By^*. \tag{4.1}$$

We will denote the solution set of split equality variational inequality problem by , that is,

$$\Omega = \{(x^*, y^*) \in VI(B_1, C) \times VI(B_2, Q) : Ax^* = By^*\}.$$

Setting $F(x, y) = \langle B_1 x, y-x\rangle$, and $G(x, y) = \langle B_2 x, y-x\rangle$, it is easy to show that F and G satisfy the conditions (A1)–(A4) as B_i is a β_i-inverse strongly accretive mapping for $i = 1, 2$. For $r > 0, x \in H_1$ and $u \in H_2$, define mappings $T_r : H_1 \to C$ and $S_r : H_2 \to Q$ as follows:

$$T_r(x) = \{z \in C : F(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \quad \forall y \in C\},$$

and

$$S_r(u) = \{z \in Q : G(z, v) + \frac{1}{r}\langle v - z, z - u\rangle \geq 0, \quad \forall v \in Q\}.$$

We know that $F(T_r) = VI(B_1, C) \neq \emptyset, F(S_r) = VI(B_2, Q) \neq \emptyset$. Thus the split equality variational inequality problem with respect to B_1 and B_2 is equivalent to the following split equality fixed point problem: to find $x^* \in F(T_r), y^* \in F(S_r)$ such that $Ax^* = By^*$. Then it follows from Theorem 3.1 that the following result holds.

Theorem 4.1. *Let H_1, H_2 and H_3 be real Hilbert spaces, C and Q be nonempty closed convex subsets of H_1 and H_2, respectively. Let $B_i(i = 1, 2)$ is a β_i-inverse strongly monotone mappings and $A : H_1 \to H_3$ and $B : H_2 \to H_3$ be two bounded linear operators with adjoint A^* and B^*, respectively. S_r and T_r be resolvent operator of the equilibrium function F and G, respectively. For any $x_0, u \in H_1$ and $y_0, v \in H_2$, the sequence $\{(x_n, y_n)\}$ is generated by*

$$\begin{aligned}
u_n &= T_r(x_n - \gamma_n A^*(Ax_n - By_n)) \\
v_n &= S_r(y_n + \gamma_n B^*(Ax_n - By_n)) \\
x_{n+1} &= \beta_n u_n + (1 - \beta_n - \alpha_n)(x_n - \gamma_n A^*(Ax_n - By_n)) + \alpha_n u \\
y_{n+1} &= \beta_n v_n + (1 - \beta_n - \alpha_n)(y_n + \gamma_n B^*(Ax_n - By_n)) + \alpha_n v
\end{aligned}$$

for all $n \in \mathbb{N} \cup \{0\}$ where $\{t_n\}$ and $\{\gamma_n\}$ are two sequence of positive real numbers and $\{\beta_n\}$ is a sequence in $(0, 1)$ satisfying

(1) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$,

(2) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$,

(3) $\beta_n + \alpha_n < 1$,

(4) $\gamma_n \in \left(\epsilon, \frac{2\|Ax_n - By_n\|^2}{\|A^*(Ax_n - By_n)\|^2 + \|B^*(Ax_n - By_n)\|^2} - \epsilon\right)$ *for some* $\epsilon > 0$.

If $\Omega = \{(x^*, y^*) \in VI(B_1, C)\times VI(B_2, Q) : Ax^* = By^*\} \neq \emptyset$, then the sequence $\{(x_n, y_n)\}$ converges strongly to a solution $(x^*, y^*) \in \Omega$ of (4.1).

4.2 Application to the split equality equilibrium problem

Let H_1, H_2 and H_3 be real Hilbert spaces and C, Q be nonempty closed and convex subset of H_1 and H_2 respectively. Suppose that $A : H_1 \to H_3$ and $B : H_2 \to H_3$ are bounded linear operators. Let $F : C \times C \to \mathbb{R}$ and $G : Q \times Q \to \mathbb{R}$ be bi-functions. A split equality equilibrium problem is to find a point $x^* \in C, y^* \in Q$ such that

$$F(x^*, z) \geq 0, G(y^*, u) \geq 0, \forall z \in C, \forall u \in Q \text{ and } Ax^* = By^* \tag{4.2}$$

We assume the following condition of F and G:

(A1) $F(x,x) = 0, \forall x \in C$ and $G(x,x) = 0, \forall x \in Q$,

(A2) $F(x,y) + F(y,x) \leq 0, \forall x,y \in C$ and $G(x,y) + G(y,x) \leq 0, \forall x,y \in Q$,

(A3) For all $x,y,z \in C, \lim_{t\downarrow 0} F(tz+(1-t)x,y) \leq F(x,y)$ and for all $x,y,z \in Q, \lim_{t\downarrow 0} G(tz+(1-t)x,y) \leq G(x,y)$,

(A4) For each $x \in C$, the function $y \mapsto F(x,y)$ is convex and lower semi-continuous and For each $x \in Q$, the function $y \mapsto F(x,y)$ is convex and lower semi-continuous.

Define the resolvent mappings $T_r^F, r > 0$ and $T_s^G, s > 0$ as

$$T_r^F(x) = \{z \in C : F(z,y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \quad \forall y \in C\},$$

and

$$T_s^G(u) = \{z \in Q : G(z,v) + \frac{1}{s}\langle v - z, z - u\rangle \geq 0, \quad \forall v \in Q\}.$$

It is well known that the resolvent mapping T_r^F and T_s^G are firmly nonexpansive mappings and hence quasi-nonexpansive mappings. Moreover it is known that if x^* and y^* solves problem (4.2), then $x^* \in F(T_r^F)$ and $y^* \in F(T_s^G)$.

Theorem 4.2. *Let H_1, H_2 and H_3 be real Hilbert spaces, C and Q be nonempty closed convex subsets of H_1 and H_2, respectively. Let $A : H_1 \to H_3$ and $B : H_2 \to H_3$ be two bounded linear operators with adjoint A^* and B^*, respectively. T_r^F and T_s^G be resolvent operator of the equilibrium function F and G, respectively. For any $x_0, u \in H_1$ and $y_0, v \in H_2$, the sequence $\{(x_n, y_n)\}$ is generated by*

$$\begin{aligned}
u_n &= T_r^F(x_n - \gamma_n A^*(Ax_n - By_n)) \\
v_n &= T_s^G(y_n + \gamma_n B^*(Ax_n - By_n)) \\
x_{n+1} &= \beta_n u_n + (1 - \beta_n - \alpha_n)(x_n - \gamma_n A^*(Ax_n - By_n)) + \alpha_n u \\
y_{n+1} &= \beta_n v_n + (1 - \beta_n - \alpha_n)(y_n + \gamma_n B^*(Ax_n - By_n)) + \alpha_n v
\end{aligned}$$

for all $n \in \mathbb{N} \cup \{0\}$ where $\{t_n\}$ and $\{\gamma_n\}$ are two sequence of positive real numbers and $\{\beta_n\}$ is a sequence in $(0,1)$ satisfying

(1) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$,

(2) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$,

(3) $\beta_n + \alpha_n < 1$,

(4) $\gamma_n \in \left(\epsilon, \frac{2\|Ax_n - By_n\|^2}{\|A^*(Ax_n - By_n)\|^2 + \|B^*(Ax_n - By_n)\|^2} - \epsilon\right)$ *for some* $\epsilon > 0$.

If $\Omega = \{(x^*, y^*) \in C \times Q : F(x^*, z) \geq 0, G(y^*, u) \geq 0, \forall z \in C, \forall u \in Q$ and $Ax^* = By^*\} \neq \emptyset$, then the sequence $\{(x_n, y_n)\}$ converges strongly to a solution $(x^*, y^*) \in \Omega$ of (4.2).

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