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Some Results for Best Proximity Pair on Banach Lattices

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Abstract As we know, most of the metric spaces we face in analysis having natural order that with this order become a lattice. In theorems related to the best proximity pair on metric spaces, it doesn't use this useful tools. In this article we have used this natural order, and we proved the theorems on existence and uniqueness of the best proximity pair on Banach lattices.

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1. Introduction

The best proximity pair problem is one of the important topics that has been written a lot of articles concerning it in recent years. It is very extensive field which has many applications in mathematics and some other sciences. If A and B are two nonemplty subsets in metric space X so,

$$d(A,B) = dist(A,B) = \inf\{d(x,y) : x \in A,\ y \in B\} > 0$$

then the mapping $T : A \to B$ having no fixed point, at that time, we are following points like $x_0 \in A$ which $d(x_0, Tx_0) = d(A,B)$.

The pair (x_0, Tx_0) is called a best proximity pair of T. Also $x_0 \in A$ is said to be a best proximity point for T. The set of the best proximity points is shown by $P_T(A,B)$, i.e.,

$$P_T(A,B) = \{x \in A : d(x,Tx) = dist(A,B)\}.$$

A sequence $\{x_n\} \subseteq A$ is a T-minimizing sequence of A if $\lim_{n\to\infty} d(x_n, Tx_n) = d(A,B)$.

In some essays the best proximity pair theorem has been discussed in metric spaces, and the problem has been solved considering the specific states of T and also the conditions on A and B. (Like compactness or convexity of A). For instance, Eldred and veeramani in [1] introduced cyclic contraction maps and discussed the best proximity problem for

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cyclic contraction map on uniformly convex Banach spaces. In [2] this problem examined for relatively nonexpansive maps. Also proximinal pointwise contraction maps defined by Anuradha and Veeramani in [3], they proved existence of best proximity point on a pair of weakly compact convex subsets of a Banach space. New results can be find in [4], [5], [6] and [7]. Kurc in [8] introduced dominated the best approximation problem in Banach lattices and in [9] he and Hudzik extended the results in [8]. In other words they conected the best approximation problem to the monotonicity problem. Some Authors in [10], [11] and [12] modify some results in nearest and farthest point and monotonity problems.

In this article we discuss the best proximity pair for general mapping $T : A \to B$. Also by attention to order's properties in Banach lattice, we consider A be a downwards directed set.

One of an important conditions on A is convexity. However, convexity is sometimes a very restrictive assumption, so there is a clear need to study the best proximity pair by not necessarily convex sets. Downwards directed sets are nonconvex sets which play an important role in some parts of mathematical.

2. Preliminaries

In this section we first introduce definitions and recall some basic results. If X is a partially ordered vector space, then X is called a vector lattice space (or a Riesz space) if $x \vee y := \sup\{x, y\}$ and $x \wedge y := \inf\{x, y\}$ both exist in X $(\forall x, y \in X)$. For any vector x in vector lattice space X define $x^+ := x \vee 0$, $x^- := x \wedge 0$ and $|x| := x \vee (-x)$. The set $X^+ = \{x \in X : x \geq 0\}$ is called a positive cone of X, and its members are called the positive elements of X. Recall that a norm $\|\cdot\|$ on a vector lattice space is said to be a lattice norm whenever $|x| \leq |y|$ implies $\|x\| \leq \|y\|$. A vector lattice space equipped with a lattice norm is known as a normed vector lattice space (or a normed Riesz space). If normed vector lattice space is also norm complete (i.e., if every norm Cauchhy sequence has a norm limit), then it is referred to as a Banach lattice. More details about Banach lattices could be find in [13],[14] and [15]. Let $(X, \leq)$ be a Banach lattice with a lattice norm $\|\cdot\|$. The norm $\|\cdot\|$ is said to be strictly monotone $(X \in STM)$ if for all $x, y \in X^+$, the conditions $x \geq y$, $y \neq 0$ and $\|x\| = \|y\|$ implies $x = y$. We say that the norm is uniformly monotone $(X \in UM)$ if for any $y_n \geq x_n \geq 0$, $\lim_{n\to\infty} \|x_n\| = \lim_{n\to\infty} \|y_n\|$ implies $\|y_n - x_n\| \to 0$. As an example, L_p-spaces with $1 \leq p < \infty$ are UM spaces, but the space L_∞ is not even an STM space.

Definition 2.1. A nonempty subset W of a Banach lattice X is said to be downwards directed set if for any two elements u and v in W, there exists an element $w \in W$ such that $w \leq u \wedge v$.

For example, let $f : X \to \mathbb{R}$ be an increasing function, then $W = \{x \in X : f(x) \leq c\}$ is a downwards directed set for all $c \in \mathbb{R}$.

Following if A and B are two nonempty subsets in Banach lattice X, the symbol $A \geq B$ means $a \geq b$ for any $a \in A$ and $b \in B$.

Definition 2.2. Let A and B be nonempty subsets of a Banach lattice X. In this case it is said that pair (A, B) having STM property if it is applied to the following property:

If $x, x' \in A$ and $y \in B$ such that $0 \leq x \leq x'$ and $\|x - y\| = \|x' - y\| = dist(A, B)$, then $x = x'$.

Remark 2.3. It is obvious that in any STM space, if $A \geq B$ (or $B \geq A$), then pair (A, B) having STM property.

Definition 2.4. Let A and B be nonempty subsets of a Banach lattice X. In this case it is said that pair (A, B) having UM property if it is applied to the following property:

If $\{x_n\}$ and $\{y_n\}$ are sequences in A and $\{y_n\}$ is a sequence in B such that $0 \leq x_n \leq x'_n$ and $\lim_{n\to\infty} \|x_n - y_n\| = \lim_{n\to\infty} \|x'_n - y_n\| = d(A, B)$, then $\lim_{n\to\infty} \|x_n - x'_n\| \to 0$.

Remark 2.5. It is clear that in any UM space, if $A \geq B$ (or $B \geq A$), then pair (A, B) having UM property.

3. Main results

In this section we pay to examine the conditions for proving the existence and uniqueness of the best proximity pair problem in Banach lattice, similar to the main result in [1], (Theorem 3.10) which it has been proved in uniformly convex Banach spaces.

Theorem 3.1. *Let X be a Banach lattice and pair (A, B) having STM property with $A \geq B$. If A is a downwards directed set then $card(P_T(A, B)) \leq 1$.*

Proof. Suppose that there exist $x, y \in A$ such that $\|x - Tx\| = \|y - Ty\| = d(A, B) = d$. Since A is a downwards directed set, there exists $w \in A$ auch that $w \leq x \wedge y$. Thus $0 \leq w - Tx \leq (x - Tx) \wedge (y - Tx)$ and we have $d \leq \|w - Tx\| \leq \|x - Tx\| = d$.

Attention to pari (A, B) has STM property, we get $w = x$ and by similarity it conclude that $w = y$ as a result $card(P_T(A, B)) \leq 1$. ∎

Theorem 3.2. *Let X be a Banach lattice and pair (A, B) having UM property with $A \geq B$. If A is a downwards directed set then any T-minimizing sequence of A is a Cauchy sequence.*

Proof. Let $\{u_n\} \subseteq A$ be a T-minimizing sequence of A, i.e., $\lim_{n\to\infty} \|u_n - Tu_n\| = d(A, B) = d$. We show that $\{u_n\}$ is a Cauchy sequence. Otherwise, there are subsequences $\{u_{n_k}\}$ and $\{u_{m_k}\}$ of $\{u_n\}$ and $\varepsilon > 0$ such that $\|u_{n_k} - u_{m_k}\| \geq \varepsilon$. By considering A is a downwards directed set, there exists $v_k \in A$ such that $v_k \leq u_{n_k} \wedge u_{m_k}$. Therefore

$$0 \leq v_k - Tu_{n_k} \leq (u_{n_k} - Tu_{n_k}) \wedge (u_{m_k} - Tu_{n_k})$$

and we have $d \leq \|v_k - Tu_{n_k}\| \leq \|u_{n_k} - Tu_{n_k}\| \to d$ as $n \to \infty$.

Attention to pair (A, B) has UM property we obtain, $\|v_k - u_{n_k}\| \to 0$ as $k \to \infty$. Likewise, we get $\|v_k - u_{m_k}\| \to 0$ as $k \to \infty$. Hence $\|u_{n_k} - u_{m_k}\| \to 0$, a contradiction. ∎

Definition 3.3 ([1])**.** Let A and B be nonempty subsets of a metric space X. A map $T : A \cup B \to A \cup B$ is a cyclic contraction map if it satisfies:

(1) $T(A) \subseteq B$ and $T(B) \subseteq A$
(2) For some $k \in (0, 1)$ we have $d(Tx, Ty) \leq kd(x, y) + (1 - k)dist(A, B)$

Note that (2) implies that T satisfies $d(Tx, Ty) \leq d(x, y)$, for all $x \in A$, $y \in B$.

Proposition 3.4 ([1])**.** *Let A and B be nonempty subsets of a metric space X. Suppose $T : A \cup B \to A \cup B$ is a cyclic contraction map. Then starting with any x_0 in $A \cup B$ we have $d(x_n, Tx_n) \to dist(A, B)$, where $x_{n+1} = Tx_n$, $n = 0, 1, 2, \ldots$.*

Theorem 3.5. *Let X be a Banach lattice and pair (A, B) having UM property with $A \geq B$. If A is a closed downwards directed set and $T : A \cup B \to A \cup B$ is a cyclic contraction map, then $card(P_T(A, B)) = 1$.*

Proof. If pair (A, B) has UM property then pair (A, B) has STM property, so based on Theorem 3.1, $cardP_T(A, B) \leq 1$.

Suppose $x_0 \in A$ and define $x_{n+1} = Tx_n$, $n = 0, 1, 2, \ldots$, in this case based on Proposition 3.4, $\{x_{2n}\}$ is a T-minimizing sequence in A. So based on Theorem 3.2, $\{x_{2n}\}$ is convergence. Thus there exists $x \in A$ such that $\lim_{n\to\infty} x_{2n} = x$. Since

$$d(A, B) \leq \|x - Tx_{2n}\| \leq \|x - x_{2n}\| + \|x_{2n} - Tx_{2n}\| \to d(A, B)$$

Therefore $\|x - Tx_{2n}\| \to d(A, B)$.

Also,

$$\|x_{2n+2} - Tx\| = \|Tx_{2n+1} - Tx\| \leq \|x_{2n+1} - x\| = \|Tx_{2n} - x\| \to d(A, B).$$

i.e., $\|x_{2n+2} - Tx\| \to d(A, B)$.

Finally $d(A, B) \leq \|x - Tx\| \leq \|x - x_{2n}\| + \|x_{2n} - Tx\| \to d(A, B)$ as $n \to \infty$, so $\|x - Tx\| = d(A, B)$. ∎

Example 3.6. Assume that A and B be points of X, and $X = M_n(\mathbb{R})$ be the vector space of $n \times n$ real matrices such that $A \leq B$ if and only if $a_{ij} \leq b_{ij}$ for $i, j \in \{1, 2, \ldots, n\}$. If $\|A\| = \sum_{i,j=1}^{n} |a_{ij}|$ then $(X, \|\cdot\|)$ is a Banach lattice. Let $\mathcal{A} = \left\{(a_{ij})_{i,j=1}^{n} : 2 \leq a_{ij} \leq 3\right\}$ and $\mathcal{B} = \left\{(b_{ij})_{i,j=1}^{n} : -1 \leq b_{ij} \leq 0\right\}$ and $T : \mathcal{A} \cup \mathcal{B} \to \mathcal{A} \cup \mathcal{B}$ defined by $T((a_{ij})_{ij=1}^{n}) = (2 - a_{ij})_{ij=1}^{n}$, thus T is a cyclic contraction map and pair $(\mathcal{A}, \mathcal{B})$ has UM property. By Theorem 3.5, $card(P_T(\mathcal{A}, \mathcal{B})) = 1$. In fact if $x_0 = (2)_{i,j=1}^{n}$, then $\|x_0 - Tx_0\| = d(\mathcal{A}, \mathcal{B}) = 2n^2$.

It's necessary to review $A_0 = \{a \in A : d(a, y) = d(A, B)$ for some $y \in B\}$ and $B_0 = \{b \in B : d(x, b) = d(A, B)$ for some $x \in A\}$. It is obvious that $A_0 \neq \emptyset$ if and only if $B_0 \neq \emptyset$.

Theorem 3.7. *Let X be a Banach lattice and pair (A, B) having UM property with $A \geq B$ and $T : A \to B$ be a continuous map. If A is a closed downwards directed set and $A_0 \neq \emptyset$ such that $T(A_0) \subseteq B_0$, then $card(P_T(A, B)) = 1$.*

Proof. We start with any $x_0 \in A_0$. Since $T(A_0) \subseteq B_0$ so there exists $x_1 \in A_0$ such that $\|x_1 - Tx_0\| = d(A, B)$. Also $T(x_1)$ is in B_0, it follows that there exists $x_2 \in A_0$ such that $\|x_2 - T(x_1)\| = d(A, B)$. Inductively, we get $\|x_{n+1} - T(x_n)\| = d(A, B)$. By a similar argument as proving Theorem 3.2, we can prove that $\{x_n\} \subseteq A$ is a convergence sequence. Let $x_n \to x \in A$. By continuity of T, we have $Tx_n \to Tx$ and as a result $\|x - Tx\| = d(A, B)$. On the other hand, pair (A, B) has UM property then pair(A, B) has STM property, so by Theorem 3.1, card$(P_T(A, B)) \leq 1$. Thus card$(P_T(A, B)) = 1$. ∎

Definition 3.8 ([3])**.** Let A and B be nonempty subsets of a normed linear space X. $T : A \cup B \to A \cup B$ is cyclic if $T(A) \subseteq B$ and $T(B) \subseteq A$. Also T is relatively nonexpansive if $\|Tx - Ty\| \leq \|x - y\|$, for any $x \in A$ and $y \in B$.

Also $T : A \cup B \to A \cup B$ is called a cyclic relatively nonexpansive map if T is cyclic and relatively nonexpansive.

Corollary 3.9. *Let X be Banach lattice and pair(A, B) having UM property with $A \geq B$. Let $T : A \cup B \to A \cup B$ be a cyclic relatively nonexpansive and continuous map. If A is a closed downwards directed set and $A_0 \neq \emptyset$, then $card(P_T(A, B)) = 1$.*

Proof. Let $x_0 \in A_0$, so there exists $y_0 \in B_0$ such that $\|x_0 - y_0\| = d(A, B)$. Therefore $d(A, B) \leq \|Tx_0 - Ty_0\| \leq \|x_0 - y_0\| = d(A, B)$. It means that $T(A_0) \subseteq B_0$. By a similar argument as proving Theorem 3.7, the proof is complete. ∎

Corollary 3.10. *Let X be Banach lattice and pair(A, B) having UM property with $A \geq B$ and $A \cap B \neq \emptyset$. Let $T : A \cup B \to A \cup B$ be a cyclic relatively nonexpansive map. If A is a closed downwards directed set and $A_0 \neq \emptyset$, then T has a unique fixed point.*

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