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The Extended Multi-Index Mittag-Leffler Functions and Their Properties Connected with Fractional Calculus and Integral Transforms

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Abstract The aim of this paper is to present the extended multi-index Mittag-Leffler type functions while using the extended Beta function and investigate several properties including integral representation, Derivatives, Beta transform, Mellin transform, Relationships between this function with the Leguerre polynomials and Whittakar functions. Further, several properties of the Riemann-Liouville fractional derivative and integral operators related to extended multi-index Mittag-Leffler functions are also investigated. Finally, various interesting special cases of these functions are also pointed out.

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1. Introduction and preliminaries

Fractional calculus generalizes the integration and differentiation of integer order to arbitrary order is being studied for past 300 years. The growing interest of researchers in this field has led to solve the real-world issues in type of fractional differential equations due to their non-local behavior and these equations are well suited to describe various phenomenon in the field of physics and engineering. Also fractional derivatives

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are capable to model various processes mathematically which exhibit the memory and hereditary properties. A number of researchers [1–21] have also investigated the structure, implementations and various directions of extensions of the fractional integration and differentiation in detail.

In 1903, Mittag-Leffler (M-L) [22] defined a function in term of a power series:

$$E_{\xi}\left(z\right)=\sum_{n=0}^{\infty}\frac{z^{n}}{\Gamma\left(\xi n+1\right)}\quad\left(\xi>0,z\in\mathbb{C}\right). \tag{1.1}$$

Wiman [23] has also given a two-index generalization of this function as:

$$E_{\xi,\tau}\left(z\right)=\sum_{n=0}^{\infty}\frac{z^{n}}{\Gamma\left(\xi n+\tau\right)}\quad\left(\xi>0,\tau>0,z\in\mathbb{C}\right). \tag{1.2}$$

In particular, the functions (1.1) and (1.2) are entire functions of order $\rho=1/\xi$ and type $\sigma=1$, (see, for example ([24], P. 118)).
Prabhakar [25] presents the generalizing series representation (1.2) as:

$$E_{\xi,\tau}^{\varsigma}\left(z\right)=\sum_{n=0}^{\infty}\frac{\left(\varsigma\right)_{n}}{\Gamma\left(\xi n+\tau\right)}\frac{z^{n}}{n!}\quad\left(\xi,\tau,\varsigma\in\mathbb{C},\Re(\xi)>0,\Re(\tau)>0\right). \tag{1.3}$$

This is an entire function of order $\left[\Re(\tau)\right]^{-1}$ ([25], p. 7) and $\left(\varsigma\right)_{n}$ denotes the well-known Pochhammer's symbol that is defined (see, ([26], p.2 and p.5)) as:

$$\left(\varsigma\right)_{n}=\frac{\Gamma\left(\varsigma+n\right)}{\Gamma\left(\varsigma\right)}=\begin{cases}1 & n=0;\ \varsigma\in\mathbb{C}\backslash\left\{0\right\}\\ \varsigma\left(\varsigma+1\right)...\left(\varsigma+\left(n-1\right)\right), & \varsigma=n\in\mathbb{N};\ \varsigma\in\mathbb{C}.\end{cases}$$

By means of the generalizing (1.3) series representation Kilbas et al. [27] defined an extension as:

$$E_{\varsigma}\left[\left(\xi,\tau\right)_{m};z\right]=E_{\varsigma}\left[\left(\xi_{1},\tau_{1}\right),...,\left(\xi_{m},\tau_{m}\right);z\right]=\sum_{n=0}^{\infty}\frac{\left(\varsigma\right)_{n}}{\prod_{j=1}^{m}\left(\xi_{j}n+\tau_{j}\right)}\frac{z^{n}}{n!}. \tag{1.4}$$

If $\tau_j\in\mathbb{R}$ ($\tau_j\neq0$), $\xi_j\in\mathbb{C}$ ($j=1,...,m$) and $z=\mathbb{C}$, then it is proved by Kilbas et al. [27] that:

(1) If $\sum_{j=1}^{m}\tau_j>0$, then the above extended Wright function is an entire function of z.
(2) If $\sum_{j=1}^{m}\tau_j>0$ and either $|z|<\sum_{j=1}^{m}|\tau_j|^{\tau_j}$ or $z=\sum_{j=1}^{m}|\tau_j|^{\tau_j}$, $\sum_{j=1}^{m}\Re(\xi_j)>\Re(\varsigma)+m/2$, then the series in $\sum_{n=0}^{\infty}\frac{(\varsigma)_n}{\prod_{j=1}^{m}(\xi_j n+\tau_j)}\frac{z^n}{n!}$ is absolutely convergent.

In addition, Formula (1.4) gives the Mellin-Barnes integral formulation for the extensive M-L function $E_{\varsigma}\left[\left(\xi,\tau\right)_{m};z\right]$. A full H-function account is available in the Mathai and Saxena [28] Kilbas and Saigo [29] and monographs.

Further, generalized multi-index M-L function are well-defined and studied by Saxena and Nishimoto [30] in the following manner:

$$E_{\varsigma,q}\left[\left(\xi,\tau\right)_{m};z\right]=E_{\left(\xi_{j},\tau_{j}\right)_{m}}^{\varsigma,q}\left[z\right]=\sum_{n=0}^{\infty}\frac{\left(\varsigma\right)_{qn}}{\prod_{j=1}^{m}\Gamma\left(\xi_{j}n+\tau_{j}\right)}\frac{z^{n}}{n!}, \tag{1.5}$$

$$\left(\xi_j, \tau_j, \varsigma, q, z \in \mathbb{C}, \Re(\tau_j) > 0\,(j = 1, 2, ..., m)\,; \Re(\sum_{j=1}^{m} \xi_j) > \max\{0, \Re(q) - 1\}\right).$$

Readers may refer to the recent research work [31–40] and the references cited therein for a detailed account of the various properties, extended version and implementations of this function.

Motivated by the established potential for application of these M-L function, we extend the generalized multi-index M-L function Eq. (1.5) by means of the extended beta function $B_p(x, y)$ and investigate other physical properties, including integral representation and differentiation laws, Mellin Transform and Beta Transform with their various special cases, Relations between the proposed function with Laguerre polynomials and Whittaker functions. Furthermore, certain relationships between the proposed function and the fractional derivatives and integrals of Riemann-Liouville (R-L) are investigated. Several specific cases of our main outcomes are also considered.

2. A Class of Extended multi-index M-L type function

For the present investigation, we extent the generalized multi-index M-L function $E^{\varsigma,q}_{(\xi_j,\tau_j)_m}[z]$ in the following way:

$$E_{(\varsigma,c);q}\left[(\xi,\tau)_m\,;(z;p)\right] = E^{(\varsigma,c);q}_{(\xi_j,\tau_j)_m}(z;p)$$

$$= \sum_{n=0}^{\infty} \frac{\mathcal{B}_p(\varsigma + nq, c - \varsigma)}{B(\varsigma, c - \varsigma)} \frac{(c)_{qn}}{\prod_{j=1}^{m} \Gamma(\xi_j n + \tau_j)} \frac{z^n}{n!}, \tag{2.1}$$

where $\xi_j, \tau_j, \varsigma, q, z \in \mathbb{C}, p \geq 0, \Re(c) > \Re(\varsigma) > 0, \Re(\tau_j) > 0\,(j = 1, 2, ..., m)\,; \Re\left(\sum_{j=1}^{m} \xi_j\right) > \max\{0, \Re(q) - 1\}$, which will be called as extended multi-index M-L type functions (EMMLF). The $\mathcal{B}p(x, y)$ is an extended beta function shown as follows in [41, 42]:

$$\mathcal{B}_p(x, y) = \int_0^1 t^{x-1}(1-t)^{y-1} e^{\frac{-p}{t(1-t)}} dt \quad (\Re(x) > 0, \Re(y) > 0, \Re(p) > 0)\,. \tag{2.2}$$

If $p = 0$, the function $\mathcal{B}_p(x, y)$ reduces into the classical beta function. Specific special cases of this function are listed below as:

(1) When $p = 0$, the EMMLF reduces into the one that has been considered by Saxena and Nishimo defined in Eq. (1.5) (see [30]).

(2) If we set $m = 1$ and $p = 0$ with $\Re(\tau) > 0$; $\Re(\xi) > \max\{0, \Re(q) - 1\}$, the EMMLF reduces in to generalized M-L function which has been considered by Srivastava and Tomovski ([43], p. 200, Eq. (1.13))

$$E_{\varsigma,q}\left[(\xi,\tau)\,;z\right] = E^{\varsigma,q}_{(\xi,\tau)}[z] = \sum_{n=0}^{\infty} \frac{(\varsigma)_{qn}}{\Gamma(\xi n + \tau)} \frac{z^n}{n!}, \tag{2.3}$$

when $q = 1$, we obtain as special case of Eq. (2.3) which is defined in Eq. (1.4).

(3) The special case of (2.1), when $\varsigma, q = 1$ and $p = 0$ yields the M-L function due to Al-Bassam and Luchko [44]

$$E_{1,1}\left[(\xi_j,\tau_j)_m\,;z\right] = E^{1,1}_{(\xi_j,\tau_j)_m}[z] = \sum_{n=0}^{\infty} \frac{z^n}{\prod_{j=1}^{m} \Gamma(\xi_j n + \tau_j)} \tag{2.4}$$

$(\xi_j, \tau_j, z \in \mathbb{C}, \Re(\tau_j) > 0, \Re(\xi_j) > 0 \, (j = 1, 2, ..., m))$.

Next, if we set $m = 2$ in Eq. (2.4) provides another type of generalization of M-L function due to Djrbashyan [24].

(4) When $\varsigma, q = 1$, $p = 0$ and ξ_j is substituted by $\frac{1}{\xi_j} \, (j = 1, 2, ..., m)$, then Eq. (2.1) reduces to the multi-index M-L function defined by Kiryakova [45]

$$E_{1,1}\left[(1/\xi_j, \tau_j)_m; z\right] = E^{1,1}_{(1/\xi_j, \tau_j)_m}[z] = \sum_{n=0}^{\infty} \frac{z^n}{\prod_{j=1}^{m} \Gamma((n/\xi_j) + \tau_j)} \tag{2.5}$$

$(\xi_j, \tau_j, z \in C, \Re(\tau_j) > 0, \Re(\xi_j) > 0 \, (j = 1, 2, ..., m))$.

(5) If we put $\varsigma = q = m = 1$ with $min\{\Re(\xi), \Re(\tau)\}$, then (2.1) reduces in to the generalized M-L function considered by Wiman [23] defined in Eq. (1.2) and also $\varsigma = \tau = q = m = 1$ found in Eq. (1.1).

3. Basic Properties of $E^{(\varsigma,c);q}_{(\xi_j,\tau_j)_m}(z; p)$

In this section, we acquire some basic properties, including integral representation, integral and differentiation properties of the extended multi-index M-L functions.

Theorem 3.1. *The extended multi-index M-L function can be represented as:*

$$E^{(\varsigma,c);q}_{(\xi_j,\tau_j)_m}(z; p) = \frac{1}{B(\varsigma, c-\varsigma)} \int_0^1 t^{\varsigma-1} (1-t)^{c-\varsigma-1} e^{\frac{-p}{t(1-t)}} E^{c,q}_{(\xi_j,\tau_j)_m}(t^q z) \, dt \tag{3.1}$$

where $p \geq 0, \Re(c) > 0, \Re(\varsigma) > 0, \Re(\tau_j) > 0 \, (j = 1, 2, ..., m); \Re\left(\sum_{j=1}^{m} \xi_j\right) > \max\{0, \Re(q) - 1\}$.

Proof. Using Eq. (2.2) in Eq. (2.1), we obtain

$$E^{(\varsigma,c);q}_{(\xi_j,\tau_j)_m}(z; p) = \sum_{n=0}^{\infty} \left\{ \int_0^1 t^{\varsigma+qn-1} (1-t)^{c-\varsigma-1} e^{\frac{-p}{t(1-t)}} dt \right\} \times \frac{(c)_{nq} \, z^n}{B(\varsigma, c-\varsigma) \prod_{j=1}^{m} \Gamma(\xi_j n + \tau_j) \, n!}.$$

Changing the order of summation and integration, and after simplification of above equation, we get

$$E^{(\varsigma,c);q}_{(\xi_j,\tau_j)_m}(z; p) = \frac{1}{B(\varsigma, c-\varsigma)} \sum_{n=0}^{\infty} \int_0^1 t^{\varsigma+qn-1} (1-t)^{c-\varsigma-1} e^{\frac{-p}{t(1-t)}} \frac{(c)_{nq} \, z^n}{\prod_{j=1}^{m} \Gamma(\xi_j n + \tau_j) \, n!} dt. \tag{3.2}$$

Using Eq. (1.5) in Eq. (3.2), we obtain the desired result Eq. (3.1). ■

Corollary 3.2. *Taking* $t = \frac{r}{1+r}$ *in Theorem 3.1, we get*

$$E^{(\varsigma,c);q}_{(\xi_j,\tau_j)_m}(z; p) = \frac{1}{B(\varsigma, c-\varsigma)} \int_0^{\infty} \frac{r^{\varsigma-1}}{(1+r)^c} e^{\frac{-p(1+r)^2}{r}} E^{c,q}_{(\xi_j,\tau_j)_m}\left(\frac{(r)^q z}{(1+r)^q}\right) dr. \tag{3.3}$$

Corollary 3.3. *Consider $t = \sin^2\theta$ in Theorem 3.1, we obtain*

$$E^{(\varsigma,c);q}_{(\xi_j,\tau_j)_m}(z;p) = \frac{2}{B(\varsigma,c-\varsigma)}\int_0^{\frac{\pi}{2}} (\sin\theta)^{2\varsigma-1}(\cos\theta)^{2c-2\varsigma-1} exp\left(\frac{-p}{\sin^2\theta\cos^2\theta}\right) \times E^{c,q}_{(\xi_j,\tau_j)_m}\left(z\sin^{2q}\theta\right) d\theta. \tag{3.4}$$

Corollary 3.4. *Recurrence relation holds in the definition of (2.1) for $j = 1$ as:*

$$E^{(\varsigma,c);q}_{(\xi,\tau)}(z;p) = \tau E^{(\varsigma,c);q}_{(\xi,\tau+1)}(z;p) + \xi z\frac{d}{dz}E^{(\varsigma,c);q}_{(\xi,\tau+1)}(z;p) \tag{3.5}$$

where $p \geq 0, \Re(c) > 0, \Re(\varsigma) > 0, \Re(\tau) > 0; \Re(\xi) > \max\{0, \Re(q) - 1\}$.

Proof. Consider the definition of (2.1) for $j = 1$, and the right side of the Eq. (3.5), we obtain

$$\begin{aligned}
&\tau E^{(\varsigma,c);q}_{(\xi,\tau+1)}(z;p) + \xi z\frac{d}{dz}E^{(\varsigma,c);q}_{(\xi,\tau+1)}(z;p)\\
&= \tau\sum_{n=0}^{\infty}\frac{\mathcal{B}_p(\varsigma+nq,c-\varsigma)}{B(\varsigma,c-\varsigma)}\frac{(c)_{qn}}{\Gamma(\xi n+\tau+1)}\frac{z^n}{n!}\\
&+ \xi\, z\frac{d}{dz}\sum_{n=0}^{\infty}\frac{\mathcal{B}_p(\varsigma+nq,c-\varsigma)}{B(\varsigma,c-\varsigma)}\frac{(c)_{qn}}{\Gamma(\xi n+\tau+1)}\frac{z^n}{n!}\\
&= \sum_{n=0}^{\infty}\frac{\mathcal{B}_p(\varsigma+nq,c-\varsigma)}{B(\varsigma,c-\varsigma)}\frac{(c)_{qn}(\xi n+\tau)}{\Gamma(\xi n+\tau+1)}\frac{z^n}{n!}\\
&= E^{(\varsigma,c);q}_{(\xi,\tau)}(z;p).
\end{aligned}$$

■

Theorem 3.5. *For the extended multi-index M-L functions, we have the following higher derivative formula:*

$$\frac{d^n}{dz^n}E^{(\varsigma,c);q}_{(\xi_j,\tau_j)_m}(z;p) = (c)_q(c+q)_q\ldots(c+(n-1)q)_q\, E^{(\varsigma+nq,c+nq);q}_{(\xi_j,\tau_j+n\xi_j)_m}(z;p). \tag{3.6}$$

Proof. Differentiating with respect to z in Eq. (2.1), we get

$$\begin{aligned}
\frac{d}{dz}E^{(\varsigma,c);q}_{(\xi_j,\tau_j)_m}(z;p) &= \sum_{n=0}^{\infty}\frac{\mathcal{B}_p(\varsigma+nq,c-\varsigma)}{B(\varsigma,c-\varsigma)}\frac{(c)_{qn}}{\prod_{j=1}^{m}\Gamma(\xi_j n+\tau_j)}\frac{nz^{n-1}}{n!}\\
&= \sum_{n=0}^{\infty}\frac{\mathcal{B}_p(\varsigma+(n-1)q+q,c-\varsigma)}{B(\varsigma,c-\varsigma)}\frac{(c)_{(n-1)q+q}}{\prod_{j=1}^{m}\Gamma(\xi_j n+\tau_j)}\frac{z^{n-1}}{(n-1)!},
\end{aligned} \tag{3.7}$$

we can write the Pochhammer symbols as

$$(c)_{q(n-1)+q} = (c+q)_{q(n-1)}(c)_q. \tag{3.8}$$

Now using Eq. (3.8) in Eq. (3.7), we obtain Eq. (3.9) as

$$\frac{d}{dz}E^{(\varsigma,c);q}_{(\xi_j,\tau_j)_m}(z;p) = (c)_q\, E^{(\varsigma+q,c+q);q}_{(\xi_j,\tau_j+\xi_j)_m}(z;p). \tag{3.9}$$

Again differentiation with with respect to z in Eq. (3.9), we get

$$\frac{d^2}{dz^2}E^{(\varsigma,c);q}_{(\xi_j,\tau_j)_m}(z;p)=(c)_q\,(c+q)_q\,E^{(\varsigma+2q,c+2q);q}_{(\xi_j,\tau_j+2\xi_j)_m}(z;p)\,. \tag{3.10}$$

Continuing this procedure n times, we obtain the desired result Eq. (3.6). ∎

Theorem 3.6. *The following differentiation holds for the extended multi-index M-L functions as*

$$\frac{d^n}{dz^n}\left\{z^{\tau_1\ldots\tau_m-1}E^{(\varsigma,c);q}_{(\xi_j,\tau_j)_m}\left(\lambda z^{\xi_1\ldots\xi_m};p\right)\right\}=z^{\tau_1\ldots\tau_m-n-1}E^{(\varsigma,c);q}_{(\xi_j,\tau_j-n)_m}\left(\lambda z^{\xi_1\ldots\xi_m};p\right). \tag{3.11}$$

Proof. Replace z by $\lambda z^{\xi_1\ldots\xi_m}$ in Eq. (2.1) and take its product with $z^{\tau_1\cdots\tau_m-1}$, after that taking differentiation with respect to z, we get

$$\begin{aligned}&\frac{d}{dz}\left\{z^{\tau_1\ldots\tau_m-1}E^{(\varsigma,c);q}_{(\xi_j,\tau_j)_m}\left(\lambda z^{\xi_1\ldots\xi_m};p\right)\right\}\\&=z^{\tau_1\ldots\tau_m-2}\sum_{n=0}^{\infty}\frac{\mathcal{B}_p\left(\varsigma+nq,c-\varsigma\right)}{B\left(\varsigma,c-\varsigma\right)}\frac{(c)_{qn}}{\prod_{j=1}^{m}\Gamma\left(\xi_jn+\tau_j\right)}\frac{\lambda^n z^{\xi_1 n\ldots\xi_m n}}{n!}.\end{aligned}$$

Further, taking differentiation with respect to z up to n times of above term, we obtain our required result. ∎

4. Integral Transform of $E^{(\varsigma,c);q}_{(\xi_j,\tau_j)_m}(z;p)$

Definition 4.1. The Mellin transform [46] of the function $f(z)$ is defined as

$$\mathcal{M}\left(f\left(z\right);s\right)=\int_0^{\infty}z^{s-1}f\left(z\right)dz=f^*\left(s\right),\left(\Re\left(s\right)>0\right), \tag{4.1}$$

then inverse Mellin transform

$$f\left(z\right)=\mathcal{M}^{-1}[f^*\left(s\right);z]=\frac{1}{2\pi i}\int_{\lambda-i\infty}^{\lambda+i\infty}f^*\left(s\right)z^{-s}ds, \tag{4.2}$$

where $\lambda>0$.

In the following theorem, we provide Mellin transform of the extended multi-index M-L functions in term of the Wright generalized hypergeometric function [47].

$${}_p\psi_q\left(z\right)={}_p\psi_q\left[\begin{array}{l}(c_1,C_1),(c_2,C_2),...,(c_p,C_p);\\(d_1,D_1),(d_2,D_2),...,(d_q,D_q);\end{array}z\right]=\sum_{n=0}^{\infty}\frac{\prod_{i=1}^{p}\Gamma\left(c_i,C_in\right)z^n}{\prod_{j=1}^{q}\Gamma\left(d_i,D_in\right)n!}, \tag{4.3}$$

where the coefficients $C_i\,(i=1,2,,...,p)$ and $D_j\,(j=1,2,...,q)$ are positive real numbers such that

$$1+\sum_{j=1}^{q}D_j-\sum_{i=1}^{p}C_i\geq 0.$$

Theorem 4.2. *The Mellin transform of the extended multi-index M-L functions is given by*

$$\mathcal{M}\left\{E_{(\xi_j,\tau_j)_m}^{(\varsigma,c);q}(z;p);s\right\} = \frac{\Gamma(s)\Gamma(s+c-\varsigma)}{\Gamma(\varsigma)\Gamma(c-\varsigma)}\,{}_2\psi_{m+1}\left[\begin{array}{ll}(c,q),(r+s,q); & \\ (\tau_j,\xi_j)_1^m,(c+2s,q); & z\end{array}\right], \tag{4.4}$$

where $\Re(c) > 0, \Re(s) > 0, \Re(\varsigma) > 0, \Re(\tau_j) > 0\,(j = 1,2,...,m)\,; \Re\left(\sum_{j=1}^m \xi_j\right) > \max\{0, \Re(q) - 1\}, p \geq 0$ *and* ${}_2\psi_{m+1}$ *is the Wright generalized hypergeometric function.*

Proof. Applied the Mellin transform of the extended multi-index M-L functions, we have

$$\mathcal{M}\left\{E_{(\xi_j,\tau_j)_m}^{(\varsigma,c);q}(z;p);s\right\} = \int_0^\infty p^{s-1}E_{(\xi_j,\tau_j)_m}^{(\varsigma,c);q}(z;p)\,dp. \tag{4.5}$$

Using equation (3.1) in right sided of equation (4.5), we get

$$\begin{aligned}&\mathcal{M}\left\{E_{(\xi_j,\tau_j)_m}^{(\varsigma,c);q}(z;p);s\right\}\\ &= \frac{1}{B(\varsigma,c-\varsigma)}\int_0^\infty p^{s-1}\left\{\int_0^1 t^{\varsigma-1}(1-t)^{c-\varsigma-1}e^{\frac{-p}{t(1-t)}}E_{(\xi_j,\tau_j)_m}^{c,q}(t^q z)\,dt\right\}dp.\end{aligned} \tag{4.6}$$

Upon Interchanging the order of integration in Eq. (4.6), which is admittable due to the conditions of the Theorem 4.2, we get

$$\begin{aligned}&\mathcal{M}\left\{E_{(\xi_j,\tau_j)_m}^{(\varsigma,c);q}(z;p);s\right\}\\ &= \frac{1}{B(\varsigma,c-\varsigma)}\int_0^1 t^{\varsigma-1}(1-t)^{c-\varsigma-1}E_{(\xi_j,\tau_j)_m}^{c,q}(t^q z)\left\{\int_0^\infty p^{s-1}e^{\frac{-p}{t(1-t)}}\,dp\right\}dt.\end{aligned} \tag{4.7}$$

Now letting $u = \frac{p}{t(1-t)}$ in Eq. (4.7), and applying the mathematical formula that $\Gamma(s) = \int_0^\infty u^{s-1}e^{-u}du$, we get

$$\begin{aligned}&\mathcal{M}\left\{E_{(\xi_j,\tau_j)_m}^{(\varsigma,c);q}(z;p);s\right\}\\ &= \frac{\Gamma(s)}{B(\varsigma,c-\varsigma)}\int_0^1 t^{\varsigma+s-1}(1-t)^{c-\varsigma+s-1}E_{(\xi_j,\tau_j)_m}^{c,q}(t^q z)\,dt.\end{aligned} \tag{4.8}$$

Using Eq. (1.5), and interchanging the order of summation and integration which is permitted for $\Re(\xi_j) > 0, \Re(\tau_j) > 0, \Re(s) > 0, \Re(c) > \Re(\varsigma) > 0, \Re(c+s-\varsigma) > 0$, we obtain

$$\begin{aligned}&\mathcal{M}\left\{E_{(\xi_j,\tau_j)_m}^{(\varsigma,c);q}(z;p);s\right\}\\ &= \frac{\Gamma(s)}{B(\varsigma,c-\varsigma)}\sum_{n=0}^\infty \frac{(c)_{qn}}{\prod_{j=1}^m \Gamma(\xi_j n+\tau_j)}\frac{(z)^n}{n!}\int_0^1 t^{\varsigma+s+qn-1}(1-t)^{s+c-\varsigma-1}\,dt.\end{aligned} \tag{4.9}$$

Using the relation between Beta function and Gamma function, we obtain

$$\begin{aligned}&\mathcal{M}\left\{E_{(\xi_j,\tau_j)_m}^{(\varsigma,c);q}(z;p);s\right\}\\ &= \frac{\Gamma(s)}{B(\varsigma,c-\varsigma)}\sum_{n=0}^\infty \frac{(c)_{qn}}{\prod_{j=1}^m \Gamma(\xi_j n+\tau_j)}\frac{z^n}{n!}\frac{\Gamma(\varsigma+s+qn)\Gamma(s+c-\varsigma)}{\Gamma(c+2s+qn)}.\end{aligned}$$

After simplification, we obtain

$$\begin{aligned}&\mathcal{M}\left\{E^{(\varsigma,c);q}_{(\xi_j,\tau_j)_m}(z;p);s)\right\}\\&=\frac{\Gamma(s)\Gamma(s+c-\varsigma)}{\Gamma(\varsigma)\Gamma(c-\varsigma)}\sum_{n=0}^{\infty}\frac{\Gamma(c+qn)\Gamma(\varsigma+s+qn)z^n}{\prod_{j=1}^{m}\Gamma(\xi_j n+\tau_j)\Gamma(c+2s+qn)n!}.\end{aligned}\tag{4.10}$$

In view of Eq. (4.3), we arrived at our result Eq. (4.4). ∎

Corollary 4.3. *Taking* $s=1$, *Theorem 4.2, we get*

$$\int_0^\infty E^{(\varsigma,c);q}_{(\xi_j,\tau_j)_m}(z;p)\,dp=\frac{\Gamma(c-\varsigma+1)}{\Gamma(\varsigma)\Gamma(c-\varsigma)}{}_2\psi_{m+1}\left[\begin{array}{l}(c,q),(r+1,q);\\(\tau_j,\xi_j)_1^m,(c+2,q);\end{array}\;z\right].\tag{4.11}$$

Corollary 4.4. *Applying the inverse Mellin transform on reft and right side of equation* (4.4)*, we gain the important complex integral representation*

$$\begin{aligned}E^{(\varsigma,c);q}_{(\xi_j,\tau_j)_m}(z;p);s)&=\frac{1}{2\pi i\Gamma(\varsigma)\Gamma(c-\varsigma)}\int_{\lambda-i\infty}^{\lambda-i\infty}\Gamma(s)\Gamma(s+c-\varsigma)\\&\times{}_2\psi_{m+1}\left[\begin{array}{l}(c,q),(\varsigma+s,q);\\(\tau_j,\xi_j)_1^m,(c+2s,q);\end{array}\;z\right]p^{-s}ds.\end{aligned}\tag{4.12}$$

5. Relations between the $E^{(\varsigma,c);q}_{(\xi_j,\tau_j)_m}(z;p)$ with Laguerre polynomial and Whittaker function

In this part, we represent the extended multiindex M-L functions in terms of Laguerre polynomials and Whittaker's function.

Theorem 5.1. *For* $\Re(c)>0,\Re(\varsigma)>0,\ \Re(\tau_j)>0\,(j=1,2,...,m);\Re\left(\sum_{j=1}^m\xi_j\right)>\max\{0,\Re(q)-1\}$, $p\geq0$. *The extended multiindex M-L functions holds true:*

$$\begin{aligned}e^{2p}E^{(\varsigma,c);q}_{(\xi_j,\tau_j)_m}(z;p)&=\frac{1}{B(\varsigma,c-\varsigma)}\sum_{a,b,k=0}^{\infty}\frac{L_a(p)L_b(p)(c)_{qk}z^k}{\prod_{j=1}^{m}\Gamma(\xi_j n+\tau_j)k!}\\&\times B(a+qk+\varsigma+1,c+b-\varsigma+1).\end{aligned}\tag{5.1}$$

Proof. We begin by recalling the valuable identity which is used in [48] as:

$$e^{\left(\frac{-p}{t(1-t)}\right)}=e^{-2p}\sum_{a,b=0}^{\infty}L_b(p)L_a(p)t^{a+1}(1-t)^{b+1};\quad(0<t<1).\tag{5.2}$$

Applying equation (5.2) in equation (3.1), we get

$$\begin{aligned}E^{(\varsigma,c);q}_{(\xi_j,\tau_j)_m}(z;p)&=\frac{1}{B(\varsigma,c-\varsigma)}\int_0^1 t^{\varsigma-1}(1-t)^{c-\varsigma-1}e^{-2p}\\&\times\sum_{a,b=0}^{\infty}L_b(p)L_a(p)t^{a+1}(1-t)^{b+1}E^{c,q}_{(\xi_j,\tau_j)_m}(t^q z)\,dt,\end{aligned}\tag{5.3}$$

and using equation (1.5) in (5.3), we get

$$E^{(\varsigma,c);q}_{(\xi_j,\tau_j)_m}(z;p)=\frac{1}{B(\varsigma,c-\varsigma)}\int_0^1 t^{\varsigma-1}(1-t)^{c-\varsigma-1}e^{-2p} \times\sum_{a,b=0}^{\infty}L_b(p)L_a(p)t^{a+1}(1-t)^{b+1}\sum_{k=0}^{\infty}\frac{(c)_{qk}}{\prod_{j=1}^{m}\Gamma(\xi_j k+\tau_j)}\frac{(t^q z)^k}{k!}dt. \tag{5.4}$$

Upon interchanging the order of integration and summation, which provide under the assumption of the theorem, we obtain

$$E^{(\varsigma,c);q}_{(\xi_j,\tau_j)_m}(z;p)=\frac{e^{-2p}}{B(\varsigma,c-\varsigma)}\sum_{a,b,k=0}^{\infty}\frac{L_b(p)L_a(p)(c)_{qk}z^k}{\prod_{j=1}^{m}\Gamma(\xi_j k+\tau_j)k!}\int_0^1 t^{a+qk+\varsigma}(1-t)^{c+b-\varsigma}dt. \tag{5.5}$$

Now using definition of Beta function

$$E^{(\varsigma,c);q}_{(\xi_j,\tau_j)_m}=\frac{e^{-2p}}{B(\varsigma,c-\varsigma)}\sum_{a,b,k=0}^{\infty}\frac{L_b(p)L_a(p)(c)_{qk}z^k}{\prod_{j=1}^{m}\Gamma(\xi_j k+\tau_j)k!}B(a+qk+\varsigma+1,c+b-\varsigma+1). \tag{5.6}$$

After simplification, we found the desired result. ■

Theorem 5.2. *For the extended multi-index M-L functions, we have*

$$e^{\frac{3p}{2}}E^{(\varsigma,c);q}_{(\xi_j,\tau_j)_m}=\frac{\Gamma(c-\varsigma+1)}{B(\varsigma,c-\varsigma)}\sum_{a,k=0}^{\infty}\frac{L_a(p)(c)_{qk}z^k}{\prod_{j=1}^{m}\Gamma(\xi_j k+\tau_j)k!}p^{\left(\frac{a+qk+\varsigma-1}{2}\right)} \times\mathcal{W}_{\frac{-1-a-qk+\varsigma-2c}{2},\frac{a+qk+\varsigma}{2}}(p). \tag{5.7}$$

Proof. Allowing for the following equality $e^{\left(\frac{-p}{t(1-t)}\right)}=e^{\left(\frac{-p}{1-t}\right)}e^{\left(\frac{-p}{t}\right)}$ and via the generating function relating to the Laguerre polynomials, we obtain

$$e^{\frac{-p}{t(1-t)}}=e^{(-p)}e^{(-p/t)}(1-t)\sum_{a=0}^{\infty}L_a(p)t^a, \tag{5.8}$$

Substituting Eq. (5.8) in to account in Eq. (3.1), we get

$$E^{(\varsigma,c);q}_{(\xi_j,\tau_j)_m}(z;p)=\frac{1}{B(\varsigma,c-\varsigma)}\int_0^1 t^{\delta+a-1}(1-t)^{c-\varsigma-1}e^{(-p)}e^{(-p/t)}(1-t)$$

$$\times\sum_{a=0}^{\infty}L_a(p)E^{c,q}_{(\xi_j,\tau_j)_m}(t^q z)dt,$$

Interchanging the order of summation and integration, we obtain

$$E^{(\varsigma,c);q}_{(\xi_j,\tau_j)_m}(z;p)=\frac{e^{-p}}{B(\varsigma,c-\varsigma)}\sum_{a,k=0}^{\infty}\frac{L_a(p)(c)_{qk}z^k}{\prod_{j=1}^{m}\Gamma(\xi_j k+\tau_j)k!} \times\int_0^1 t^{a+qk+\varsigma-1}(1-t)^{c-\varsigma}e^{(-p/t)}dt. \tag{5.9}$$

Using the following integral representation [49]:

$$\int_0^1 t^{\mu-1}(1-t)^{\nu-1} e^{\frac{-p}{t}} dt = \Gamma(\nu) p^{\frac{\mu-1}{2}} e^{\frac{-p}{2}} \mathcal{W}_{\frac{1-\mu-2\nu}{2},\frac{\mu}{2}}(p) \quad (\Re(\nu)>0, \Re(p)>0),$$

in Eq. (5.9), we found the desired result. ∎

6. Fractional Calculus Approach of $E^{(\varsigma,c);q}_{(\xi_j,\tau_j)_m}(z;p)$

In this section, we derive a little useful EMMLF properties associated with the right-sided Riemann-Liouville (R-L) fractional integral operator I^{ϑ}_{a+} and the right sided R-L fractional derivative operator D^{ϑ}_{a+}, defined for $\vartheta \in \mathbb{C}, (\Re(\vartheta)>0)$, $x>0$ (See, for details [50, 51]):

$$\left(I^{\vartheta}_{a+} f\right)(x) = \frac{1}{\Gamma(\vartheta)} \int_a^x \frac{f(t)}{(x-t)^{1-\vartheta}} dt, \tag{6.1}$$

and

$$\left(D^{\vartheta}_{a+} f\right)(x) = \left(\frac{d}{dx}\right)^{\ell} \left(I^{\ell-\vartheta}_{a+} f\right)(x) \quad \ell = [\Re(\vartheta)+1]. \tag{6.2}$$

where $[\Re(\vartheta)]$ is the integral part of $\Re(\vartheta)$. The above is a generalization of the R-L fractional derivative operator (6.2) by implementing a right-hand R-L fractional derivative operator $D^{\vartheta,\sigma}_{a+}$ of order $0<\vartheta<1$ and $0 \le \sigma \le 1$ of Hilfer [52]:

$$\left(D^{\vartheta,\sigma}_{a+} f\right)(x) = \left(I^{\sigma(1-\vartheta)}_{a+} \frac{d}{dx}\right)\left(I^{(1-\sigma)(1-\vartheta)}_{a+} f\right)(x). \tag{6.3}$$

The extension of Eq. (6.3) yields the R-L fractional derivative operator D^{ϑ}_{a+} when $\sigma=0$.

Theorem 6.1. *Let $\vartheta, \lambda, \xi_j, \tau_j, \varsigma \in \mathbb{C}$ be such that $\Re(\vartheta)>0$, $p \ge 0$ and the conditions given in Eq. (2.1) is satisfied, for $x>a$, the following relation holds:*

$$\left(I^{\vartheta}_{a+}\left\{\prod_{j=1}^{m}(z-a)^{\tau_j-1} E^{(\varsigma,c);q}_{(\xi_j,\tau_j)_m}\left(\lambda(z-a)^{\xi_j};p\right)\right\}\right)(x)$$

$$= (x-a)^{\vartheta+\tau_1\ldots\tau_m-1} E^{(\varsigma,c);q}_{(\xi_j,\tau_j+\vartheta)_m}\left(\lambda(x-a)^{\xi_1\ldots\xi_m};p\right). \tag{6.4}$$

$$\left(D^{\vartheta}_{a+}\left\{\prod_{j=1}^{m}(z-a)^{\tau_j-1} E^{(\varsigma,c);q}_{(\xi_j,\tau_j)_m}\left(\lambda(z-a)^{\xi_j};p\right)\right\}\right)(x)$$

$$= (x-a)^{\tau_1\ldots\tau_m-\vartheta-1} E^{(\varsigma,c);q}_{(\xi_j,\tau_j-\vartheta)_m}\left(\lambda(x-a)^{\xi_1\ldots\xi_m};p\right). \tag{6.5}$$

$$\left(D^{\vartheta,\sigma}_{a+}\left\{\prod_{j=1}^{m}(z-a)^{\tau_j-1} E^{(\varsigma,c);q}_{(\xi_j,\tau_j)_m}\left(\lambda(z-a)^{\xi_j};p\right)\right\}\right)(x)$$

$$= (x-a)^{\tau_1\ldots\tau_m-\vartheta-1} E^{(\varsigma,c);q}_{(\xi_j,\tau_j-\vartheta)_m}\left(\lambda(x-a)^{\xi_1\ldots\xi_m};p\right). \tag{6.6}$$

Proof. By virtue of the formulas Eq. (6.1) and Eq. (2.1), the term by term fractional integration and use of the relation [51]:

$$\left(I_{a+}^{\vartheta}(z-a)^{\tau-1}\right)(x)=\frac{\Gamma(\tau)}{\Gamma(\vartheta+\tau)}(x-a)^{\vartheta+\tau-1}\quad(\tau,\vartheta\in\mathbb{C},\Re(\vartheta)>0,\Re(\tau)>0) \tag{6.7}$$

yield for $x>a$,

$$\begin{aligned}&\left(I_{a+}^{\vartheta}\left\{\prod_{j=1}^{m}(z-a)^{\tau_j-1}E_{(\xi_j,\tau_j)_m}^{(\varsigma,c);q}\left(\lambda(z-a)^{\xi_j};p\right)\right\}\right)(x)\\&=\left(I_{a+}^{\vartheta}\left\{\sum_{n=0}^{\infty}\frac{\mathcal{B}_p(\varsigma+nq,c-\varsigma)}{B(\varsigma,c-\varsigma)}\frac{(c)_{qn}}{\prod_{j=1}^{m}\Gamma(\xi_j n+\tau_j)}\frac{\lambda^n(z-a)^{\tau_1\ldots\tau_m+(\xi_1\ldots\xi_m)n-1}}{n!}\right\}\right)(x)\\&=(x-a)^{\vartheta+\tau_1\ldots\tau_m-1}E_{(\xi_j,\tau_j+\vartheta)_m}^{(\varsigma,c);q}\left(\lambda(x-a)^{\xi_1\ldots\xi_m};p\right).\end{aligned}\tag{6.8}$$

Subsequent, by Eq. (6.5) and Eq. (2.1), we find that

$$\begin{aligned}&\left(D_{a+}^{\vartheta}\left\{\prod_{j=1}^{m}(z-a)^{\tau_j-1}E_{(\xi_j,\tau_j)_m}^{(\varsigma,c);q}\left(\lambda(z-a)^{\xi_j};p\right)\right\}\right)(x)\\&=\left(\frac{d}{dx}\right)^{\ell}\left(I_{a+}^{\ell-\vartheta}\left\{\prod_{j=1}^{m}(z-a)^{\tau_j-1}E_{(\xi_j,\tau_j)_m}^{(\varsigma,c);q}\left(\lambda(z-a)^{\xi_j};p\right)\right\}\right)(x)\\&=\left(\frac{d}{dx}\right)^{\ell}\left((x-a)^{\tau_1\ldots\tau_m+\ell-\vartheta-1}E_{(\xi_j,\tau_j+\ell-\vartheta)_m}^{(\varsigma,c);q}\left(\lambda(x-a)^{\xi_1\ldots\xi_m};p\right)\right)(x).\end{aligned}\tag{6.9}$$

Applying Eq. (3.11), we are led to the desired result Eq. (6.5). Lastly, by Eq. (6.3) and Eq. (2.1), we becomes

$$\begin{aligned}&\left(D_{a+}^{\vartheta,\sigma}\left\{\prod_{j=1}^{m}(z-a)^{\tau_j-1}E_{(\xi_j,\tau_j)_m}^{(\varsigma,c);q}\left(\lambda(z-a)^{\xi_j};p\right)\right\}\right)(x)\\&=\left(D_{a+}^{\vartheta,\sigma}\left\{\sum_{n=0}^{\infty}\frac{\mathcal{B}_p(\varsigma+nq,c-\varsigma)}{B(\varsigma,c-\varsigma)}\frac{(c)_{qn}}{\prod_{j=1}^{m}\Gamma(\xi_j n+\tau_j)}\frac{\lambda^n(z-a)^{\tau_1\ldots\tau_m+(\xi_1\ldots\xi_m)n-1}}{n!}\right\}\right)(x)\\&=\sum_{n=0}^{\infty}\frac{\mathcal{B}_p(\varsigma+nq,c-\varsigma)}{B(\varsigma,c-\varsigma)}\frac{(c)_{qn}}{\prod_{j=1}^{m}\Gamma(\xi_j n+\tau_j)}\frac{\lambda^n}{n!}\left(D_{a+}^{\vartheta,\sigma}\left\{(z-a)^{\tau_1\ldots\tau_m+(\xi_1\ldots\xi_m)n-1}\right\}\right)(x).\end{aligned}\tag{6.10}$$

Using the familiar relation of Srivastava and Tomovski [43]:

$$\left(D_{a+}^{\vartheta,\sigma}\left\{(z-a)^{\tau-1}\right\}\right)(x)=\frac{\Gamma(\tau)}{\Gamma(\tau-\vartheta)}(x-a)^{\tau-\vartheta-1}\tag{6.11}$$

$$(x>a;0<\vartheta<1;0\leq\sigma\leq1,\Re(\tau)>0).$$

In Eq. (6.10), we are led to the result Eq. (6.6). ■

The classical ϑ fractional R-L derivative is typically defined by

$$D_z^{\vartheta}\{f(z)\} = \frac{1}{\Gamma(-\vartheta)}\int_0^z (z-t)^{-\vartheta-1} f(t)\,dt \quad (\Re(\vartheta) < 0), \tag{6.12}$$

where the path of integration is a line from 0 to z in the complex t-plane. For the case $\ell - 1 < \Re(\vartheta) < \ell\,(\ell = 1, 2, ...)$, it is defined below as

$$\begin{aligned} D_z^{\vartheta}\{f(z)\} &= \frac{d^{\ell}}{dz^{\ell}} D_z^{\vartheta-\ell}\{f(z)\} \\ &= \frac{d^{\ell}}{dz^{\ell}}\left\{\frac{1}{\Gamma(\ell-\vartheta)}\int_0^z (z-t)^{\ell-\vartheta-1} f(t)\,dt\right\}. \end{aligned}$$

Özarslan and Özergin [48] have established the extended R-L fractional derivative operator as follows:

$$D_z^{\vartheta,p}\{f(z)\} = \frac{1}{\Gamma(-\vartheta)}\int_0^z (z-t)^{-\vartheta-1} e^{\frac{-pz^2}{t(z-t)}} f(t)\,dt \quad (\Re(\vartheta) < 0, \Re(p) > 0), \tag{6.13}$$

and for $\ell - 1 < \Re(\vartheta) < \ell\,(\ell = 1, 2, ...)$,

$$\begin{aligned} D_z^{\vartheta,p}\{f(z)\} &= \frac{d^{\ell}}{dz^{\ell}} D_z^{\vartheta-\ell}\{f(z)\} \\ &= \frac{d^{\ell}}{dz^{\ell}}\left\{\frac{1}{\Gamma(\ell-\vartheta)}\int_0^z (z-t)^{\ell-\vartheta-1} e^{\frac{-pz^2}{t(z-t)}} f(t)\,dt\right\}. \end{aligned}$$

If we set $p = 0$, then we obtain the classical R-L fractional derivative operator.

Theorem 6.2. *Let $p \geq 0, \Re(\lambda) > 0, \Re(\xi_j) > 0, \Re(\tau_j) > 0.$ Then*

$$D_z^{\lambda-c,p}\left\{z^{\lambda-1} E_{(\xi_j,\tau_j)_m}^{c,q}(z^q)\right\} = \frac{z^{c-1}}{\Gamma(c-\lambda)} B(\lambda, c-\lambda)\, E_{(\xi_j,\tau_j)_m}^{(\varsigma,c);q}(z^q; p) \tag{6.14}$$

Proof. Replacing ϑ by $\lambda - c$ in the definition of the extended fractional derivative operator (6.13), we get

$$\begin{aligned} &D_z^{\lambda-c,p}\left\{z^{\lambda-1} E_{(\xi_j,\tau_j)_m}^{c,q}(z^q)\right\} \\ &= \frac{1}{\Gamma(c-\lambda)}\int_0^z t^{\lambda-1}(z-t)^{c-\lambda-1} E_{(\xi_j,\tau_j)_m}^{c,q}(t^q)\, e^{\frac{-pz^2}{t(z-t)}}\,dt \\ &= \frac{z^{c-\lambda-1}}{\Gamma(c-\lambda)}\int_0^z t^{\lambda-1}\left(1-\frac{t}{z}\right)^{c-\lambda-1} E_{(\xi_j,\tau_j)_m}^{c,q}(t^q)\, e^{\frac{-pz^2}{t(z-t)}}\,dt, \end{aligned}$$

Substitute, $u = t/z$ in above equation, we get

$$\begin{aligned} D_z^{\lambda-c,p}\left\{z^{\lambda-1} E_{(\xi_j,\tau_j)_m}^{c,q}(z^q)\right\} = \frac{z^{c-1}}{\Gamma(c-\lambda)}\int_0^1 & u^{\lambda-1}(1-u)^{c-\lambda-1} \\ & \times E_{(\xi_j,\tau_j)_m}^{c,q}(u^q z^q)\, e^{\frac{-p}{u(1-u)}}\,du. \end{aligned}$$

Relating this result with equation (3.1), we obtain

$$D_z^{\lambda-c,p}\left\{z^{\lambda-1} E_{(\xi_j,\tau_j)_m}^{c,q}(z^q)\right\} = \frac{z^{c-1}}{\Gamma(c-\lambda)} B(\lambda, c-\lambda)\, E_{(\xi_j,\tau_j)_m}^{(\varsigma,c);q}(z^q; p).$$

■

7. Concluding remark and discussion

The properties, integral transform, representation in terms of Laguerre polynomials, Whittaker function and fractional calculus of the newly defined extended multi-index M-L type functions are investigated here. Various special cases of the paper's related results may be analyzed by taking appropriate values of the relevant parameters. For example, if we set $m = 1$ and $q = 1$ in (2.1), we obtain the unswervingly result due to Mittal et al. [53] and Özarslan and Yilmaz [31] respectively. For several other special cases, we refer to [25, 54, 55] and have left the findings to interested readers.

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References

[1] M.K. Bansal, D. Kumar, J. Singh, K.S. Nisar, On the solutions of a class of integral equations pertaining to incomplete H-function and incomplete H(bar)-function, Mathematics 8 (2020) 819.

[2] Y. Singh, V. Gill, J. Singh, D. Kumar, K. Ilyas, Computable Generalization of Fractional Kinetics Equation with Special Functions, Journal of King Saud University-Science 33 (1) (2021) 101221.

[3] J. Singh, D. Kumar, M.K. Bansal, Solution of Nonlinear Differential Equation and Special Functions, Mathematical Methods in the Applied Sciences 43 (5) (2020) 2106-2116.

[4] J. Singh, D. Kumar, S.D. Purohit, A.M. Mishra, M. Bohra, An efficient numerical approach for fractional multi-dimensional diffusion equations with exponential memory, Numerical Methods for Partial Differential Equations 37 (2) (2021) 1631-1651.

[5] S. Yadav, D. Kumar, J. Singh, D. Baleanu, Analysis and Dynamics of Fractional Order Covid-19 Model with Memory Effect, Results in Physics 24 (2021) 104017.

[6] P. Agarwal, A. Berdyshev, E. Karimov, Solvability of a non-local problem with integral transmitting condition for mixed type equation with Caputo fractional derivative, Results in Mathematics 71 (3) (2017) 1235-1257.

[7] P. Agarwal, J.J. Nieto, Some fractional integral formulas for the Mittag-Leffler type function with four parameters, Open Mathematics 31 (1) (2015) 537-546.

[8] A.A. Alderremy, K.M. Saad, P. Agarwal, S. Aly, S. Jain, Certain new models of the multi space-fractional Gardner equation, Physica A: Statistical Mechanics and its Applications 545 (2020) 23806.

[9] P. Agarwal, M. Chand, D. Baleanu, D. ÓRegan, S. Jain, On the solutions of certain fractional kinetic equations involving k-Mittag-Leffler function, Adv. Difference Equ. 1 (249) (2018) 1-13.

[10] P. Agarwal, S.K. Ntouyas, S. Jain, M. Chand, G. Singh, Fractional kinetic equations involving generalized k-Bessel function via Sumudu transform, Alexandria engineering journal 57 (3) (2018) 1937-1942.

[11] S. Jain, R.P. Agarwal, P. Agarwal, P. Singh, Certain Unified Integrals Involving a Multivariate Mittag–Leffler Function, Axioms 10 (2) (2021) 81.

[12] J. Choi, P. Agarwal, A note on fractional integral operator associated with multiindex Mittag-Leffler functions, Filomat 30 (7) (2016) 1931-1939.

[13] P. Agarwal, Q. Al-Mdallal, Y.J. Cho, S. Jain, Fractional differential equations for the generalized Mittag-Leffler function, Adv. Difference Equ. (1) (2018) 1-8.

[14] S. Jain, P. Agarwal, A. Kilicman, Pathway fractional integral operator associated with 3m-parametric Mittag-Leffler functions, Int. J. Appl. Comput. Math. 4 (5) (2018) 1-7.

[15] G. Rahman, P. Agarwal, S. Mubeen, M. Arshad, Fractional integral operators involving extended Mittag–Leffler function as its Kernel. Boletén de la Sociedad Matemática Mexicana 24 (2) (2018) 381-392.

[16] M.F. Yassen, A.A. Attiya, P. Agarwal, Subordination and superordination properties for certain family of analytic functions associated with Mittag–Leffler function, Symmetry 12 (10) (2021) 1724.

[17] P. Agarwal, M. Chand, S. Rani, T.M. Rassias, Certain feynman type integrals involving generalized k-Mittag-Leffler function and general class of polynomials, Appl. Anal. Discrete Math. 13 (3) (2019) 746-773.

[18] G.S. Khammash, P. Agarwal, J. Choi, Extended k-Gamma and k-Beta functions of matrix arguments, Mathematics 8 (10) (2020) 1715.

[19] P. Agarwal, R.P. Agarwal, M. Ruzhansky, Special Functions and Analysis of Differential Equations,CRC Press, 2020.

[20] P. Agarwal, J.E. Restrepo, An extension by means of ω-weighted classes of the generalized Riemann-Liouville k-fractional integral inequalities, J. Math. Inequalities 14 (2020) 35-46.

[21] R. Goyal, S. Momani, P. Agarwal, M.T. Rassias, An Extension of Beta Function by Using Wiman's Function, Axioms 10 (3) (2021) 187.

[22] G.M. Mittag-Leffler, Sur la nouvelle fonction $E_\xi(x)$, C.R. Acad. Sci. 137 (1903) 554-558.

[23] A. Wiman, $\ddot{U}$ber der Fundamentalsatz in der Theorie der Funktionen $E_\alpha(x)$, Acta Math. 29 (1905) 191-201.

[24] M.M. Dzherbashyan, On the integral transforms generated by the generalized Mittag-Leffler function, Izv. An arm. SSR 13 (3) (1960) 21-63.

[25] T.R. Prabhakar, A singular integral equation with a generalized Mittag-Leffler function in the kernel, Yokohama Math. J. 19 (1971) 7-15.

[26] H.M. Srivastava, J. Choi, Zeta and q-Zeta functions and associated series and integrals, Elsevier Science Publishers, Amsterdam, London and New York, 2012.

[27] A.A. Kilbas, M. Saigo, J.J. Trujullo, On the generalized Wright function, Frac. Calc. Appl. Anal. 5 (2002) 437-466.

[28] A.M. Mathai, R.K. Saxena, The H-functions with Applications in Statistics and Other Disciplines, Halsted Press, John Eastern Limited, New - Delhi; Wiley & Sons, New York, 1978.

[29] A.A. Kilbas, M. Saigo, Theory and Applications, Ser. Analytic Methods and Special Functions, Vol. 9, CRC Press, London and NewYork, 2004.

[30] R.K. Saxena, K. Nishimoto, N-fractional calculus of generalized Mittag-Leffler functions, J. Fract. Calc. 37 (2010) 43-52.

[31] M.A. Özarslan, B. Yılmaz, The extended Mittag-Leffler function and its properties, J. Inequal. Appl. 2014 (2014) 85.

[32] D.L. Suthar, S.D. Purohit, R.K. Parmar, Generalized fractional calculus of the multiindex Bessel function, Math. Nat. Sci. 1 (2017) 26-32.

[33] D.L. Suthar, H. Habenom, H.Tadesse, Generalized fractional calculus formulas for a product of Mittag-Leffler function and multivariable polynomials, Int. J. Appl. Comput. Math. 4 (1) (2018) 1-12.

[34] D.L. Suthar, T. Tsagye, Riemann-Liouville fractional integrals and differential formula involving Multiindex Bessel-function, Math. Sci. Lett. 6 (3) (2017) 1-5.

[35] R.K. Parmar, A Class of Extended Mittag-Leffler Functions and Their Properties Related to Integral Transforms and Fractional Calculus, Mathematics 3 (2015) 1069-1082.

[36] H.M. Srivastava, R.K. Parmar, P. Chopra, A class of extended fractional derivative operators and associated generating relations involving hypergeometric functions, Axioms 1 (2012) 238-258.

[37] K.S. Nisar, D.L. Suthar, R. Agarwal, S.D. Purohit, Fractional calculus operators with Appell function kernels applied to Srivastava polynomials and extended Mittag-Leffler function, Adv. Difference Equ. (2020) 148: 2020.

[38] D.L. Suthar, H. Amsalu, K. Godifey, Certain integrals involving multivariate Mittag-Leffler function, J. Inequal. Appl. (2019) 2019:208.

[39] D.L. Suthar, M. Andualem, B. Debalkie, A study on generalized multivariable Mittag-Leffler function via generalized fractional calculus operators, Journal of Mathematics 2019, Article ID 9864737, 7 pages.

[40] K. Tilahun, K. Tadessee, D.L. Suthar, The Extended Bessel-Maitland Function and Integral Operators Associated with Fractional Calculus, Journal of Mathematics, Article ID 7582063 (2020) 1-8.

[41] M.A. Chaudhry, A. Qadir, H.M. Srivastava, R.B. Paris, Extended hypergeometric and confluent hypergeometric functions, Appl. Math. Comput. 159 (2004) 589-602.

[42] M.A. Chaudhry, S.M. Zubair, On a Class of Incomplete Gamma Functions with Applications, Champman and Hall (CRC Press company): Boca Raton, FL, USA; London, UK; New York, USA; Washington, DC, USA, 2002.

[43] H.M. Srivastava, Z. Tomovski, Fractional calculus with an integral operator containing generalized Mittag-Leffler function in the kernel. Appl. Math. Comput. 211 (1) (2009) 198-210.

[44] M.A. Al-Bassam , Y.F. Luchko, On generalized fractional calculus and its application to the solution of integro-differential equations, J. Fract. Calc. 7 (1995) 69-88.

[45] V.S. Kiryakova, Multi-index Mittag-Leffler functions related to Gelfond-Leontier operators and Laplace type integral transforms. Fract. Calc. Appl. Anal. 2 (1999) 445-462.

[46] I.N. Sneddon, The use of Integral Transform, New Delhi, Tata McGraw Hill, 1979.

[47] H.M. Srivastava, P.W. Karlson, Multiple Gaussian Hypergeometric Series, Halsted Press (Ellis Horwood Limited, Chichester), John Wiley and Sons, New York, Chichester, Brisbane and Toronto, 1985.

[48] M.A. Özarslan, E. Özergin, Some generating relations for extended hypergeometric functions via generalized fractional derivative operator, Math. Comput. Model. 52 (2010) 1825-1833.

[49] M.A. Özarslan, Some remarks on extended hypergeometric, extended confluent hypergeometric and extended Appell's functions, J. Comput. Anal. Appl. 14 (6) (2012) 1148-1153.

[50] A.A. Kilbas, H.M. Srivastava, J.J. Trujullo, Theory and Applications of Fractional Differential Equations; North-Holland Mathematical Studies; Elsevier (North-Holland) Science Publishers. Amsterdam, the Netherland; London, UK; New York, NY, USA, 2006.

[51] S.G. Samko, A.A. Kilbas, O.I. Marichev, Fractional Integrals and Derivatives: Theory and Applications; Gordon and Breach: Yverdon, Switzerland, 1993.

[52] R. Ed Hilfer, Applications of Fractional Calculus in Physics, World Scientific Publishing Company: Singapore, Singapore; Hackensack, NJ, USA; London, UK; Hong Kong, China, 2000.

[53] E. Mittal, R.M. Pandey, S. Joshi , On extension of Mittag-Leffler function. Appl. Appl. Math. 11 (1) (2016) 307-316.

[54] R.K. Saxena, J. Ram, D.L. Suthar, Generalized fractional calculus of the generalized Mittag-Leffler functions, J. Indian Acad. Math. 31 (1) (2009) 165-172.

[55] S.D. Purohit, S.L, Kalla, D.L. Suthar, Fractional integral operators and the multiindex Mittag-Leffler functions, Sci. Ser. A Math. Sci. (N.S.) 21 (2011) 87-96.