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Coincidence Best Proximity Point Theorems for (α, g)-Geraghty Contractive Mappings in Metric Spaces without an Isometry of a Mapping g

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Abstract In this work, we introduce new concepts of a pair of mappings, called (α, g)-proximal admissible, triangular (α, g)-proximal admissible and (α, g)-Geraghty contractive mappings. By using these types of mappings, we prove the existence and uniqueness of a coincidence best proximity point in complete metric spaces without an isometry of the mapping g. Moreover, we give an example for our main result. And also, Our main result is a generalization of some well-known results in the literature.

MSC: 46N40; 46T99; 47H10; 54H25

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1. Preliminaries and Introduction

It is well known that the concept of best proximity point is derived from the concept of fixed point, which is concerned with non-self mapping. Let (X, d) be a metric space. In 1922, Banach [1] introduced and gave the concept of self-mapping as follows:

a mapping $J : X \to X$ is said to be *contraction* if there exits a constant $k \in [0, 1)$ such that

$$d(Jx, Jy) \leq kd(x, y), \text{ for all } x, y \in X.$$

Moreover, he presented the theorem which was stated as follows:

if X is a complete metric space and $J : X \to X$ is a contraction, then J has a unique fixed point in X.

After that, contraction and the above theorem received a lot of attention from researchers. Because such research can be applied to many fields, resulting in such research is very famous and researchers call this theorem "the Banach Contraction Principle". In 1973,

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Geraghty [2] defined a contractive mapping based on the class $\mathcal{F}$ of mappings $\beta : [0, \infty) \to [0, 1)$ such that

$$\lim_{n\to\infty} \beta(t_n) = 1 \implies \lim_{n\to\infty} t_n = 0. \tag{1.1}$$

Definition 1.1 ([2])**.** Let (X, d) be a metric space. A mapping $J : X \to X$ is called *Geraghty contraction* if there exists $\beta \in \mathcal{F}$ such that for all $x, y \in X$,

$$d(Jx, Jy) \le \beta(d(x, y))d(x, y).$$

Moreover, the author [2] sproved the existence and uniqueness of fixed point for self-mapping of Geraghty contractions.

Theorem 1.2 ([2])**.** *Let (X, d) be a complete metric space and $J : X \to X$ be Geraghty contraction. Then J has a unique fixed point $x \in X$, and $\{J^n x\}$ converges to x.*

In 2012, Samet et al. [3] introduced the following definition.

Definition 1.3 ([3])**.** Let $J : X \to X$ be a mapping and $\alpha : Y \times Y \to \mathbb{R}$ be a function. Then J is said to be *α-admissible* if for all $x, y \in X$,

$$\alpha(x, y) \ge 1 \text{ implies } \alpha(Jx, Jy) \ge 1.$$

One year later, Karapınar et al. [4] gave the following definition.

Definition 1.4 ([4])**.** An α-admissible mapping $J : X \to X$ is said to be *triangular α-admissible* if for all $x, y, z \in X$,

$$\alpha(x, y) \ge 1 \text{ and } \alpha(x, z) \ge 1 \text{ implies } \alpha(x, z) \ge 1.$$

Next, Let (X, d) be a metric space and Y, Z be nonempty subsets of X. We give the meaning of the sets Y_0 and Z_0 as follows:

$$d(Y, Z) := \inf\{d(x, y) \ : \ x \in Y \text{ and } y \in Z\},$$
$$Y_0 := \{x \in Y \ : \ d(x, y) = d(Y, Z) \text{ for some } y \in Z\},$$
$$Z_0 := \{y \in Z \ : \ d(x, y) = d(Y, Z) \text{ for some } x \in Y\}.$$

Let Y and Z be nonempty subsets of X and let $J : Y \to Z$ be a non-self mapping. A point $x \in Y$ such that $d(x, Jx) = d(Y, Z)$ is called *a best proximity point of J*.

Definition 1.5. Let $J : Y \to Z$ and $g : Y \to Y$ be mappings. An element $x \in Y$ is said to be a *coincidence best proximity point of the pair* (g, J) if

$$d(gx, Jx) = d(Y, Z).$$

In 2012, Caballero et al. [5] studied the best proximity point for a pair (Y, Z) of subsets of a metric space (X, d) and the authors obtained a generalization of main result of Geraghty [2] in the context of a non-self mapping with the P-property, which is first introduced by Raj [6].

Definition 1.6 ([6])**.** Let (Y, Z) be a pair of nonempty subsets of a metric space (X, d) with $Y_0 \neq \emptyset$. Then the pair (Y, Z) is said to have the *P-property* if

$$\left.\begin{aligned} d(x_1, y_1) = d(Y, Z) \\ d(x_2, y_2) = d(Y, Z) \end{aligned}\right\} \implies d(x_1, x_1) = d(y_1, y_2),$$

where $x_1, x_2 \in Y$ and $y_1, y_2 \in Z$.

Definition 1.7 ([5]). Let (Y, Z) be a pair of nonempty subsets of a metric space (X, d). A mapping $J : Y \to Z$ is said to be a *Geraghty contraction* if there exists $\beta \in \mathcal{F}$ such that

$$d(Jx, Jy) \leq \beta(d(x, y))d(x, y), \tag{1.2}$$

for any $x, y \in Y$

In 2018, Komal et al. [7] proved the existence of the best proximity coincidence point for α-Geraghty contractions in a complete metric space with the P-property.

Definition 1.8 ([7]). Let (X, d) be a metric space and Y, Z be non-empty subsets of X, and let $\alpha : Y \times Y \to [0, +\infty)$ be a function. A mapping $J : Y \to Z$ is called *α-Geraghty contraction* if there exists $\beta \in \mathcal{F}$ such that for all $x, y \in Y$,

$$\alpha(x, y)d(Jx, Jy) \leq \beta(d(x, y))d(x, y).$$

Theorem 1.9 ([7]). *Let Y and Z be two nonempty closed subsets of a complete metric space (X, d) such that A_0 is nonempty and $g : Y \to Y$ is an isometry such that $Y_0 \subseteq g(Y_0)$, let $\alpha : Y \times Y \to [0, +\infty)$ be a function. Define a map $J : Y \to Z$ satisfying the following conditions:*

(i) *J is continuous α-Geraghty contraction;*
(ii) *J be an α-proximal admissible;*
(iii) *for each $x, y \in Y_0$ satisfying $d(x, Jy) = d(Y, Z)$ and $\alpha(y, x) \geq 1$;*
(iv) *$J(Y_0) \subseteq Z_0$ and the pair (Y, Z) has the P-property.*

Then there exists x in Y such that $d(gx, Jx) = d(Y, Z)$.

Moreover, motivated by Geraghty, Ayari [8] defined a class $\mathcal{B}$ of all mappings $\beta : [0, \infty) \to [0, 1]$ such that

$$\lim_{n\to\infty} \beta(t_n) = 1 \implies \lim_{n\to\infty} t_n = 0.$$

By using $\beta \in \mathcal{B}$, Ayari [8] introduced α-proximal Geraghty non-self mappings which is a generalization of the definition of Geraghty [2]. In addition, the author proved the existence and uniqueness of a best proximity point for such mappings.

Definition 1.10 ([9]). Given a mapping $J : Y \to Z$ and an isometry $g : Y \to Y$, the mapping J is said to *preserve isometric distance with respect to g* if

$$d(Jgx_1, Jgx_2) = d(Jx_1, Jx_2)$$

for all x_1 and x_2 in Y.

Lemma 1.11 ([10]). *Let (Y, Z) be a pair of nonempty closed subsets of a complete metric space (X, d) such that Y_0 is nonempty and (Y, Z) has the P-property. Then (Y_0, Z_0) is a closed pair of subsets of X.*

In addition to the above research, there are many researchers who are interested in studying best proximity point theorems, which can see details from references [6, 10–29].

Moreover, in Theorem 1.9, which is created by Komal et al. [7], the authors stated that g is an isometry. The main aim of this paper is to study coincidence best proximity point results for (α, g)-Geraghty contractive mappings on a complete metric space. By using this type of mappings, we prove the existence and uniqueness of a coincidence best proximity point for such mapings without an isometry of the mapping g under certain condition in a complete metric space.

2. Main Results

In this section, we prove the existence and uniqueness of a coincidence best proximity point in a complete metric space without an isometry of the mapping g.

Definition 2.1. Let (Y, Z) be a pair of nonempty subsets of a metric space (X, d), and let $\alpha : Y \times Y \to [0, +\infty)$ be a function and $g : Y \to Y$ be a self mapping. A nonself mapping $J : Y \to Z$ is said to be an *(α, g)-proximal admissible* if for all $x_1, x_2, u_1, u_2 \in Y$,

$$\left.\begin{aligned} &\alpha(gx_1, gx_2) \geq 1 \\ &d(gu_1, Jx_1) = d(Y, Z) \\ &d(gu_2, Jx_2) = d(Y, Z) \end{aligned}\right\} \implies \alpha(gu_1, gu_2) \geq 1.$$

Definition 2.2. Let $\alpha : Y \times Y \to [0, +\infty)$ be a function and $g : Y \to Y$ be a self mapping. An (α, g)-proximal admissible mapping $J : Y \to Z$ is siad to be *triangular (α, g)-proximal admissible* if $\alpha(gx, gy) \geq 1$ and $\alpha(gy, gz) \geq 1$ implies $\alpha(gx, gz) \geq 1$.

Definition 2.3. Let (Y, Z) be a pair of nonempty subsets of a metric space (X, d), and let $\alpha : Y \times Y \to [0, +\infty)$ be a function and $g : Y \to Y$ be a self mapping. A nonself mapping $J : Y \to Z$ is said to be an *(α, g)-Geraghty contractive mapping* if there exists $\beta \in \mathcal{B}$ such that for all $x, y \in Y$,

$$\alpha(gx, gy)d(Jx, Jy) \leq \beta(d(gx, gy))d(gx, gy). \tag{2.1}$$

Theorem 2.4. *Let Y and Z be non-empty closed subsets of a complete metric space (X, d) such that Y_0 is non-empty and the pair (Y, Z) has the P-property. Let $g : Y \to Y$ be a self mapping with $Y_0 \subseteq g(Y_0)$, and let $\alpha : Y \times Y \to [0, \infty)$ and $J : Y \to Z$ satisfy the following conditions:*

(i) *J is an (α, g)-Geraghty contractive mapping with $J(Y_0) \subseteq Z_0$;*
(ii) *J is a triangular (α, g)-proximal admissible;*
(iii) *There exist $x_0, x_1 \in Y_0$ such that $d(gx_1, Jx_0) = d(Y, Z)$ and $\alpha(gx_0, gx_1) \geq 1$;*

Then, it can establish a sequence $\{gx_n\}$ in Y_0 such that

$$d(gx_{n+1}, Jx_n) = d(Y, Z), \text{ for each } n \in \mathbb{N} \cup \{0\},$$

and then the sequence $\{gx_n\}$ converges to gx^*, for some $x^* \in Y_0$.

Proof. From the condition (iii), there are $x_0, x_1 \in Y_0$ such that

$$d(gx_1, Jx_0) = D \text{ and } \alpha(gx_0, gx_1) \geq 1.$$

Since $x_1 \in Y_0$ and $J(Y_0) \subseteq Z_0$, $Jx_1 \in Z_0$. There exits $u_1 \in Y_0$ such that $d(u_1, Jx_1) = D$. So, there exists $x_2 \in Y_0$ such that $u_1 = gx_2$ since $Y_0 \subseteq g(Y_0)$. It implies that

$$d(gx_2, Jx_1) = D.$$

Since $d(gx_1, Jx_0) = D = d(gx_2, Jx_1)$, $\alpha(gx_0, gx_1) \geq 1$ and J is an (α, g) proximal admissible, we have

$$\alpha(gx_1, gx_2) \geq 1.$$

Again, since $x_2 \in Y_0$ and $J(Y_0) \subseteq Z_0$, $Jx_2 \in Z_0$. There exits $u_2 \in Y_0$ such that $d(u_2, Jx_2) = D$. So, there exists $x_3 \in Y_0$ such that $u_2 = gx_3$ since $Y_0 \subseteq g(Y_0)$. It implies that

$$d(gx_3, Jx_2) = D.$$

Since $d(gx_2, Jx_1) = D = d(gx_3, Jx_2)$, $\alpha(gx_1, gx_2) \geq 1$ and J is an (α, g) proximal admissible, we have

$$\alpha(gx_2, gx_3) \geq 1.$$

Similarly to the above method, we get a sequence $\{gx_n\}$ in Y_0 satisfying

$$d(gx_{n+1}, Jx_n) = D \text{ and } \alpha(gx_n, gx_{n+1}) \geq 1, \text{ for each } n \in \mathbb{N} \cup \{0\}. \tag{2.2}$$

Since the pair (Y, Z) has the P-property, we have that

$$d(gx_n, gx_{n+1}) = d(Jx_{n-1}, Jx_n), \tag{2.3}$$

for each $n \in \mathbb{N}$. Next, we claim that

$$\lim_{n\to\infty} d(gx_{n-1}, gx_n) = 0. \tag{2.4}$$

From (2.2), (2.3) and J is an (α, g)-Geraghty contractive mapping, for each $n \in \mathbb{N}$, we obtain that

$$\begin{aligned} d(gx_n, gx_{n+1}) &= d(Jx_{n-1}, Jx_n) \\ &\leq \alpha(gx_{n-1}, gx_n) d(Jx_{n-1}, Jx_n) \\ &= \beta(d(gx_{n-1}, gx_n)) d(gx_{n-1}, gx_n) \\ &\leq d(gx_{n-1}, gx_n). \end{aligned} \tag{2.5}$$

That is, the sequence $\{d(gx_{n-1}, gx_n)\}$ is non-increasing. So, there exists $\gamma \geq 0$ such that

$$\lim_{n\to\infty} d(gx_{n-1}, gx_n) = \gamma. \tag{2.6}$$

If $\gamma = 0$, then (2.4) holds. Suppose that $\gamma > 0$, it is clear that $gx_{n-1} \neq gx_n$, for all $n \in \mathbb{N}$, and so $d(gx_{n-1}, gx_n) > 0$ for all $n \in \mathbb{N}$. By (2.5), we obtain that

$$\frac{d(gx_n, gx_{n+1})}{d(gx_{n-1}, gx_n)} \leq \beta(d(gx_{n-1}, gx_n)) \leq 1, \text{ for each } n \in \mathbb{N}. \tag{2.7}$$

From (2.6) and (2.7), we get that

$$\lim_{n\to\infty} \beta(d(gx_{n-1}, gx_n)) = 1.$$

By the condition of the function β, we can conclude that

$$\lim_{n\to\infty} d(gx_{n-1}, gx_n) = \gamma = 0,$$

which is a contradiction. Thus $\gamma = 0$, i.e., (2.4) holds. Next, we will show that the sequence $\{gx_n\}$ is a Cauchy sequence. Suppose that $\{gx_n\}$ is not a Cauchy sequence. Then there are subsequences $\{gx_{n_k}\}$ and $\{gx_{m_k}\}$ such that $m_k > n_k \geq k$ for each $k \in \mathbb{N}$, we have that

$$d(gx_{n_k}, gx_{m_k}) \geq \varepsilon. \tag{2.8}$$

Additionally, we can choose the smallest m_k satisfying (2.8) and $d(gx_{n_k}, gx_{m_k-1}) < \varepsilon$. By the triangle inequality, for each $k \in \mathbb{N}$, we have that

$$\begin{aligned} \varepsilon &\leq d(gx_{n_k}, gx_{m_k}) \\ &\leq d(gx_{n_k}, gx_{m_k-1}) + d(gx_{m_k-1}, gx_{m_k}) \\ &< \varepsilon + d(gx_{m_k-1}, gx_{m_k}). \end{aligned}$$

From (2.4), and by taking the limit as $k \to \infty$ in above inequality, we have that

$$\lim_{k\to\infty} d(gx_{n_k}, gx_{m_k}) = \varepsilon. \tag{2.9}$$

Consider

$$d(gx_{n_k}, gx_{m_k}) \le d(gx_{n_k}, gx_{n_k+1}) + d(gx_{n_k+1}, gx_{m_k+1}) + d(gx_{m_k+1}, gx_{m_k}) \tag{2.10}$$

and

$$d(gx_{n_k+1}, gx_{m_k+1}) \le d(gx_{n_k+1}, gx_{n_k}) + d(gx_{n_k}, gx_{m_k}) + d(gx_{m_k}, gx_{m_k+1}). \tag{2.11}$$

From (2.10), (2.11) and $d(gx_{n_k}, gx_{m_k}) \ge \varepsilon$, it implies that $d(gx_{n_k+1}, gx_{m_k+1}) \ge \varepsilon$. Again, using (2.11), we get that

$$\lim_{k\to\infty} d(gx_{n_k+1}, gx_{m_k+1}) = \varepsilon. \tag{2.12}$$

According to the fact that $\{gx_{m_k}\}$ and $\{gx_{n_k}\}$ are subsequences of $\{gx_n\}$ and using (2.2), for each $k \in \mathbb{N}$

$$d(gx_{n_k+1}, Jx_{n_k}) = D = d(gx_{m_k+1}, Jx_{m_k}).$$

Since the pair (Y, Z) has the P-property, we get that

$$d(gx_{n_k+1}, gx_{m_k+1}) = d(Jx_{n_k}, Jx_{m_k}), \text{ for each } k \in \mathbb{N}.$$

Again, from (2.2) and J is a triangular (α, g)-admissible, we obtain that

$$\alpha(gx_{n_k}, gx_{m_k}) \ge 1, \text{ for each } k \in \mathbb{N}.$$

Hence, for each $k \in \mathbb{N}$,

$$\begin{aligned} d(gx_{n_k+1}, gx_{m_k+1}) &= d(Jx_{n_k}, Jx_{m_k}) \\ &\le \alpha(gx_{n_k}, gx_{m_k}) d(Jx_{n_k}, Jx_{m_k}) \\ &\le \beta(d(gx_{n_k}, gx_{m_k})) d(gx_{n_k}, gx_{m_k}) \\ &\le d(gx_{n_k}, gx_{m_k}) \end{aligned}$$

because J is an (α, g)-Geraghty contractive mapping. From (2.8), we have that $d(gx_{n_k}, gx_{m_k}) > 0$. Then, we conclude that

$$\frac{d(gx_{n_k+1}, gx_{m_k+1})}{d(gx_{n_k}, gx_{m_k})} \le \beta(d(gx_{n_k}, gx_{m_k})) \le 1.$$

Using (2.9) and (2.12), we obtain that

$$1 = \frac{\varepsilon}{\varepsilon} \le \lim_{k\to\infty} \beta(d(gx_{n_k}, gx_{m_k})) \le 1,$$

that is, $\lim_{k\to\infty} \beta(d(gx_{n_k}, gx_{m_k})) = 1$. By the definition of the function β, we can conclude that

$$\lim_{n\to\infty} d(gx_{n_k}, gx_{m_k}) = 0 < \varepsilon,$$

which contradicts (2.9). Hence, the sequence $\{gx_n\}$ is a Cauchy sequence in Y_0. It implies that there exists $x^* \in Y_0$ such that $gx_n \to gx^*$ as $n \to \infty$ because Y_0 is a closed subset of a complete metric space (X, d) and $Y_0 \subseteq g(Y_0)$. ■

Next, we will give the first coincidence best proximity point theorem for an (α, g)-Geraghty contractive mapping in a complete metric space without an isometry of a mapping g.

Theorem 2.5. *Let X, Y, Z, Y_0, g, J be as in Theorem 2.4 and suppose that all hypotheses are true. Assume that for any sequences $\{gx_n\}$ in Y such that $\alpha(gx_n, gx_{n+1}) \geq 1$ for all $n \in \mathbb{N}$, if $gx_n \to gx^*$ for some $x^* \in Y$, then there exists a subsequence $\{gx_{n_k}\}$ of $\{gx_n\}$ such that $\alpha(gx_{n_k}, gx^*) \geq 1$ for all $n \in \mathbb{N}$. Then (g, J) has a coincidence best proximity point, i.e., there exists a point $x^* \in Y$ such that*

$$d(gx^*, Jx^*) = d(Y, Z).$$

Morover, if g is one-to-one and $\alpha(gx^*, gy^*) \geq 1$ for any coincidence best proximity point $x^*, y^* \in Y$, then (g, J) has a unique coincidence best proximity point.

Proof. By Theorem 2.4, we can establish a sequence $\{gx_n\}$ in Y_0 such that

$$d(gx_{n+1}, Jx_n) = d(Y, Z) \text{ and } \alpha(gx_n, gx_{n+1}) \geq 1, \text{ for each } n \in \mathbb{N} \cup \{0\}.$$

Moreover, the sequence $\{gx_n\}$ converges to gx^*, for some $x^* \in Y_0$. By the assumption, we get that there exists a subsequence $\{gx_{n_k}\}$ of $\{gx_n\}$ such that $\alpha(gx_{n_k}, gx^*) \geq 1$, for each $k \in \mathbb{N}$. Sincs J is an (α, g)-Geraghty contractive mapping,

$$\begin{aligned} d(Jx_{n_k}, Jx^*) &\leq \alpha(gx_{n_k}, gx^*) d(Jx_{n_k}, Jx^*) \\ &\leq \beta(d(gx_{n_k}, gx^*)) d(gx_{n_k}, gx^*) \\ &\leq d(gx_{n_k}, gx^*). \end{aligned}$$

By the triangular inequality, we obtain that

$$\begin{aligned} d(gx^*, Jx^*) &\leq d(gx^*, gx_{n_k+1}) + d(gx_{n_k+1}, Jx_{n_k}) + d(Jx_{n_k}, Jx^*) \\ &\leq d(gx^*, gx_{n_k+1}) + D + d(gx_{n_k}, gx^*), \end{aligned}$$

for each $k \in \mathbb{N}$. Taking the limit as $k \to \infty$ in above inequality, we get that $d(gx^*, Jx^*) \leq D$. By the fact that $gx^* \in Y$ and $Jx^* \in Z$, we get that $D \leq d(gx^*, Jx^*)$. It implies that

$$d(gx^*, Jx^*) = D,$$

that is, x^* is a coincidence best proximity point of the pair (g, J).

Next, we will show that (g, J) has a unique coincidence best proximity point. Suppose that there exists a coincidence best proximity point $y^* \in Y$ such that $x^* \neq y^*$ and

$$d(gy^*, Jy^*) = D.$$

From the pair (Y, Z) has the P-property and $d(gx^*, Jx^*) = D = d(gy^*, Jy^*)$, we have that

$$d(gx^*, gy^*) = d(Jx^*, Jy^*).$$

Since J is an an (α, g)-Geraghty contractive mapping and $\alpha(gx^*, gy^*) \geq 1$,

$$\begin{aligned} d(gx^*, gy^*) &= d(Jx^*, Jy^*) \\ &\leq \alpha(gx^*, gy^*) d(Jx^*, Jy^*) \\ &\leq \alpha(gx^*, gy^*) d(gx^*, gy^*) \\ &\leq \beta(d(gx^*, gy^*)) d(gx^*, gy^*) \\ &\leq d(gx^*, gy^*). \end{aligned}$$

Since g is one-to-one and $x^* \neq y^*$, we obain that $d(gx^*, gy^*) > 0$, and so $\beta(d(gx^*, gy^*)) = 1$. It implies that $d(gx^*, gy^*) = 0$, i.e., $gx^* = gy^*$. Again, since g is one-to-one, $x^* = y^*$, which is a contradiction. Therefore, (g, J) has a unique coincidence best proximity point. ∎

Now, we give an example to illustrate Theorem 2.5, where g is not an isometry.

Example 2.6. Consider $X = \mathbb{R}^2$, with the usual metric d. Let $Y = \{(0, x) : x \in [0, \infty)\}$ and $Z = \{(1, y) : y \in [0, \infty)\}$. Obviously, $d(Y, Z) = 1$, $Y_0 = Y$ and $Z_0 = Z$. Define $J : Y \to Z$ by

$$J(0, x) = (1, \ln(x + 1)), \text{ for all } (0, x) \in Y,$$

and let $g : Y \to Y$ be defined by

$$g(0, x) = \begin{cases} (0, 2), & \text{if } x = 1, \\ (0, 3), & \text{if } x = 2, \\ (0, 1), & \text{if } x = 3, \\ (0, x), & \text{otherwise.} \end{cases}$$

It is easy to see that $J(Y_0) \subseteq Z_0$, g is one-to-one and $Y_0 \subseteq g(Y_0)$. Moreover, It is easy to verify that the pair (Y, Z) has the P-property.

Let $\alpha : \mathbb{R}^2 \times \mathbb{R}^2 \to [0, \infty)$ be a function defined by

$$\alpha((a_1, b_2), (a_2, b_2)) = \begin{cases} 1, & \text{if } a_1 = a_2 = 0 \text{ and } 0 \leq b_1, b_2 < \infty, \\ 0, & \text{otherwise.} \end{cases}$$

Note that the conditions (ii) and (iii) in Theorem 2.4 is true. Next, we will show that J is an (α, g)-Geraghty contractive mapping. Let

$$\beta(t) = \begin{cases} \dfrac{\ln(1 + t)}{t}, & \text{if } t \neq 0, \\ 1, & \text{if } t = 0. \end{cases}$$

Note that $\beta \in \mathcal{B}$. By the definition of the function β, we can see that $\alpha((0, a), (0, b)) = 1$ for all $(0, a), (0, b) \in Y$. Let $(0, a), (0, b) \in Y$. If $a = b$, then we are done. Suppose that $a \neq b$ and $b < a$. Hence

$$\begin{aligned}
\alpha(g(0, a), g(0, b))\, d(J(0, a), J(0, b)) &= 1 \cdot d(J(0, a), J(0, b)) \\
&= |\ln(a + 1) - \ln(b + 1)| \\
&= \left| \ln\left(\frac{a + 1}{b + 1}\right) \right| \\
&= \left| \ln\left(\frac{(b + 1) + (a - b)}{b + 1}\right) \right| \\
&= \left| \ln\left(1 + \frac{a - b}{b + 1}\right) \right| \\
&\leq \ln(1 + |a - b|) \\
&= \frac{\ln(1 + |a - b|)}{|a - b|} \cdot |a - b| \\
&= \frac{\ln(1 + d(g(0, a), g(0, b)))}{d(g(0, a), g(0, b))} \cdot d(g(0, a), g(0, b)) \\
&= \beta(d(g(0, a), g(0, b)))d(g(0, a), g(0, b)).
\end{aligned}$$

Similarly to the above inequality, we can also conclude the case $a < b$. Therefore, J is an (α, g)-Geraghty contractive mapping. Finally, it is clear that for any sequences $\{gx_n\}$ in Y such that $\alpha(gx_n, gx_{n+1}) \geq 1$ for all $n \in \mathbb{N}$, if $gx_n \to gx^*$ for some $x^* \in Y$, then there exists a subsequence $\{gx_{n_k}\}$ of $\{gx_n\}$ such that $\alpha(gx_{n_k}, gx^*) \geq 1$ for all $n \in \mathbb{N}$. Moreover, $(0, 0)$ is the unique best proximity coincidence point of the pair (g, J).

Definition 2.7. Given mappings $J : Y \to Z$ and $g : Y \to Y$, the mapping J is called *preserve distance with respect to* g if

$$d(Jgx_1, Jgx_2) = d(Jx_1, Jx_2)$$

for all x_1 and x_2 in Y.

Theorem 2.8. *Let X, Y, Z, Y_0, g, J be as in Theorem 2.4 and suppose that all hypotheses are true. Assume that J is continuous and preserves distance with respect to g. Then (g, J) has a coincidence best proximity point, i.e., there exists a point $x^* \in Y$ such that*

$$d(gx^*, Jx^*) = d(Y, Z).$$

Morover, if g is one-to-one and $\alpha(gx^*, gy^*) \geq 1$ for any coincidence best proximity point $x^*, y^* \in Y$, then (g, J) has a unique coincidence best proximity point.

Proof. By Theorem 2.4, we can establish a sequence $\{gx_n\}$ in Y_0 such that

$$d(gx_{n+1}, Jx_n) = d(Y, Z) \text{ and } \alpha(gx_n, gx_{n+1}) \geq 1, \text{ for each } n \in \mathbb{N} \cup \{0\}.$$

Moreover, the sequence $\{gx_n\}$ converges to gx^*, for some $x^* \in Y_0$. Since J is continuous, $Jgx_n \to Jgx^*$ as $n \to \infty$. This is,

$$d(Jgx_n, Jgx^*) \to 0 \text{ as } n \to \infty.$$

But $d(Jgx_n, Jgx^*) = d(Jx_n, Jx^*)$ since J is preserves distance with respect to g. It implies that

$$d(Jx_n, Jx^*) \to 0 \text{ as } n \to \infty.$$

Using $d(gx_{n+1}, Jx_n) = D$ for all $n \in \mathbb{N} \cup \{0\}$, we get that

$$d(gx^*, Jx^*) = D.$$

that is, x^* is a coincidence best proximity point of the pair (g, J).

By the proof of Theorem 2.5, we can conclude that (g, J) has a unique coincidence best proximity point. This completes the proof. ∎

3. Some Particular Cases

As results of our main theorems, we obtian some results which is the specific case of our main results.

Definition 3.1 ([30]). Let (Y, Z) be a pair of nonempty subsets of a metric space (X, d), and let $\alpha : Y \times Y \to [0, +\infty)$ be a function. A nonself mapping $J : Y \to Z$ is said to be an *α-proximal admissible* if for all $x_1, x_2, u_1, u_2 \in Y$,

$$\left.\begin{aligned} \alpha(x_1, x_2) &\geq 1 \\ d(u_1, Jx_1) &= d(Y, Z) \\ d(u_2, Jx_2) &= d(Y, Z) \end{aligned}\right\} \implies \alpha(u_1, u_2) \geq 1.$$

Definition 3.2. Let $\alpha : Y \times Y \to [0, +\infty)$ be a function. An α-proximal admissible mapping $J : Y \to Z$ is siad to be *triangular α-proximal admissible* if $\alpha(x, y) \geq 1$ and $\alpha(y, z) \geq 1$ implies $\alpha(x, z) \geq 1$.

In Theorems 2.5 and 2.8, if g is the identity mapping, we obtain the following corollary as follows.

Corollary 3.3. *Let Y and Z be non-empty closed subsets of a complete metric space (X, d) such that Y_0 is non-empty and the pair (Y, Z) has the P-property. Assume that $\alpha : Y \times Y \to [0, \infty)$ and $J : Y \to Z$ satisfy the following conditions:*

(i) *J is an α-Geraghty contraction with $J(Y_0) \subseteq Z_0$;*
(ii) *J is a triangular α-proximal admissible;*
(iii) *There exist $x_0, x_1 \in Y_0$ such that $d(x_1, Jx_0) = d(Y, Z)$ and $\alpha(x_0, x_1) \geq 1$;*
(iv) *Either (a) or (b) is true;*
  (a) J is continuous;
  (b) For any sequences $\{x_n\}$ in Y such that $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N}$, if $x_n \to x^*$ for some $x^* \in Y$, then there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\alpha(x_{n_k}, x^*) \geq 1$ for all $n \in \mathbb{N}$.

Then J has a best proximity point, i.e., there exists a point $x^* \in Y$ such that

$$d(x^*, Jx^*) = d(Y, Z).$$

Moreover, if $\alpha(x^*, y^*) \geq 1$ for any best proximity point $x^*, y^* \in Y$, then J has a unique best proximity point.

Definition 3.4. Let (Y, Z) be a pair of nonempty subsets of a metric space (X, d) and $g : Y \to Y$ be a self mapping. A nonself mapping $J : Y \to Z$ is said to be an *Geraghty contractive mapping with respect to g* if there exists $\beta \in \mathcal{B}$ such that for all $x, y \in Y$,

$$d(Jx, Jy) \leq \beta(d(gx, gy))d(gx, gy).$$

By taking $\alpha(x, y) = 1$ for all $x, y \in Y$, we immediately obtain the following corollary as follows.

Corollary 3.5. *Let Y and Z be non-empty closed subsets of a complete metric space (X, d) such that Y_0 is non-empty and the pair (Y, Z) has the P-property. Assume that $g : Y \to Y$ be a self mapping with $Y_0 \subseteq g(Y_0)$ and $J : Y \to Z$ satisfy the following conditions:*

(i) *J is an Geraghty contractive mapping with respect to g such that $J(Y_0) \subseteq Z_0$;*
(ii) *There exist $x_0, x_1 \in Y_0$ such that $d(gx_1, Jx_0) = d(Y, Z)$;*
(iii) *g is one-to-one.*

Then (g, J) has a unique coincidence best proximity point, i.e., there exists a unique point $x^* \in Y$ such that

$$d(gx^*, Jx^*) = d(Y, Z).$$

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