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A Note on Interaction between Groups and Convergence Spaces

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Abstract This note is devoted to present the idea to explore groups from the viewpoint of convergence spaces by identifying their Cayley graphs with a finitely generated pretopological space. The notion of continuity for groups is explored with some examples.

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1. Introduction

Convergence spaces, based on the concept of filters, generalize topological spaces. A filter $\mathcal{F}$ is a family of proper ($\phi \notin \mathcal{F}$) subsets of a set X, which is closed under supersets (i.e., isotone ($F_1 \supset F_2 \in \mathcal{F} \Rightarrow F_1 \in \mathcal{F}$)) and under finite intersection ($F_1 \in \mathcal{F}, F_2 \in \mathcal{F} \Rightarrow F_1 \cap F_2 \in \mathcal{F}$). Following [1] we denote by $\mathbb{F}X$ the set of all filters in a set X. Some basic examples of filters include the collection of all supersets of a given nonempty subset, the collection of all cofinite subsets of an infinite set, and the collection of all the neighbourhoods of a point of a topological space.

A family $\mathcal{A}$ of subsets of a set is said to be isotone if $A \in \mathcal{A}$ and $A \subset B$ implies $B \in \mathcal{A}$. Isotonisation (written $\mathcal{A}^{\uparrow}$) of $\mathcal{A}$ is the least isotone family that contains $\mathcal{A}$. For $\mathcal{F} \in \mathbb{F}(X)$ and $\mathcal{G} \in \mathbb{F}(Y)$ define,

$$\mathcal{F} \times \mathcal{G} = \{F \times G : F \in \mathcal{F} \text{ and } G \in \mathcal{G}\}.$$

The isotonisation of $\mathcal{F} \times \mathcal{G}$ is called the product filter.

Based on the concept of a filter the convergence spaces are defined as follows:

Definition 1.1. Let X be a set, and let λ be an arbitrary relation between X and $\mathbb{F}X$. This relation is called a convergence on that set if for $\mathcal{F}_1, \mathcal{F}_2$ in $\mathbb{F}X$ and x in X the following conditions hold:
(i) **Centred:** $x^{\uparrow} \in \lambda(x)$,
(ii) **Isotone**: If $\mathcal{F}_1 \in \lambda(x)$ and $\mathcal{F}_1 \leq \mathcal{F}_2$, then $\mathcal{F}_2 \in \lambda(x)$ and

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(iii) **Finitely deep**: If $\mathcal{F}_1, \mathcal{F}_2 \in \lambda(x)$ then, $\mathcal{F}_1 \cap \mathcal{F}_2 \in \lambda(x)$.
A *convergence space* is the pair (X, λ).

This relation is also denoted by $\mathcal{F} \underset{\lambda}{\to} x$ (or $\mathcal{F} \to x$, if no ambiguity is possible) and we say $\mathcal{F}$ *converges* to x, or x is the *limit* of $\mathcal{F}$, whenever $\mathcal{F} \in \lambda(x)$. The convergence is called *Hausdorff* if every filter on X converge to at most one point. Further, a function f between two convergence spaces X and Y is *continuous* at x if $f(\mathcal{F})$ converge to $f(x)$ in Y whenever $\mathcal{F}$ converge to a point x in X. The function is continuous if it is continuous at every point of X.

For a detailed account and other terms related to the convergence spaces we refer the reader to [1, 2].

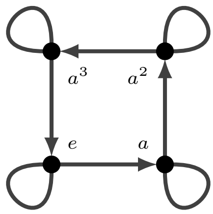
By means of convergence spaces, discrete and continuous models of computation are unified [3] and, in turn, the definition of differential is extended to discrete structures [4–6]. Since the category of convergence spaces includes the category of reflexive directed graphs, groups, via their respective Cayley graphs, are representable by convergence spaces. In this paper we present the idea to represent the discrete groups as convergence spaces.

2. Groups with Convergence

Definition 2.1. [4, Definition 3.20] Let Γ be a subset of a group G such that each element of G is a product of elements of Γ and no element of Γ is redundant, i.e., no element of Γ can be written as product of its other elements. The edge colored Cayley digraph (directed graph) for G generated by Γ is the directed graph C such that the vertex set of C is G and the edge set of C is $E = \{(g, g\gamma) : g \in G\ ,\ \gamma \in \Gamma\}$. The edges are colored by $j : E \to \Gamma$. Further, in addition to the condition that $\gamma \in \Gamma$ if another condition $\gamma = id$ is added to the definition of Cayley graph we obtain a reflexive Cayley digraph as follows: The reflexive Cayley digraph for G generated by Γ is the reflexive digraph C such that the vertex set of C is G and the edge set of C is $\{(g, g\gamma) : g \in G$ and $\gamma = id$ or $\gamma \in \Gamma\}$.

Next, we present some examples, how the convergence space representation of certain groups can be obtained. It is noteworthy that the use of graph neighborhoods [4] rather than convergence of arbitrary filters (see [7] for details about the finitely generated pretopological space) is sufficient for studying reflective digraphs, but as we are more interested in some general ideas we use filter convergence for the convergence space representations. Further, the convergence structure defined via Cayley graphs is not necessarily compatible (see. [8] for details regarding convergence groups) with group structure and is not necessarily Hausdroff. So this kind of approach provides a view towards "groups with convergence" in place of "convergence groups" (The latter are those groups with a convergence structure compatible with the algebraic structure of groups (see [9, 10])).

Example 2.2 (The Cyclic group of order four, Z_4)**.** Consider the group Z_4 with elements $\{e, a, a^2, a^3\}$. In view of definition 2.1 and [5, Definition 2.1] the reflexive directed graph of Z_4 with respect to the generator $\{a\}$ is:



The graph neighbourhood of vertices is given as

$$gnbd(e) = \{e, a\}; \quad gnbd(a) = \{a, a^2\}; \quad gnbd(a^2) = \{a^2, a^3\}; \quad gnbd(a^3) = \{a^3, e\}.$$

And the convergence (denoted λ_{Z_4}) generated by this digraph is defined as follows:
The only convergent filters are:

$$\{e\}^{\uparrow} \to \{e, a^3\}; \quad \{a\}^{\uparrow} \to \{e, a\}; \quad \{a^2\}^{\uparrow} \to \{a, a^2\}; \quad \{a^3\}^{\uparrow} \to \{a^2, a^3\}$$
$$\{e, a\}^{\uparrow} \to \{e\}; \quad \{e, a^3\}^{\uparrow} \to \{a^3\}; \quad \{a, a^2\}^{\uparrow} \to \{a\}; \quad \{a^2, a^3\}^{\uparrow} \to \{a^2\}.$$

As the Cayley graph of the finite cyclic group is the cycle graph so, as presented in this example, the convergence representation can be obtained for any finite cyclic group.

In view of the above example, it is easy to show that for a reflexive digraph representation of the group G (corresponding to a fixed generator set) a convergence structure λ corresponding to the algebraic structure of the group can be obtained. We call the group with this convergence structure (G, λ) as a *group with convergence*.

Remark 2.3. If the convergence structure is compatible with the group structure then notation of groups with convergence and convergence groups coincide.

Now we define a continuous function between two abstract groups G_1 and G_2.

Definition 2.4. Let $f : G_1 \to G_2$ be a function and $\mathcal{F}$ be a filter on G_1. The image of a filter $\mathcal{F}$ under f is the filter generated by $\{f(F) : F \in \mathcal{F}\}$.

Definition 2.5. A function $f : G_1 \to G_2$ is said to be *continuous* if it is a continuous function between the convergence spaces (G_1, λ_1) and (G_2, λ_2).

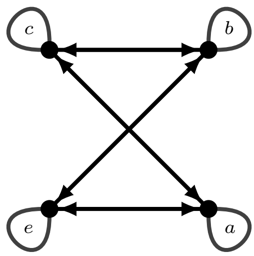
It should be noted that continuity is equivalent to preservation of graph neighborhoods; in other words, a function between two groups is continuous if and only if it is a graph morphism.

Example 2.6. Consider the group Z_4 with the convergence λ_{Z_4}. The identity function $i : Z_4 \to Z_4$ is continuous.

Example 2.7. The non-cyclic abelian group, Klein four-group (V_4) Let us consider the case that $V_4 = \{e, a, b, c\}$ is generated by $\{a, b\}$. The product of elements of the generator set with the elements of the group is given as:

Generator	e	a	b	c
a	a	e	c	b
b	b	c	e	a

In view of definition 2.1 the reflexive diagraph of V_4 is:



The graph neighbourhood of vertices is given as:

$$gnbd(e) = \{e, a, b\}; \quad gnbd(a) = \{a, e, c\}; \quad gnbd(b) = \{c, e, b\}; \quad gnbd(c) = \{b, a, c\}.$$

The convergence (denoted λ_{V_4}) generated (the filters except these do not converge) by this graph is:

$$\{e\}^{\uparrow} \to \{e, a, c\}; \quad \{a\}^{\uparrow} \to \{e, a, b\}; \quad \{b\}^{\uparrow} \to \{e, b, c\};$$

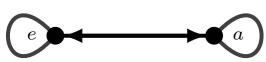
$$\{c\}^{\uparrow} \to \{a, b, c\}; \quad \{e, a\}^{\uparrow} \to \{e, a\}; \quad \{e, b\}^{\uparrow} \to \{e, c\};$$

$$\{e, c\}^{\uparrow} \to \{a, c\}; \quad \{a, b\}^{\uparrow} \to \{e, b\}; \quad \{a, c\}^{\uparrow} \to \{a, b\};$$

$$\{b, c\}^{\uparrow} \to \{b, c\}; \quad \{e, a, b\}^{\uparrow} \to \{e\}; \quad \{e, a, c\}^{\uparrow} \to \{a\};$$

$$\{e, b, c\}^{\uparrow} \to \{c\}; \quad \{a, b, c\}^{\uparrow} \to \{b\}.$$

Now let us consider another representation of V_4. The directed reflexive diagraph representation is



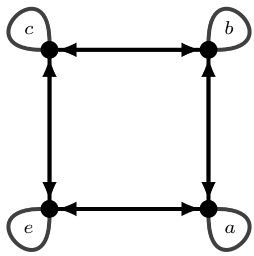
The graph neighbourhood of vertices is given by:

$$gnbd(e) = \{e, a\}; \quad gnbd(a) = \{e, a\}.$$

The convergence (denoted λ_{Z_2}) generated by this graph is given by

$$\{e\}^{\uparrow} \to \{e, a\}; \quad \{a\}^{\uparrow} \to \{e, a\}; \quad \{e, a\}^{\uparrow} \to \{e, a\};$$

Now the product $Z_2 \oplus Z_2$ (see [11]) will have Cayley graph representation



The graph neighbourhood of vertices is given as

$$gnbd(e) = \{e, a, c\}; \quad gnbd(a) = \{a, b, e\}; \quad gnbd(b) = \{a, b, c\}; \quad gnbd(c) = \{b, c, e\}.$$

The convergence (denoted $\lambda_{Z_2 \oplus Z_2}$) generated by this graph is

$$\{e\}^{\uparrow} \to \{e, a, c\}; \quad \{a\}^{\uparrow} \to \{e, a, c\}; \quad \{b\}^{\uparrow} \to \{a, b, c\};$$

$$\{c\}^{\uparrow} \to \{e, b, c\}; \quad \{e, a\}^{\uparrow} \to \{e, a\}; \quad \{e, b\}^{\uparrow} \to \{a, c\};$$

$$\{e, c\}^{\uparrow} \to \{e, c\}; \quad \{a, b\}^{\uparrow} \to \{a, b\}; \quad \{a, c\}^{\uparrow} \to \{e, b\};$$

$$\{b, c\}^{\uparrow} \to \{b, c\}; \quad \{e, a, b\}^{\uparrow} \to \{a\}; \quad \{e, a, c\}^{\uparrow} \to \{e\};$$

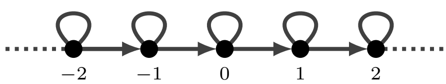
$$\{e, b, c\}^{\uparrow} \to \{c\}; \quad \{a, b, c\}^{\uparrow} \to \{b\}.$$

Example 2.8. Consider the identity map $i : (V_4, \lambda_{V_4}) \to (V_4, \lambda_{Z_2 \oplus Z_2})$. We have, $\{a\}^{\uparrow} \xrightarrow[\lambda_{V_4}]{} \{e, a, b\}$ and $\{a\}^{\uparrow} \xrightarrow[\lambda_{Z_2 \oplus Z_2}]{} \{e, a, c\}$. So, this map is not continuous. Similarly, the identity map $i : (V_4, \lambda_{Z_2 \oplus Z_2}) \to (V_4, \lambda_{V_4})$ is also not continuous. Hence, we can conclude that the convergence generated by λ_{V_4} and $\lambda_{Z_2 \oplus Z_2}$ are different.

Next we present some elementary examples of the convergence structure associated with the infinite groups.

Example 2.9. An infinite cyclic group (Additive group of integers $\mathbb{Z}$)

Consider the additive group of integers with the generator $\{1\}$. The reflexive Cayley digraph of $\mathbb{Z}$ generated by $\{1\}$ is:



The graph neighbourhood of vertices is

$$\ldots\ gnbd(-4) = \{-4, -3\}; \quad gnbd(-3) = \{-3, -2\}; \quad gnbd(-2) = \{-2, -1\}$$

$$gnbd(-1) = \{-1, 0\}; \quad gnbd(0) = \{0, 1\}; \quad gnbd(1) = \{1, 2\}$$

$$gnbd(2) = \{2, 3\}; \quad gnbd(3) = \{3, 4\}; \quad gnbd(4) = \{4, 5\}\ \ldots.$$

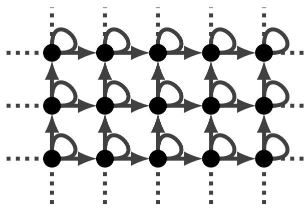
In general the graph neighbourhood of any vertex is given as:

$$\forall m \in \mathbb{Z}, \ \ gnbd(m) = \{m, m + 1\}.$$

So, the convergence generated by this diagraph is:
For $m \in \mathbb{Z}$, the convergent filters are:

$$\{m\}^{\uparrow} \to \{m-1, m\}$$
$$\{m, m+1\}^{\uparrow} \to \{m\}.$$

Example 2.10. An infinite non-cyclic group ($\mathbb{Z} \oplus \mathbb{Z}$) Consider the group $\mathbb{Z} \oplus \mathbb{Z}$ with generator set $\{(1,0),(0,1)\}$. The reflexive Cayley diagraph of this group is:



The graph neighbourhood of any vertex of this graph is:

$$\forall m, n \in \mathbb{Z}, \ \ gnbd((m,n)) = \{(m+1,n),(m,n),(m,n+1)\}.$$

The convergence generated by the Cayley graph for $\mathbb{Z} \oplus \mathbb{Z}$ is given by the following convergent filters:

$$\{(m,n)\}^{\uparrow} \to \{(m-1,n),(m,n),(m,n-1)\};$$
$$\{(m,n),(m+1,n)\}^{\uparrow} \to \{(m,n)\};$$
$$\{(m,n),(m,n+1)\}^{\uparrow} \to \{(m,n)\};$$
$$\{(m+1,n),(m,n),(m,n+1)\}^{\uparrow} \to \{(m,n)\}.$$

3. Conclusion

This note is devoted to present how the abstract groups can be given a convergence space representation (groups with convergence) and hence, this gives us an idea to explore the algebraic properties of the groups from a convergence space point of view.

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