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A Hybrid Algorithm for Pseudo-contractive Mappings

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Abstract In this paper, we present a hybrid technique for pseudo-contractive mappings in Hilbert spaces and prove a convergence theorem.

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1. Introduction

Let C be a nonempty closed convex subset of a real Hilbert space H. Let T be a self-mapping of C and $F(T)$ be a set of all fixed point of T.

A mapping $T : C \to C$ is said to be strict pseudo-contraction [1] if there exists a constant $0 \leq k < 1$ such that

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + k\|(I - T)x - (I - T)y\|^2, \text{ for all } x, y \in C.$$

If $k = 1$, then T is said to be a pseudo-contraction, i.e,

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + \|(I - T)x - (I - T)y\|^2,$$

is equivalent to,

$$\langle (I - T)x - (I - T)y, x - y\rangle \geq 0, \text{ for all } x, y \in C.$$

A mapping $T : C \to C$ is said to be nonexpansive, if $\|Tx-Ty\| \leq \|x-y\|$, for all $x, y \in C$. And $T : C \to C$ is L-Lipschitzian mapping if there exists a contant $L \geq 0$ such that

$$\|Tx - Ty\| \leq L\|x - y\|, \text{ for all } x, y \in C.$$

Example 1.1. [2] Let $H = (\mathbb{R}^2, \langle .,.\rangle)$, where $\mathbb{R}$ be the set of real numbers and $\langle .,.\rangle$ be defined by the dot product on H. If $x = (x_1, x_2) \in H$, define $x^{\perp} = (x_2, -x_1)$, let $C = \{x \in H : \|x\| = \sqrt{{x_1}^2 + {x_2}^2} \leq 5\}$ and $T : C \to C$ be defined by $Tx = x + x^{\perp}$. Then C is a nonempty closed convex subset of a real Hilbert H and T is a Lipschitz with $L = \sqrt{2}$ and pseudo-contractive and $F(T) = \{(0, 0)\}$.

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Nonexpansive mapping is extended by the class of stringent pseudo-contractions. It's worth noting that T is nonexpansive only if it's a 0-strict pseudo-contraction. The stringent pseudo-contractive mapping falls under the pseudo-contractive mapping category. Nonexpansive mappings, on the other hand, entail L-Lipschitzian mappings. Iterative approaches for nonexpansive mappings have recently received a lot of attention; see [3–6] and the references therein for further information. Although Browder and Petryshyn began their work in 1967, iterative approaches for strictly pseudo-contractive mappings are significantly less established than those for nonexpansive mappings. In addressing inverse problems, rigorously pseudo-contractive mappings, on the other hand, offer more potent applications than nonexpansive mappings; see Scherzer [7]. As a result, developing iterative methods for strictly pseudo-contractive mappings is intriguing.

The hybrid iterative approach for pseudo-contractive mapping in Hilbert spaces was introduced by Y. H. Yao et al. [8] in 2009:

Let C be a nonempty closed convex subset of a real Hilbert space H. Let $T : C \to C$ be a pseudo-contractive mapping. Assume that $\{\alpha_n\}$ is a sequence in (0,1). Let $x_0 \in H$. For $C_1 = C$ and $x_1 = P_{C_1}x_0$, define a sequence $\{x_n\}$ of C as follow:

$$\langle 1\rangle \begin{cases} y_n = (1-\alpha_n)x_n + \alpha_n T x_n, \\ C_{n+1} = \{z \in C_n : \|\alpha_n(I-T)y_n\|^2 \leq 2\alpha_n\langle x_n - z, (I-T)y_n\rangle\}, \\ x_{n+1} = P_{C_{n+1}}x_0, n \geq 1. \end{cases}$$

They proved that if C is a nonempty closed convex subset of a real Hilbert space H. Let $T : C \to C$ be a L-Lipschitz pseudo-contractive mapping such that $F(T) \neq \emptyset$. Assume $\{\alpha_n\}$ is a sequence in $[a, b]$ for some $a, b \in (0, \frac{1}{1+L})$, then the sequence $\{x_n\}$ generated by $\langle 1\rangle$ converges strongly to $P_{F(T)}x_0$.

Tang et al. [9] applied the hybrid algorithm $\langle 1\rangle$ to the Ishikawa iterative process [10] as follows in 2011:

Let C be closed convex subset of a real Hilbert space H. Let $T : C \to C$ be pseudo-contractive mapping. Assume that $\{\alpha_n\}$, $\{\beta_n\}$ are sequences in [0,1]. Let $x_0 \in H$. For $C_1 = C$ and $x_1 = P_{C_1}x_0$, define a sequence $\{x_n\}$ of C as follow:

$$\langle 2\rangle \begin{cases} y_n = (1-\alpha_n)x_n + \alpha_n T z_n, \\ z_n = (1-\beta_n)x_n + \beta_n T x_n, \\ C_{n+1} = \{z \in C_n : \|\alpha_n(I-T)y_n\|^2 \leq 2\alpha_n\langle x_n - z, (I-T)y_n\rangle \\ \qquad\qquad +2\alpha_n\beta_n L\|x_n - Tx_n\|\|y_n - x_n + \alpha_n(I-T)y_n\|\}, \\ x_{n+1} = P_{C_{n+1}}x_0, n \geq 1. \end{cases}$$

They proved that if C be a nonempty closed convex subset of a real Hilbert space H. Let $T : C \to C$ be a L-Lipschitz pseudo-contractive mapping with $L \geq 1$ such that $F(T) \neq \emptyset$. Assume that $\{\alpha_n\}, \{\beta_n\}$ are sequence in $(0, 1)$ satisfying: (i) $b \leq \alpha_n < \alpha_n(L + 1)(1 + \beta_n L) < a < 1$ for some $a, b \in (0, 1)$; (ii) $\lim_{n\to\infty} \beta_n = 0$, then the sequence $\{x_n\}$ generated by $\langle 2\rangle$ converges strongly to $P_{F(T)}x_0$.

In 2011, Phuengrattana and Suantai [11] introduced the SP-iteration process is defined by the sequence $\{x_n\}$,

$$\langle 3\rangle \begin{cases} x_1 \in C, \\ z_n = (1-\gamma_n)x_n + \gamma_n T x_n, \\ y_n = (1-\beta_n)z_n + \beta_n T z_n, \\ x_{n+1} = (1-\alpha_n)y_n + \alpha_n T y_n, \quad n \geq 1, \end{cases}$$

where $\{\alpha_n\}, \{\beta_n\}$ and $\{\gamma_n\}$ are sequence in $[0, 1]$.

If $\gamma_n = \beta_n = 0$ for all n, then the iteration $\langle 3 \rangle$ reduce to Mann iteration [12],

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n T x_n, \quad n \geq 1,$$

where $\{\alpha_n\} \subseteq [0,1]$.

We are motivated and inspired by the work of Phuengrattana and Suantai [11] and Tang et al. [9]. We generalize the hybrid algorithm $\langle 2 \rangle$ to the SP-iterative process $\langle 3 \rangle$, which is defined by the following.

Let C be a nonempty closed convex subset of a real Hilbert space H. Let $T : C \to C$ be a pseudo-contractive mapping. Assume that $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$ are sequences in [0,1]. Let $x_0 \in H$. For $C_1 = C$ and $x_1 = P_{C_1}x_0$ define a sequence $\{x_n\}$ of C as follow:

$$\langle 4 \rangle \begin{cases} z_n = (1-\gamma_n)x_n + \gamma_n T x_n, \\ y_n = (1-\beta_n)z_n + \beta_n T z_n, \\ w_n = (1-\alpha_n)y_n + \alpha_n T y_n, \\ C_{n+1} = \{z \in C_n : \|\alpha_n(I-T)w_n\|^2 \leq 2\alpha_n\langle x_n - z, (I-T)w_n\rangle \\ \quad +2(\gamma_n + \beta_n(1+\gamma_n L))(1+\alpha_n(1+L))\|x_n - Tx_n\|\|w_n - x_n + \alpha_n(I-T)w_n\| \\ x_{n+1} = P_{C_{n+1}}x_0, n \geq 1. \end{cases}$$

Further, we prove the strong convergence of the hybrid algorithm $\langle 4 \rangle$ for Lipschitz pseudo-contractive mappings in Hilbert spaces.

2. Preliminaries

We'll collect some helpful results in this section, which will be used in the next phase.

Let H be a real Hilbert space with inner product $\langle .,. \rangle$ and norm $||.||$ and let C be a closed convex subset of H.

Recall that, the nearest point projection P_C form H onto C assigns to each $x \in H$ its nearest point denote $P_C x$ in C, that is, $P_C x$ is the unique point in with the property :

$$\|x - P_C x\| \leq \|x - y\|, \quad \text{for all}\, y \in C.$$

We use the following notations:
(i) $\rightharpoonup$ for weak convergence and $\to$ for strong convergence;
(ii) $\omega_w(x_n) = \{x : \exists x_{n_j} \rightharpoonup x\}$ denotes the weak $\omega - limit$ set of $\{x_n\}$.
The following Lemmas are well known.

Lemma 2.1. *Let C be a closed convex subset of a real Hilbert space H. Given $x \in H$ and $z \in C$, then $z = P_C x$ if and only if there hold the relation $\langle x - z, y - z \rangle \leq 0$, for all $y \in C$.*

Lemma 2.2. *Let H be a real Hilbert space, then for all $x, y \in H$, we have*

$$\|x - y\|^2 = \|x\|^2 - \|y\|^2 - 2\langle x - y, y \rangle.$$

Lemma 2.3. [13] *Let H be a real Hilbert space, C a closed convex subset of H and $T : C \to C$ a continuous pseudo-contractive mapping, then*

(i) $F(T)$ closed convex subset of C.

(ii) $I - T$ is demiclosed at zero, i.e., if $\{x_n\}$ is a sequence in C such that $x_n \rightharpoonup z$ and $(I-T)x_n \to 0$, then $(I-T)z = 0$.

Lemma 2.4. [14] *Let C be a closed convex subset of a real Hilbert space H. Let $\{x_n\}$ be a sequence in H and $u \in H$. Let $q = P_C u$. If $\{x_n\}$ is such that $\omega_\omega(x_n) \subseteq C$ and satisfies the condition : $\|x_n - u\| \leq \|u - q\|, \forall n \geq 1$. Then $\{x_n\}$ converges strongly to q.*

3. Main Results

For pseudo-contractive mappings in Hilbert spaces, we argue that the modified hybrid algorithms $\langle 4\rangle$ have a strong convergence theorem.

Theorem 3.1. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $T : C \to C$ be a L-Lipschitz pseudo-contractive mapping with $F(T) \neq \emptyset$. Assume that $\{\alpha_n\}, \{\beta_n\}$ and $\{\gamma_n\}$ are sequences in $(0,1)$ satisfying:*
(i) $b \leq (L+1)[\gamma_n + (\beta_n + (1+\beta_n L)\alpha_n)(1+\gamma_n L)] < a < 1$, for some $a, b \in (0,1)$;
(ii) $\lim_{n\to\infty} \beta_n = 0$, $\lim_{n\to\infty} \gamma_n = 0$.
Then the sequence $\{x_n\}$ generated by $\langle 4\rangle$ converges strongly to $P_{F(T)}x_0$.

Proof. By Lemma 2.3 (i), we get that $F(T)$ is closed and convex. Then $P_{F(T)}$ is well defined. It is easy to see that C_n is closed and convex.

Next, we will show that $F(T) \subseteq C_n$ for all n.
Let $p \in F(T)$, by Lemma 2.2 and $\langle (I-T)x - (I-T)y, x-y\rangle \geq 0$, for all $x, y \in C$, then

$$\begin{aligned}
& \|x_n - p - \alpha_n(I-T)w_n\|^2 \\
= & \|x_n - p\|^2 - \|\alpha_n(I-T)w_n\|^2 - 2\alpha_n\langle (I-T)w_n, x_n - p - \alpha_n(I-T)w_n\rangle \\
= & \|x_n - p\|^2 - \|\alpha_n(I-T)w_n\|^2 - 2\alpha_n\langle (I-T)w_n, x_n - w_n - \alpha_n(I-T)w_n\rangle \\
& - 2\alpha_n\langle (I-T)w_n - (I-T)p, w_n - p\rangle \\
\leq & \|x_n - p\|^2 - \|\alpha_n(I-T)w_n\|^2 - 2\alpha_n\langle (I-T)w_n, x_n - w_n - \alpha_n(I-T)w_n\rangle \\
= & \|x_n - p\|^2 - \|x_n - w_n + w_n - x_n + \alpha_n(I-T)w_n\|^2 \\
& - 2\alpha_n\langle (I-T)w_n, x_n - w_n - \alpha_n(I-T)w_n\rangle \\
= & \|x_n - p\|^2 - \|x_n - w_n\|^2 - \|w_n - x_n + \alpha_n(I-T)w_n\|^2 \\
& - 2\langle x_n - w_n, w_n - x_n + \alpha_n(I-T)w_n\rangle \\
& + 2\alpha_n\langle (I-T)w_n, w_n - x_n + \alpha_n(I-T)w_n\rangle \\
\leq & \|x_n - p\|^2 - \|x_n - w_n\|^2 - \|w_n - x_n + \alpha_n(I-T)w_n\|^2 \\
& + 2|\langle x_n - w_n - \alpha_n(I-T)w_n, w_n - x_n + \alpha(I-T)w_n\rangle|.
\end{aligned} \tag{3.1}$$

We consider the last item of (3.1), then

$$\begin{aligned}
& |\langle x_n - w_n - \alpha_n(I-T)w_n, w_n - x_n + \alpha_n(I-T)w_n\rangle| \\
= & |\langle x_n - y_n + \alpha_n y_n - \alpha_n Ty_n - \alpha_n(I-T)w_n, w_n - x_n + \alpha_n(I-T)w_n\rangle| \\
= & |\langle x_n - y_n + \alpha_n(y_n - Ty_n) - \alpha_n(I-T)w_n, w_n - x_n + \alpha_n(I-T)w_n\rangle| \\
\leq & |\langle x_n - y_n, w_n - x_n + \alpha_n(I-T)w_n\rangle| \\
& + \alpha_n|\langle y_n - Ty_n - (I-T)w_n, w_n - x_n + (I-T)w_n\rangle| \\
= & |\langle x_n - z_n + \beta_n z_n - \beta_n Tz_n, w_n - x_n + \alpha_n(I-T)w_n\rangle| \\
& + \alpha_n|\langle z_n - \beta_n z_n + \beta_n Tz_n - Ty_n - (I-T)w_n, w_n - x_n + \alpha_n(I-T)w_n\rangle| \\
\leq & |\langle x_n - z_n, w_n - x_n + \alpha_n(I-T)w_n\rangle| + \beta_n|\langle z_n - Tz_n, w_n - x_n + \alpha_n(I-T)w_n\rangle| \\
& + \alpha_n[|\langle z_n - \beta_n z_n + \beta_n Tz_n - w_n, w_n - x_n + \alpha_n(I-T)w_n\rangle| \\
& + |\langle Tw_n - Ty_n, w_n - x_n + \alpha_n(I-T)w_n\rangle|] \\
\leq & |\langle x_n - z_n, w_n - x_n + \alpha_n(I-T)w_n\rangle| + \beta_n|\langle z_n - Tz_n, w_n - x_n + \alpha_n(I-T)w_n\rangle| \\
& + \alpha_n\beta_n|\langle Tz_n - z_n, w_n - x_n + \alpha_n(I-T)w_n\rangle| \\
& + \alpha_n|\langle z_n - w_n, w_n - x_n + \alpha_n(I-T)w_n\rangle| \\
& + \alpha_n|\langle Tw_n - Ty_n, w_n - x_n + \alpha_n(I-T)w_n\rangle|
\end{aligned}$$

$$
\begin{aligned}
&\le \|x_n - z_n\|\|w_n - x_n + \alpha_n(I-T)w_n\| + \beta_n\|z_n - Tz_n\|\|w_n - x_n + \alpha_n(I-T)w_n\| \\
&\quad + \alpha_n\beta_n\|Tz_n - z_n\|\|w_n - x_n + \alpha_n(I-T)w_n\| \\
&\quad + \alpha_n\|z_n - w_n\|\|w_n - x_n + \alpha_n(I-T)w_n\| \\
&\quad + \alpha_n L\|w_n - y_n\|\|w_n - x_n + \alpha_n(I-T)w_n\| \\
&= \|x_n - z_n\|\|w_n - x_n + \alpha_n(I-T)w_n\| \\
&\quad + (\beta_n + \alpha_n\beta_n)\|Tz_n - z_n\|\|w_n - x_n + \alpha_n(I-T)w_n\| \\
&\quad + \alpha_n\|z_n - w_n\|\|w_n - x_n + \alpha_n(I-T)w_n\| \\
&\quad + \alpha_n L\|w_n - y_n\|\|w_n - x_n + \alpha_n(I-T)w_n\| \\
&\le \|x_n - z_n\|\|w_n - x_n + \alpha_n(I-T)w_n\| \\
&\quad + (\beta_n + \alpha_n\beta_n)\|Tz_n - z_n\|\|w_n - x_n + \alpha_n(I-T)w_n\| \\
&\quad + \alpha_n[\|z_n - x_n\| + \|x_n - w_n\|]\|w_n - x_n + \alpha_n(I-T)w_n\| \\
&\quad + \alpha_n L[\|w_n - x_n\| + \|x_n - y_n\|]\|w_n - x_n + \alpha_n(I-T)w_n\| \\
&= (1+\alpha_n)\|x_n - z_n\|\|w_n - x_n + \alpha_n(I-T)w_n\| \\
&\quad + (\beta_n + \alpha_n\beta_n)\|Tz_n - z_n\|\|w_n - x_n + \alpha_n(I-T)w_n\| \\
&\quad + (\alpha_n + \alpha_n L)\|x_n - w_n\|\|w_n - x_n + \alpha_n(I-T)w_n\| \\
&\quad + \alpha_n L\|x_n - y_n\|\|w_n - x_n + \alpha_n(I-T)w_n\| \\
&\le (1+\alpha_n)\|x_n - z_n\|\|w_n - x_n + \alpha_n(I-T)w_n\| \\
&\quad + (\beta_n + \alpha_n\beta_n)\|Tz_n - z_n\|\|w_n - x_n + \alpha_n(I-T)w_n\| \\
&\quad + (\alpha_n + \alpha_n L)\|x_n - w_n\|\|w_n - x_n + \alpha_n(I-T)w_n\| \\
&\quad + \alpha_n L[\|x_n - z_n\| + \|z_n - y_n\|]\|w_n - x_n + \alpha_n(I-T)w_n\| \\
&= (1+\alpha_n)\gamma_n\|x_n - Tx_n\|\|w_n - x_n + \alpha_n(I-T)w_n\| \\
&\quad + (\beta_n + \alpha_n\beta_n)\|Tz_n - z_n\|\|w_n - x_n + \alpha_n(I-T)w_n\| \\
&\quad + (\alpha_n + \alpha_n L)\|x_n - w_n\|\|w_n - x_n + \alpha_n(I-T)w_n\| \\
&\quad + \alpha_n L[\gamma_n\|x_n - Tx_n\| + \beta_n\|z_n - Tz_n\|]\|w_n - x_n + \alpha_n(I-T)w_n\| \\
&= \alpha_n(1+L)\|x_n - w_n\|\|w_n - x_n + \alpha_n(I-T)w_n\| \\
&\quad + (\gamma_n + \alpha_n\gamma_n + \alpha_n\gamma_n L)\|x_n - Tx_n\|\|w_n - x_n + \alpha_n(I-T)w_n\| \\
&\quad + (\beta_n + \alpha_n\beta_n + \alpha_n\beta_n L)\|Tz_n - z_n\|\|w_n - x_n + \alpha_n(I-T)w_n\| \\
&\le \alpha_n(1+L)\|x_n - w_n\|\|w_n - x_n + \alpha_n(I-T)w_n\| \\
&\quad + (\gamma_n + \alpha_n\gamma_n + \alpha_n\gamma_n L)\|x_n - Tx_n\|\|w_n - x_n + \alpha_n(I-T)w_n\| \\
&\quad + (\beta_n + \alpha_n\beta_n + \alpha_n\beta_n L)[\|Tz_n - Tx_n\| + \|Tx_n - z_n\|]\|w_n - x_n + \alpha_n(I-T)w_n\| \\
&\le \alpha_n(1+L)\|x_n - w_n\|\|w_n - x_n + \alpha_n(I-T)w_n\| \\
&\quad + (\gamma_n + \alpha_n\gamma_n + \alpha_n\gamma_n L)\|x_n - Tx_n\|\|w_n - x_n + \alpha_n(I-T)w_n\| \\
&\quad + (\beta_n + \alpha_n\beta_n + \alpha_n\beta_n L)[\gamma_n L\|x_n - Tx_n\| + \|x_n - Tx_n\|]\|w_n - x_n + \alpha_n(I-T)w_n\| \\
&\le \alpha_n(1+L)\|x_n - w_n\|\|w_n - x_n + \alpha_n(I-T)w_n\| \\
&\quad + [\gamma_n + \alpha_n\gamma_n + \alpha_n\gamma_n L + \beta_n\gamma_n L + \alpha_n\beta_n\gamma_n L + \alpha_n\beta_n\gamma_n L^2 \\
&\quad + \beta_n + \alpha_n\beta_n + \alpha_n\beta_n L]\|x_n - Tx_n\|\|w_n - x_n + \alpha_n(I-T)w_n\| \\
&= \alpha_n(1+L)\|x_n - w_n\|\|w_n - x_n + \alpha_n(I-T)w_n\| \\
&\quad + (\gamma_n + \beta_n(1+\gamma_n L))(1+\alpha_n(1+L))\|x_n - Tx_n\|\|w_n - x_n + \alpha_n(I-T)w_n\|
\end{aligned}
$$

$$\begin{aligned}
&\leq \frac{\alpha_n(L+1)}{2}(\|x_n-w_n\|^2+\|w_n-x_n+\alpha_n(I-T)w_n\|^2)\\
&\quad +(\gamma_n+\beta_n(1+\gamma_nL))(1+\alpha_n(1+L))\|x_n-Tx_n\|\|w_n-x_n+\alpha_n(I-T)w_n\| \qquad (3.2)
\end{aligned}$$

Substituting (3.2) into (3.1), we obtain

$$\begin{aligned}
&\|x_n-p-\alpha_n(I-T)w_n\|^2\\
\leq\ &\|x_n-p\|^2-\|x_n-w_n\|^2-\|w_n-x_n+\alpha_n(I-T)w_n\|^2\\
&+\alpha_n(L+1)(\|x_n-w_n\|^2+\|w_n-x_n+\alpha_n(I-T)w_n\|^2)\\
&+2(\gamma_n+\beta_n(1+\gamma_nL))(1+\alpha_n(1+L))\|x_n-Tx_n\|\|w_n-x_n+\alpha_n(I-T)w_n\|\\
\leq\ &\|x_n-p\|^2\\
&+2(\gamma_n+\beta_n(1+\gamma_nL))(1+\alpha_n(1+L))\|x_n-Tx_n\|\|w_n-x_n+\alpha_n(I-T)w_n\| \qquad (3.3)
\end{aligned}$$

Since

$$\|x_n-p-\alpha_n(I-T)w_n\|^2=\|x_n-p\|^2-2\alpha_n\langle x_n-p,(I-T)w_n\rangle+\|\alpha_n(I-T)w_n\|^2 \qquad (3.4)$$

Therefore, from (3.3) and (3.4) we get

$$\begin{aligned}
\|\alpha_n(I-T)w_n\|^2 \leq\ &2\alpha_n\langle x_n-p,(I-T)w_n\rangle\\
&+2(\gamma_n+\beta_n(1+\gamma_nL))(1+\alpha_n(1+L))\|x_n-Tx_n\|\|w_n-x_n+\alpha_n(I-T)w_n\|,
\end{aligned}$$

which implies that $p\in C_n$. So $F(T)\subseteq C_n$ for all n.

From the definition of $\{x_n\}$ that $x_n=P_{C_n}x_0$. This implies that $\|x_n-x_0\|\leq\|z-x_0\|$, for all $z\in C_n$.

Since $F(T)\subseteq C_n$, we have $\|x_n-x_0\|\leq\|p-x_0\|$, for any $p\in F(T)$. In particular,

$$\|x_n-x_0\|\leq\|q-x_0\|, \qquad q=P_{F(T)}x_0. \qquad (3.5)$$

Hence $\{x_n\}$ is bounded. Since T is L-Lipschitz continuous, then $\{Tx_n\}$, $\{w_n\}$, $\{Tw_n\}$, $\{z_n\}$, $\{Tz_n\}$, $\{y_n\}$, $\{Ty_n\}$ are all bounded.

From $x_n=P_{C_n}x_0$ and $x_{n+1}=P_{C_{n+1}}x_0\in C_{n+1}\subseteq C_n$, we have

$$\langle x_n-x_0,x_{n+1}-x_n\rangle\geq 0. \qquad (3.6)$$

By Lemma 2.2 and (3.6), we obtain

$$\begin{aligned}
\|x_{n+1}-x_n\|^2 &= \|x_{n+1}-x_0-(x_n-x_0)\|^2\\
&= \|x_{n+1}-x_0\|^2-\|x_n-x_0\|^2-2\langle x_{n+1}-x_n,x_n-x_0\rangle\\
&\leq \|x_{n+1}-x_0\|^2-\|x_n-x_0\|^2, \qquad (3.7)
\end{aligned}$$

which implies that $\|x_n-x_0\|\leq\|x_{n+1}-x_0\|$, for all n.

Then $\{\|x_n-x_0\|\}$ is a nondecreasing sequence, and notice that $\{\|x_n-x_0\|\}$ is also bounded.

Hence, $\lim_{n\to\infty}\|x_n-x_0\|$ exists. At the same time, letting $n\to\infty$ in the right side of inequality (3.7), we have $\lim_{n\to\infty}\|x_{n+1}-x_n\|=0$. Since $x_{n+1}\in C_{n+1}\subseteq C_n$, we have

$$\begin{aligned}
\|\alpha_n(I-T)w_n\|^2 \leq\ &2\alpha_n\|x_n-x_{n+1}\|\|(I-T)w_n\|\\
&+2(\gamma_n+\beta_n(1+\gamma_nL))(1+\alpha_n(1+L))\|x_n-Tx_n\|\|w_n-x_n+\alpha_n(I-T)w_n\|.
\end{aligned}$$

And since $\lim_{n\to\infty}\beta_n = 0$, $\lim_{n\to\infty}\gamma_n = 0$, it imply that $\|w_n - Tw_n\| \to 0$ *as* $n \to \infty$. And consider,

$$\begin{aligned}
\|x_n - Tx_n\| &\leq \|x_n - w_n\| + \|w_n - Tw_n\| + \|Tw_n - Tx_n\| \\
&\leq (L+1)\|x_n - w_n\| + \|w_n - Tw_n\| \\
&\leq (L+1)[\|x_n - y_n\| + \alpha_n\|y_n - Ty_n\|] + \|w_n - Tw_n\| \\
&\leq (L+1)[\gamma_n\|x_n - Tx_n\| + \beta_n\|z_n - Tz_n\| \\
&\quad + \alpha_n[\|z_n - Tz_n\| + \beta_n L\|z_n - Tz_n\|]] + \|w_n - Tw_n\| \\
&= (L+1)[\gamma_n\|x_n - Tx_n\| + (\beta_n + \alpha_n + \alpha_n\beta_n L)\|z_n - Tz_n\|] + \|w_n - Tw_n\| \\
&\leq (L+1)[\gamma_n\|x_n - Tx_n\| + (\beta_n + \alpha_n + \alpha_n\beta_n L)[L\|z_n - x_n\| + \|Tx_n - z_n]] \\
&\quad + \|w_n - Tw_n\| \\
&\leq (L+1)[\gamma_n\|x_n - Tx_n\| + (\beta_n + \alpha_n + \alpha_n\beta_n L)[\gamma_n L\|x_n - Tx_n\| + \|x_n - Tx_n]] \\
&\quad + \|w_n - Tw_n\| \\
&= (L+1)[\gamma_n + (\beta_n + \alpha_n + \alpha_n\beta_n L)\gamma_n L + (\beta_n + \alpha_n + \alpha_n\beta_n L)]\|x_n - Tx_n]] \\
&\quad + \|w_n - Tw_n\| \\
&= (L+1)[\gamma_n + (\beta_n + (1+\beta_n L)\alpha_n)(1+\gamma_n L)]\|x_n - Tx_n\|] + \|w_n - Tw_n\|
\end{aligned}$$

Since $0 < b \leq (L+1)[\gamma_n + (\beta_n + (1+\beta_n L)\alpha_n)(1+\gamma_n L)] < a < 1$, we have

$$\|x_n - Tx_n\| \leq \tfrac{1}{1-a}\|w_n - Tw_n\| \to 0 \text{ as } n \to \infty.$$

By Lemma 2.3 (ii), $I - T$ is demiclosed at zero. Together with the fact that $\{x_n\}$ is bounded, which guarantees that every weak limit point of $\{x_n\}$ is a fixed point of T. That is $\omega_w(x_n) \subseteq F(T)$. Therefore, by inequality (3.5) and Lemma 2.4, we know $\{x_n\}$ converges strongly to $q = P_{F(T)}x_0$. This completes the proof. ∎

If T in Theorem 3.1 is a nonexpansive mapping, then we obtain the following corollary.

Corollary 3.2. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $T : C \to C$ be a nonexpansive mapping such that $F(T) \neq \emptyset$. Assume that $\{\alpha_n\}, \{\beta_n\}$ and $\{\gamma_n\}$ are sequences in $(0,1)$ satisfying:*
(i) $b \leq 2(\gamma_n + \alpha_n + 2\alpha_n\beta_n)(1+\gamma_n) < a < 1$, for some $a, b \in (0,1)$;
(ii) $\lim\limits_{n\to\infty}\beta_n = 0$, $\lim\limits_{n\to\infty}\gamma_n = 0$.
Let $x_0 \in H$. For $C_1 = C$ and $x_1 = P_{C_1}x_0$. Then the sequence $\{x_n\}$ generated by

$$\begin{cases}
z_n = (1-\gamma_n)x_n + \gamma_n Tx_n, \\
y_n = (1-\beta_n)z_n + \beta_n Tz_n, \\
w_n = (1-\alpha_n)y_n + \alpha_n Ty_n, \\
C_{n+1} = \{z \in C_n : \|\alpha_n(I-T)w_n\|^2 \leq 2\alpha_n\langle x_n - z, (I-T)w_n\rangle \\
\qquad\qquad +2(\beta_n + \gamma_n + \beta_n\gamma_n)(1+2\alpha_n)\|x_n - Tx_n\|\|w_n - x_n + \alpha_n(I-T)w_n\| \\
x_{n+1} = P_{C_{n+1}}x_0, n \geq 1,
\end{cases}$$

converges strongly to $P_{F(T)}x_0$.

If $\gamma_n = 0$ for all n in Theorem 3.1, the we get the following corollary.

Corollary 3.3. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $T : C \to C$ be a L-Lipschitz pseudo-contractive mapping with $F(T) \neq \emptyset$. Assume that $\{\alpha_n\}, \{\beta_n\}$ are sequences in $(0,1)$ satisfying:*
(i) $b \leq (L+1)(\beta_n + (1+\beta_n L)\alpha_n) < a < 1$, for some $a, b \in (0,1)$;
(ii) $\lim\limits_{n\to\infty}\beta_n = 0$.

Let $x_0 \in H$. For $C_1 = C$ and $x_1 = P_{C_1}x_0$. Then the sequence $\{x_n\}$ generated by

$$\begin{cases} y_n = (1-\beta_n)x_n + \beta_n T x_n, \\ w_n = (1-\alpha_n)y_n + \alpha_n T y_n, \\ C_{n+1} = \{z \in C_n : \|\alpha_n(I-T)w_n\|^2 \le 2\alpha_n\langle x_n - z, (I-T)w_n\rangle \\ \qquad\qquad +2\beta_n(1+\alpha_n(1+L))\|x_n - Tx_n\|\|w_n - x_n + \alpha_n(I-T)w_n\| \\ x_{n+1} = P_{C_{n+1}}x_0, n \ge 1, \end{cases}$$

converges strongly to $P_{F(T)}x_0$.

If T in corollary 3.3 is a nonexpansive mapping, then we obtain the following corollary.

Corollary 3.4. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $T : C \to C$ be a a nonexpansive mapping with $F(T) \neq \emptyset$. Assume that $\{\alpha_n\}, \{\beta_n\}$ are sequences in $(0,1)$ satisfying:*
(i) $b \le 2(\beta_n + (1+\beta_n)\alpha_n) < a < 1$, for some $a, b \in (0,1)$;
(ii) $\lim_{n\to\infty} \beta_n = 0$.
Let $x_0 \in H$. For $C_1 = C$ and $x_1 = P_{C_1}x_0$. Then the sequence $\{x_n\}$ generated by

$$\begin{cases} y_n = (1-\beta_n)x_n + \beta_n T x_n, \\ w_n = (1-\alpha_n)y_n + \alpha_n T y_n, \\ C_{n+1} = \{z \in C_n : \|\alpha_n(I-T)w_n\|^2 \le 2\alpha_n\langle x_n - z, (I-T)w_n\rangle \\ \qquad\qquad +2\beta_n(1+2\alpha_n)\|x_n - Tx_n\|\|w_n - x_n + \alpha_n(I-T)w_n\| \\ x_{n+1} = P_{C_{n+1}}x_0, n \ge 1, \end{cases}$$

converges strongly to $P_{F(T)}x_0$.

Recall that a mapping A is said to be monotone or accretive, if $\langle Ax - Ay, x - y\rangle \ge 0$, for all $x, y \in H$. The pseudo-contractive mapping is strongly related to the monotone mapping. It is well known that A is monotone or accretive mapping if and only if $(I - A)$ is pseudo-contractive mapping. Hence, the fixed points of pseudo-contractive mapping actually is the zero of monotone or accretive mapping. Due to Theorem 3.1 and Corollary 3.3, respectively. We have two the following corollaries.

Corollary 3.5. *Let $A : H \to H$ be a L-Lipschitz monotone mapping with $A^{-1}(0) \neq \emptyset$. Assume that $\{\alpha_n\}, \{\beta_n\}$ and $\{\gamma_n\}$ are sequences in $(0,1)$ satisfying:*
(i) $b \le (L+1)[\gamma_n + (\beta_n + (1+\beta_n L)\alpha_n)(1+\gamma_n L)] < a < 1$, for some $a, b \in (0,1)$;
(ii) $\lim_{n\to\infty} \beta_n = 0$, $\lim_{n\to\infty} \gamma_n = 0$.
Let $x_0 \in H$. For $C_1 = C$ and $x_1 = P_{C_1}x_0$. Then the sequence $\{x_n\}$ generated by

$$\begin{cases} z_n = x_n + \alpha_n A x_n, \\ y_n = z_n + \beta_n A z_n, \\ w_n = y_n + \gamma_n A y_n, \\ C_{n+1} = \{z \in C_n : \|\alpha_n A w_n\|^2 \le 2\alpha_n\langle x_n - z, Aw_n\rangle \\ \qquad\qquad +2(\gamma_n + \beta_n(1+\gamma_n L))(1+\alpha_n(1+L))\|Ax_n\|\|w_n - x_n + \alpha_n A w_n\| \\ x_{n+1} = P_{C_{n+1}}x_0, n \ge 1, \end{cases}$$

converges strongly to $P_{A^{-1}(0)}x_0$.

Corollary 3.6. *Let $A : H \to H$ be a L-Lipschitz monotone mapping with $A^{-1}(0) \neq \emptyset$. Assume that $\{\alpha_n\}, \{\beta_n\}$ are sequences in $(0,1)$ satisfying:*
(i) $b \le (L+1)(\beta_n + (1+\beta_n L)\alpha_n) < a < 1$, for some $a, b \in (0,1)$;
(ii) $\lim_{n\to\infty} \beta_n = 0$.

Let $x_0 \in H$. For $C_1 = C$ and $x_1 = P_{C_1}x_0$. Then the sequence $\{x_n\}$ generated by

$$\begin{cases} y_n = x_n + \beta_n A x_n, \\ w_n = y_n + \gamma_n A y_n, \\ C_{n+1} = \{z \in C_n : \|\alpha_n A w_n\|^2 \leq 2\alpha_n \langle x_n - z, A w_n\rangle \\ \qquad\qquad +2\beta_n(1+\alpha_n(1+L))\|Ax_n\|\|w_n - x_n + \alpha_n A w_n\| \\ x_{n+1} = P_{C_{n+1}} x_0, n \geq 1, \end{cases}$$

converges strongly to $P_{A^{-1}(0)}x_0$.

4. Conclusion

In this research, we have constructed a new hybrid algorithm $\langle 4\rangle$ for L-Lipschitz pseudo-contractive mappings and have established a strong convergence theorem. From Theorem 3.1, if $\beta_n = \gamma_n = 0$ for all n, then the hybrid algorithm $\langle 4\rangle$ reduces to the hybrid algorithm $\langle 1\rangle$. So Theorem 3.1 of Yao et al. [8] is a spacial case of our Theorem 3.1.

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