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Fixed Point Theorems for Discontinuous Mappings of Kannan and Bianchini Type in Distance Spaces

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Abstract In this paper, we establish fixed point theorems for two classes of discontinuous mappings (Kannan type mappings and Bianchini type mappings) in the very general setting of a space with a distance, thus extending some results in the paper [M.M. Choban, Fixed point of mappings defined on spaces with distance, Carpathian J. Math. 32 (2) (2016) 173–188]. We also indicate some particular cases of our main results and present some examples to illustrate the theoretical results and show that our generalizations are effective.

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1. Introduction

It is well known that the fixed point theorem of Banach, for contraction mappings, is one of the pivotal results in analysis. It has been used in many different fields of mathematics but has one major drawback due to the fact that any Banach contraction T is continuous. So, Banach contraction mapping principle can be applied only to operator equations involving continuous operators [1, 2].

A natural question arises: could we find contractive conditions which will imply the existence of a fixed point in a complete metric space but will not imply continuity?

Kannan [3] proved the following result giving an affirmative answer to the above question.

Theorem 1.1. [1, 3] *If (X, d) is a complete metric space and the mapping $T : X \to X$ satisfies*

$$d(Tx, Ty) \leq a \cdot [\, d(x, Tx) + d(y, Ty)\,], \quad \forall x, y \in X \tag{1.1}$$

where $0 < a < \frac{1}{2}$, then T has a unique fixed point.

Bianchini [4] obtained an extension of Theorem 1.1, as follows.

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Theorem 1.2. [4, 5] *Let (X, d) be a metric space, and $T : X \to X$ be a Bianchini mapping, i.e., there exists $h \in [0, 1)$ such that*

$$d(Tx, Ty) \leq h \cdot \max\{d(x, Tx), d(y, Ty)\}, \quad \forall x, y \in X. \tag{1.2}$$

Then T has a unique fixed point.

It is easy to see that every Kannan mapping is a Bianchini mapping but the converse is not more true, as shown by the next example.

Example 1.3. [5] Let $X = [0, 1]$ be endowed with the usual metric. Then the function $f(x) = x/3,\ 0 \leq x < 1$, satisfies Bianchini condition (1.2), but it does not satisfy Kannan condition (1.1).

In a recent paper [6], Choban stated and proved a fixed point theorem in a space with distance, which is a significant extension of the well known Banach's contraction mapping principle. He proved that in a H-distance space with a mapping T, there exists a comparison function λ such that $d(Tx, Ty) \leq \lambda(d(x, y))$, for all $x, y \in X$, then T has a unique fixed point.

So we try to answer to the following questions:

(1) Is it possible to establish a fixed point theorem for Kannan mappings in a distance space?

(2) Is it possible to establish a fixed point theorem for Bianchini mappings in a distance space?

The main aim of this paper is to give partial but positive answers to both these questions.

2. Preliminaries

For a mapping $T : X \to X$ we denote, the set of fixed points of T:

$$Fix(T) = \{x \in X : T(x) = x\}. \tag{2.1}$$

We recall in this section the basic notions and results on spaces with a distance, mainly taken from [6].

Let X be a non-empty set and $d : X \times X \to \mathbb{R}$ be a mapping such that for all $x, y \in X$ we have:

(i_m) $d(x, y) \geq 0$

(ii_m) $d(x, y) + d(y, x) = 0$ if and only if $x = y$.

Then, according to [6], (X, d) is called a *distance space* and d is called a *distance* on X.

Definition 2.1. Let (X, d) be a distance space, $\{x_n : n \in \mathbb{N} = \{1, 2, ...\}\}$ be a sequence in X and let $x \in X$. We say that the sequence $\{x_n : n \in \mathbb{N}\}$ is:

(i) *Convergent* to x if and only if $\lim\limits_{n\to\infty} d(x, x_n) = 0$. We denote this by $x_n \to x$ or $x = \lim\limits_{n\to\infty} x_n$;

(ii) *Convergent* if it converges to some point in X;

(iii) *Cauchy* or *Fundamental* if $\lim\limits_{n,m\to\infty} d(x_n, x_m) = 0$.

A distance space (X, d) is *complete* if every Cauchy sequence in X converges to some point in X.

Definition 2.2. Let X be a non-empty set and d be a distance on X. Then:

(iii_m) (X, d) is called a *symmetric space* and d is called a *symmetric* on X if, for all $x, y \in X$, we have: $d(x, y) = d(y, x)$;

(iv_m) (X, d) is called a *quasimetric space* and d is called a *quasimetric* on X if, for all $x, y, z \in X$, we have: $d(x, z) \leq d(x, y) + d(y, z)$;

(X, d) is called a *metric space* and d is called a *metric* if d is a *symmetric* and *quasimetric* simultaneously.

We say that a distance d on a space (X, d) is *balanced* if for every Cauchy sequence $\{x_n : n \in \mathbb{N}\}$ convergent to x in X and any point $y \in X$ we have $d(y, x) = \lim_{n\to\infty} d(y, x_n)$.

Definition 2.3. Let X be a non-empty set and $d(x, y)$ be a distance on X with the following property:

(**N**) For each point $x \in X$ and any $\epsilon > 0$ there exists $\delta = \delta(x, \epsilon) > 0$ such that from $d(x, y) \leq \delta$ and $d(y, z) \leq \delta$ it follows $d(x, z) \leq \epsilon$. Then (X, d) is called an *N-distance space* and d is called an *N-distance* on X. If d is a symmetric, then we say that d is an *N-symmetric*.

(**F**) for any $\epsilon > 0$ there exists $\delta = \delta(\epsilon) > 0$ such that from $d(x, y) \leq \delta$ and $d(y, z) \leq \delta$ it follows $d(x, z) \leq \epsilon$, then d is called an *F-distance* or a *Fréchet distance* and (X, d) is called an *F-distance space.* If d is a symmetric and an *F-distance* on a space X, then we say that d is an *F-symmetric.*

Remark 2.4. [6] If (X, d) is an *F-symmetric space*, then any convergent sequence is a Cauchy sequence. For *N-symmetric spaces* and for quasimetric spaces this assertion is not more true.

A distance space (X, d) is called an *H-distance space* if for any two distinct points $x, y \in X$ there exists $\delta = \delta(x, y) > 0$ such that $B(x, d, \delta) \cap B(y, d, \delta) = \emptyset$.

Remark 2.5. [6–8] Let (X, d) be a distance space. Then (X, d) is an H-distance space if and only if any convergent sequence has a unique limit point.

Lemma 2.6. [6] *Let (X, d) be a distance space and the space $(X, \tau(d))$ is Hausdorff. Then d is an H-distance and $\tau(d)$ is the topology induced by d.*

Proposition 2.7. [6] *Let (X, d) be an H-distance space and $T : X \to X$ be a continuous mapping. Then*

i. $Fix(T)$ is closed.

ii. If for some point $x \in X$, the Picard iteration $O(T, x)$ is convergent, then the set of fixed points $Fix(T)$ of the mapping T is non-empty.

Definition 2.8. Fix a distance space (X, d) and a mapping $T : X \to X$. We say that the space (X, d) is *T-bounded* if, for each $x \in X$, there exists a positive number $\lambda(x)$ such that

$$d(T^n(x), x) + d(x, T^n(x)) \leq \lambda(x),$$

for all $n \in \mathbb{N}$. The space (X, d) is called *weakly T-bounded* if for each $x \in X$ there exist a positive number $\lambda(x)$ and $p = p(x) \in \mathbb{N}$ such that

$$d(T^n(x), T^{p-1}(x)) + d(T^{p-1}(x), T^n(x)) \leq \lambda(x),$$

for each $n \geq p$.

Proposition 2.9. [6] *Let (X,d) be a distance space and the mapping $T : X \to X$ be a contraction. If the space (X,d) is weakly T-bounded, then:*

(1) For each point $x \in X$ the Picard sequence $O(T,x)$ is a Cauchy;
(2) The mapping T has a unique fixed point provided (X,d) is a complete H-distance space;
(3) The mapping T has a unique fixed point provided (X,d) is a complete balanced distance space.

A function $\lambda : [0,\infty) \to [0,\infty)$ is called a *comparison function* if it satisfies the following conditions:

(i) λ is increasing;
(ii) $\lim\limits_{n\to\infty} \lambda^n(t) = 0$ for each $t \in [0,\infty)$.

Remark 2.10. [6] If $\lambda : [0,\infty) \to [0,\infty)$ is a *comparison function* then it satisfies the following conditions: $\lambda(0) = 0$ and $\lambda(t) < t$, for each $t \in [0,\infty)$.

Proposition 2.11. [6] *Let (X,d) be a distance space and $T : X \to X$ be a mapping such that the space X,d is weakly T-bounded. If there exists a comparison function λ such that $d(Tx,Ty) \leq \lambda(d(x,y))$, for all $x,y \in X$, then:*

(1) For each point $x \in X$ the Picard sequence $O(T,x)$ is Cauchy;
(2) The mapping T has a unique fixed point provided (X,d) is a complete H-distance space.
(3) The mapping T has a unique fixed point provided (X,d) is a complete balanced distance space.

Corollary 2.12. [6] *Let (X,d) be a bounded complete H-distance space or a bounded balanced distance space, $T : X \to X$ be a mapping and suppose there exists a comparison function λ such that:*

$$d(Tx,Ty) \leq \lambda(d(x,y)), \text{ for all } x,y \in X.$$

Then T has a unique fixed point. Moreover, for each $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that $d\left(T^n(x), T^m(x)\right) < \varepsilon$, for all $x \in X$ and $n,m \geq n_0$.

3. Main Results

In this section, we present the following main result and its proof.

Theorem 3.1. *Let (X,d) be a complete H-distance space and let $T : X \to X$ be a mapping for which there exists $0 < a < \frac{1}{2}$ such that*

$$d(Tx,Ty) \leq a \cdot [\, d(x,Tx) + d(y,Ty) \,], \quad \forall x,y \in X. \tag{3.1}$$

Then the Picard iteration at the any point $x \in X$ is convergent. If, additionally, the limit $\bar{x}$ of the Picard sequence is a fixed point of T, then $\bar{x}$ is the unique fixed point of T.

Proof. Let $x_0 \in X$ and consider the Picard sequence $\{x_n\}$, generated by $x_{n+1} = Tx_n$, $n \geq 0$. We prove that

$$d(x_n, x_{n+1}) \leq \left(\frac{a}{1-a}\right)^n \cdot d(x_0, x_1)\ , \quad n = 0,1,\ldots \tag{3.2}$$

First, we note that $a \in \left(0, \frac{1}{2}\right) \implies \dfrac{a}{1-a} \in (0,1)$. Inequality (3.2) is obviously true for $n = 0$.

Now, we take $x := x_0$, $y := x_1$ in (3.1) and obtain

$$d(Tx_0, Tx_1) \leq a \cdot [\ d(x_0, Tx_0) + d(x_1, Tx_1)\ ],$$

that is,

$$d(x_1, x_2) \leq a \cdot [\ d(x_0, x_1) + d(x_1, x_2)\ ]$$

which yields

$$d(x_1, x_2) \leq \left(\frac{a}{1-a}\right)^1 \cdot d(x_0, x_1),$$

and so (3.2) holds for $n = 1$.

Next, we take $x := x_{n-1}$, $y := x_n$ in (3.1) and obtain

$$d(x_n, x_{n+1}) \leq a \cdot [\ d(x_{n-1}, x_n) + d(x_n, x_{n+1})\ ]$$

and we have

$$d(x_n, x_{n+1}) \leq \left(\frac{a}{1-a}\right) \cdot d(x_{n-1}, x_n). \tag{3.3}$$

Suppose (3.2) holds for $n = k$, that is,

$$d(x_k, x_{k+1}) \leq \left(\frac{a}{1-a}\right)^k \cdot d(x_0, x_1)$$

and prove that (3.2) also holds for $n = k + 1$.

Indeed, in view of (3.3)

$$d(x_{k+1}, x_{k+2}) \leq \frac{a}{1-a} \cdot d(x_k, x_{k+1}) \leq \frac{a}{1-a} \cdot \left(\frac{a}{1-a}\right)^k \cdot d(x_0, x_1)$$

i.e.,

$$d(x_{k+1}, x_{k+2}) \leq \left(\frac{a}{1-a}\right)^{k+1} \cdot d(x_0, x_1).$$

Then, by mathematical induction we have that (3.2) holds for all $n \geq 0$.

To prove that $\{x_n\}$ is a Cauchy sequence, we take $x := x_{n+p-1}$ and $y := x_{n-1}$ in (3.1) and so we have

$$d(Tx_{n+p-1}, Tx_{n-1}) \leq a \cdot \ [\ d(x_{n+p-1}, x_{n+p}) + d(x_{n-1}, x_n)\ ] \iff$$

$$d(x_{n+p}, x_n) \leq a \cdot \ [\ d(x_{n+p-1}, x_{n+p}) + d(x_{n-1}, x_n)\ ]$$

and by using (3.2) we have

$$\begin{aligned} d(x_{n+p},x_n) &\leq a\cdot\left[\left(\frac{a}{1-a}\right)^{n+p-1}\cdot d(x_0,x_1)+\left(\frac{a}{1-a}\right)^{n}\cdot d(x_0,x_1)\right] \\ &= a\cdot\left[\left(\frac{a}{1-a}\right)^{n+p-1}+\left(\frac{a}{1-a}\right)^{n}\right]\cdot d(x_0,x_1). \end{aligned}$$

We obtain

$$d(x_{n+p},x_n)\leq a\cdot\left(\frac{a}{1-a}\right)^{n}\cdot\left[\left(\frac{a}{1-a}\right)^{p-1}+1\right]\cdot d(x_0,x_1), \tag{3.4}$$

by which we immediately conclude that $\{x_n\}$ is a Cauchy sequence.

Since (X,d) is complete, it follows that $\{x_n\}$ is convergent, which proves the first part of the theorem.

Now, if we denote $\bar{x}=\lim_{n\to\infty}x_n\in Fix(T)$, then the uniqueness is immediate. Indeed, suppose that T would have two fixed points $\bar{x},\bar{y}\in Fix(T)$, $\bar{x}\neq\bar{y}$. Then

$$d(\bar{x},\bar{y})=d(T\bar{x},T\bar{y})\leq a\ [\ d(\bar{x},T\bar{x})+d(\bar{y},T\bar{y})\ ]=0,$$

a contradiction. Therefore, $\bar{x}\neq\bar{y}$, and hence $\bar{x}$ is the unique fixed point of T. ∎

Remark 3.2.

(1) If T is continous then the limit $\bar{x}$ of the Picard sequence is always a fixed point of T. Indeed, we have:

$$\bar{x}=\lim_{n\to\infty}x_{n+1}=\lim_{n\to\infty}T(x_n)=T(\lim_{n\to\infty}x_n)=T(\bar{x}),$$

i.e., $\bar{x}$ is a fixed point of T;

(2) In general, the limit $\bar{x}$ of the Picard sequence is not a fixed point of T;

(3) If d is actually a quasimetric, then Theorem 3.1 reduces to the well known Kannan fixed point theorem in metric spaces [3].

Example 3.3. Let $X=\{0,1\}\cup\{2^{-n}:n\in\mathbb{N}\}$. Consider on X the F-symmetric d, defined as:

$$\begin{aligned} d(x,x)&=0\\ d(0,x)=d(x,0)&=0\\ d(1,2^{-1})=d(2^{-1},1)&=1\\ d(1,2^{-n})=d(2^{-n},1)&=5,\quad n\neq 0,1\\ d(2^{-1},2^{-n})=d(2^{-n},2^{-1})&=3^{-2},\quad n\neq 0,1\\ d(2^{-n},2^{-n-1})=d(2^{-n-1},2^{-n})&=3^{2},\quad n\neq 0,1 \text{ and}\\ d(2^{-m},2^{-n})&=\left|2^{-m}-2^{-n}\right|,\quad m+1\neq n,\ \forall m,n\in\mathbb{N}. \end{aligned}$$

Note that d is not a metric, because the triangle inequality is not satisfied:

$$1=(1,2^{-1})>d(1,0)+d(0,2^{-1})=0.$$

Now we consider the mapping $T:X\to X$, where $T(0)=0$, $T(1)=2^{-1}$ and $T(2^{-n})=2^{-n-1}$. Condition (3.1) is obviously satisfied for the following cases:

(1) $x = 0, y = 1$ and
(2) $x = 0, y = 2^{-n}$, since the left hand side term of the inequality is zero.

We now check the remaining cases:

(3) $x = 1, y = 2^{-n}$, when condition (3.1) with $a = \frac{1}{4}$, reduces to

$$d(T(1), T(2^{-n})) \leq \frac{1}{4} \cdot [d(1, T(1)) + d(2^{-n}, T(2^{-n}))] \iff \frac{1}{9} \leq \frac{5}{2}.$$

(4) $x = 2^{-m}, y = 2^{-n}$, when condition (3.1) with $a = \frac{1}{4}$, reduces to

$$d(T(2^{-m}), T(2^{-n})) \leq \frac{1}{4} \cdot [d(2^{-m}, T(2^{-m})) + d(2^{-n}, T(2^{-n}))]$$

if and only if $1 < \frac{9}{2}$, if $m \neq n$

If $m = n$, then $d(T(2^{-m}), T(2^{-n})) = 0$ and (3.1) it is also true. Therefore, T is a Kannan contraction, i.e.,

$$d(T(x), T(y)) \leq \frac{1}{4} \cdot [d(x, T(x)) + d(y, T(y)], \forall x, y \in X,$$

the Picard iteration is a convergent Cauchy sequence and $Fix(T) = \{0\}$.

The next Corollary is a generalization of Kannan fixed point theorem in metric spaces [3].

Corollary 3.4. *Let (X, d) be a complete symmetric H-distance space and let $T : X \to X$ be a mapping for which there exists $0 < a < \frac{1}{2}$ such that (3.1) holds. Then:*

(1) The Picard iteration at the point x is convergent
(2) If, additionally, the limit $\bar{x}$ of the Picard sequence is a fixed point of T, then $\bar{x}$ is the unique fixed point of T.

Proof. If (X, d) is a complete symmetric H-distance space, then it is a complete H-distance and conclusion follows by Theorem 3.1. ∎

Theorem 3.5. *Let (X, d) be a complete H-distance space and let $T : X \to X$ be a mapping. Assume that there exists a number h, $0 < h < 1$ such that*

$$d(Tx, Ty) \leq h \cdot \max\{d(x, Tx), d(y, Ty)\}, \ \forall x, y \in X. \tag{3.5}$$

Then the Picard iteration of the point x is convergent. If, additionally the limit $\bar{x}$ of the Picard sequence is a fixed point of T, then $\bar{x}$ is the unique fixed point of T.

Proof. Let $x_0 \in X$ and define $\{x_n\}$ by $x_{n+1} = Tx_n$, $n \geq 0$. We prove that

$$d(x_n, x_{n+1}) \leq h \cdot \max\{d(x_{n-1}, x_n), d(x_n, x_{n+1})\}, \ n \geq 0. \tag{3.6}$$

We have two cases:

Case 3.6. If $\max\{d(x_{n-1}, x_n), d(x_n, x_{n+1})\} = d(x_n, x_{n+1})$ then by (3.6) it follows $d(x_n, x_{n+1}) \leq h \cdot \ d(x_n, x_{n+1}) < d(x_n, x_{n+1})$, which is a contradiction.

Case 3.7. If $\max\{d(x_{n-1}, x_n), d(x_n, x_{n+1})\} = d(x_{n-1}, x_n)$ then we obtain

$$d(x_n, x_{n+1}) \leq h \cdot \ d(x_{n-1}, x_n). \tag{3.7}$$

By (3.7), we have

$$d(x_n, x_{n+1}) \leq h^n \cdot d(x_0, x_1), \ n \geq 0.$$

To prove that $\{x_n\}$ is a Cauchy sequence, we take $x := x_{n+p-1}$ and $y := x_{n-1}$ in (3.5). Therefore, we have

$$d(Tx_{n+p-1}, Tx_{n-1}) \leq h \cdot \max\{d(x_{n+p-1}, x_{n+p}), d(x_{n-1}, x_n)\}$$

$$d(x_{n+p}, x_n) \leq h \cdot \max\{d(x_{n+p-1}, x_{n+p}), d(x_{n-1}, x_n)\},$$

and by using (3.7) we have

$$\begin{aligned} d(x_{n+p}, x_n) &\leq h \cdot \max\{h^{n+p-1} \cdot d(x_0, x_1), h^{n-1} \cdot d(x_0, x_1)\} \\ &= h \cdot d(x_0, x_1) \max\{h^{n+p-1}, h^{n-1}\} \\ &= h \cdot h^{n-1} \cdot d(x_0, x_1) \\ &= h^n \cdot d(x_0, x_1), \end{aligned}$$

by which we immediately conclude that $\{x_n\}$ is a Cauchy sequence.

Since (X, d) is complete, it follows that $\{x_n\}$ is convergent, which proves the first part of the theorem. Now, if we denote $\bar{x} = \lim_{n\to\infty} x_n \in Fix(T)$, then the uniqueness is immediate. Indeed, suppose that T would have two fixed points $\bar{x}, \bar{y} \in Fix(T)$, $\bar{x} \neq \bar{y}$.

$$d(\bar{x}, \bar{y}) = d(T\bar{x}, T\bar{y}) \leq h \cdot \max\{ \ d(\bar{x}, T\bar{x}), d(\bar{y}, T\bar{y})\} = 0,$$

a contradiction. So $\bar{x} \neq \bar{y}$, and hence $\bar{x}$ is the unique fixed point of T. ■

Remark 3.8.

(1) If T is continuous then the limit $\bar{x}$ of the Picard sequence is always a fixed point of T. Indeed, we have:

$$\bar{x} = \lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} T(x_n) = T(\lim_{n\to\infty} x_n) = T(\bar{x}),$$

i.e., $\bar{x}$ is a fixed point of T.

(2) In general the limit $\bar{x}$ of the Picard sequence is not a fixed point of T.

(3) If d is a quasimetric, Theorem 3.5 reduces to the well known Bianchini fixed point theorem in metric spaces [4].

Example 3.9. Let $X = \{0, 1\} \cup \{2^{-n} : n \in \mathbb{N}\}$. Consider on X the F-symmetric d, defined as:

$$\begin{aligned} d(x, x) &= 0 \\ d(0, x) &= d(x, 0) = 0 \\ d(1, 2^{-2}) &= d(2^{-2}, 1) = 2^{-1} \\ d(1, 2^{-n}) &= d(2^{-n}, 1) = 5, \quad n \neq 0, 1, 2 \\ d(2^{-1}, 2^{-n-1}) &= d(2^{-n-1}, 2^{-1}) = 2^{-1}, \quad n \neq 0, 1 \\ d(2^{-2}, 2^{-n-1}) &= d(2^{-n-1}, 2^{-2}) = 2^{-1}, \quad n \neq 0, 1, 2 \\ d(2^{-n}, 2^{-n-1}) &= 1 \quad \text{and} \\ d(2^{-m}, 2^{-n}) &= \left|2^{-m} - 2^{-n}\right|, \quad m + 1 \neq n, \forall m, n \in \mathbb{N}. \end{aligned}$$

Note that d is not a metric, because the triangle inequality is not satisfied:

$$2^{-1} = d(1, 2^{-2}) > d(1, 0) + d(0, 2^{-2}) = 0.$$

Now we consider the mapping $T : X \to X$, where $T(0) = 0$, $T(1) = 2^{-2}$, $T(2^{-n}) = 2^{-n-1}$. Condition (3.5) is obviously satisfied for the following cases:

(1) $x = 0, y = 1$ and
(2) $x = 0, y = 2^{-n}$, since the left hand side term of the inequality is zero.

We now check the remaining cases:

(3) $x = 1, y = 2^{-n}$, when condition (3.5) with $h = \frac{1}{2}$, reduces to

$$d(T(1), T(2^{-n})) \leq \frac{1}{2} \cdot \max\{d(1, T(1)),\ d(2^{-n}, T(2^{-n}))\} \iff \frac{1}{2} \leq \frac{1}{2}.$$

(4) $x = 2^{-m}, y = 2^{-n}$, when condition (3.5) with $h = \frac{1}{2}$, reduces to

$$d(T(2^{-m}), T(2^{-n})) \leq \frac{1}{2} \cdot \max\{d(2^{-m}, T(2^{-m})),\ d(2^{-n}, T(2^{-n}))\} \iff$$

$\frac{1}{2} \leq \frac{1}{2}$, $m \neq n$.

If $m = n$, then $d(T(2^{-m}), T(2^{-n})) = 0$ and (3.5) it is also true. Therefore, T is a Bianchini contraction, i.e.,

$$d(T(x), T(y)) \leq \frac{1}{2} \cdot \max\{d(x, T(x)),\ d(y, T(y))\}, x, y \in X,$$

the Picard iteration is a convergent Cauchy sequence and $Fix(T) = \{0\}$.

The next Corollary is a generalization of Bianchini fixed point theorem in metric spaces [4].

Corollary 3.10. *Let (X, d) be a complete symmetric H-distance space and let $T : X \to X$ be a mapping. Assume that there exists a number h, $0 < h < 1$ such that (3.5) holds. Then:*

(i) The Picard iteration of the point x is convergent
(ii) If, additionally the limit $\bar{x}$ of the Picard sequence is a fixed point of T, then $\bar{x}$ is the unique fixed point of T

Proof. If (X, d) is a complete symmetric H-distance space, then it is a complete H-distance and conclusion follows by Theorem 3.5. ∎

4. Conclusion

By working in the general setting of a complete H-distance space, we obtained significant generalizations of Kannan and Bianchini fixed point theorems in usual metric spaces. We note that, so far, we were not able to obtain a Banach type fixed point theorem in distance spaces which are not quasimetric spaces.

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