Thai Journal of Mathematics
Special Issue (Annual Meeting in Mathematics, 2008): 103–115

www.math.science.cmu.ac.th/thaijournal
Online ISSN 1686-0209

Mathematical Modeling of Fiber Reinforced Structures by Homogenization

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Abstract : We present another proof of a study of Bellieud and Bouchitté that we expect to be more suitable to treat more general geometrical and physical cases. We consider the homogenization of the quasi-linear elliptic problem

$$\begin{aligned} -\operatorname{div}\sigma_\varepsilon = f,\ \ \sigma_\varepsilon = a_\varepsilon\left|\nabla u_\varepsilon\right|^{p-2}\nabla u_\varepsilon \quad & \text{on } \Omega \\ u_\varepsilon = u_0 \quad & \text{on } \Gamma_0 \\ \sigma_\varepsilon\cdot n = g \quad & \text{on } \Gamma_1 \end{aligned}$$

where Ω is a bounded cylindrical open subset of $\mathbb{R}^3$ and $1 < p < +\infty$. The fibers occupy a set of thin parallel cylinders periodically distributed in Ω. The conductivity coefficient a_ε is ε-periodic and takes very high values in the fibers.

Keywords : Variational convergence, Homogenization

1 Introduction

Let $p \in (1, +\infty)$, we consider the homogenization of the elliptic problem

$$\begin{aligned} -\operatorname{div}\sigma_\varepsilon = f,\ \ \sigma_\varepsilon = a_\varepsilon\left|\nabla u_\varepsilon\right|^{p-2}\nabla u_\varepsilon \quad & \text{on } \Omega \\ u_\varepsilon = u_0 \quad & \text{on } \Gamma_0 \\ \sigma_\varepsilon\cdot n = g \quad & \text{on } \Gamma_1 \end{aligned} \tag{1.1}$$

where $\Omega := \omega \times (0, L)$ with $L > 0$ and ω is a bounded domain of $\mathbb{R}^2$ with smooth boundary and containing the origin of coordinates. The homogenization study of (1.1) consists in examining the behavior of the sequence of the solution (u_ε) as ε tends to zero. The conductivity coefficient a_ε is ε-periodic and satisfies a uniform lower bound, Γ_0 is an open subset of $\partial\Omega$ with Hausdorff measure $H^2(\Gamma_0)$ strictly positive, $\Gamma_1 = \partial\Omega\backslash\Gamma_0$, and n is the unit exterior normal on $\partial\Omega$. The boundary data u_0 is Lipschitz, while $(f, g) \in L^{p'}(\Omega) \times L^{p'}(\Gamma_1)$, $p' = p/(p-1)$.

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The problem (1.1) is related to the minimization problem

$$(\mathcal{P}_\varepsilon) \quad \min\left\{F_\varepsilon(w) - L(w) : w \in W^{1,p}_{\Gamma_0}(\Omega)\right\},$$

where

$$\begin{aligned} W^{1,p}_{\Gamma_0}(\Omega) &:= \left\{\, w \in W^{1,p}(\Omega) : w = u_0 \text{ on } \Gamma_0 \,\right\}, \\ F_\varepsilon(w) &:= \int_\Omega a_\varepsilon \phi_p(\nabla w)\, dx, \\ \phi_p(\xi) &:= \frac{1}{p}|\xi|^p, \ \forall\, \xi \in \mathbb{R}^n, n = 1,2,3, \\ L(w) &:= \int_\Omega f w\, dx + \int_{\Gamma_1} g w\, dH^2. \end{aligned} \tag{1.2}$$

We are interested in the asymptotic behavior of $(\mathcal{P}_\varepsilon)$ as $\varepsilon \to 0$. We present another proof of a study of Bellieud and Bouchitté [2] that we expect to be more suitable to treat more general geometrical and physical cases.

The bases of the cylindrical domain Ω are denoted by $\omega_0 = \omega \times \{0\}$ and $\omega_L = \omega \times \{L\}$. For each ε, we consider a periodic distribution of cells $(Y^i_\varepsilon)_{i \in I_\varepsilon}$ such that $Y^i_\varepsilon := (\varepsilon i_1, \varepsilon i_2) + (-\varepsilon/2, \varepsilon/2)^2$, and $I_\varepsilon := \left\{i \in \mathbb{Z}^2 \ : \ Y^i_\varepsilon \subset \omega\right\}$. Let $(D^i_{r_\varepsilon})_{i \in I_\varepsilon}$ be the family of disks of $\mathbb{R}^2$ centered at $\hat{x}^i_\varepsilon := (\varepsilon i_1, \varepsilon i_2)$ of radius $r_\varepsilon \ll \varepsilon$, $T^i_\varepsilon := D^i_{r_\varepsilon} \times (0, L)$ and $T_\varepsilon := \cup_{i \in I_\varepsilon} T^i_\varepsilon$. The set of thin parallel cylinders T_ε represents the fibers (see Figure 1 and Figure 2). The conductivity coefficient a_ε is

$$a_\varepsilon(x) = \begin{cases} 1 & \text{if } x \in \Omega \backslash T_\varepsilon, \\ \lambda_\varepsilon & \text{otherwise.} \end{cases}$$

We make the assumptions

$$r_\varepsilon \to 0, \quad \frac{r_\varepsilon}{\varepsilon} \to 0, \quad \lambda_\varepsilon \to +\infty, \quad k_\varepsilon := \lambda_\varepsilon \frac{r^2_\varepsilon}{\varepsilon^2} \to k, \ k \geq 0 \quad \text{as } \varepsilon \to 0. \tag{1.3}$$

In [2], it was shown that the asymptotic limit of $(\mathcal{P}_\varepsilon)$ is

$$\min\left\{\Phi(u,v) - L(u) : (u,v) \in (L^p(\Omega))^2\right\},$$

where

$$\Phi(u,v) = \begin{cases} \int_\Omega \phi_p(\nabla u)\, dx + \frac{k\pi}{p} \int_\Omega \left|\frac{\partial v}{\partial x_3}\right|^p dx + \frac{2\pi\gamma}{p} \int_\Omega |v - u|^p\, dx, \\ \qquad\qquad \text{if } \begin{cases} (u,v) \in W^{1,p}_{\Gamma_0}(\Omega) \times L^p(\omega, W^{1,p}(0,L)), \\ v = u_0 \text{ on } \Gamma_0 \cap (\omega_0 \cup \omega_L), \end{cases} \\ +\infty \quad \text{otherwise,} \end{cases} \tag{1.4}$$

and

$$[0, +\infty] \ni \gamma = \begin{cases} \lim\limits_{\varepsilon \to 0} \varepsilon^{-2} |\log r_\varepsilon|^{-1} & \text{if } p = 2, \\ \lim\limits_{\varepsilon \to 0} \left|\frac{2-p}{p-1}\right|^{p-1} r_\varepsilon^{2-p} \varepsilon^{-2} & \text{if } p \neq 2. \end{cases} \tag{1.5}$$

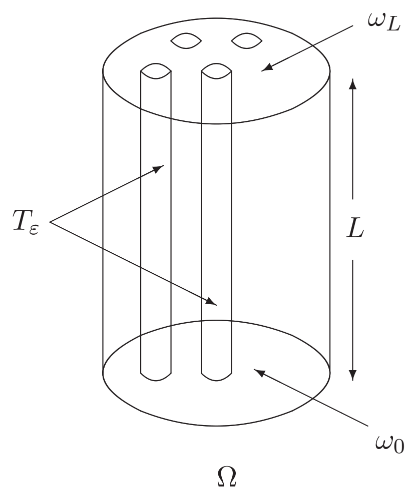

Figure 1: the domain $\Omega = \omega \times (0, L)$ occupied by a composite material

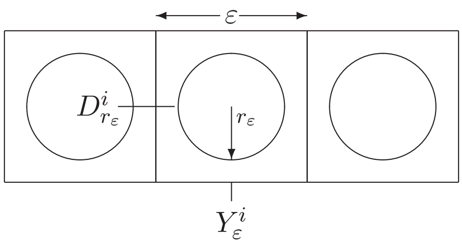

Figure 2: the circular cross section of the fiber, $Y^i_\varepsilon \subset \omega$

Here, the boundary data u_0 is assumed to be Lipschitz in order to ensure that the infimum value of problem $(\mathcal{P}_\varepsilon)$ remains finite as $\varepsilon \to 0$. In case $k = +\infty$, we add further assumption

$$k_\varepsilon r_\varepsilon \to 0, \quad \text{as } \varepsilon \to 0. \tag{1.6}$$

The conditions

$$k > 0 \quad \text{and} \quad \{\, \gamma > 0 \quad \text{or} \quad \omega_0 \subset \Gamma_0 \quad \text{or} \quad \omega_L \subset \Gamma_0 \,\} \tag{1.7}$$

guarantee that the functional Φ is coercive in $W^{1,p}(\Omega) \times L^p(\omega, W^{1,p}(0, L)$.

We are concerned with the extension of this result to more general cross sections of the fibers and more general energy density than ϕ_p. The aim of this paper is therefore to provide another proof that we expect to be more suitable to treat such general cases. The steps of the proof in [2] are to successively establish:

(i) a compactness property of the sequence (u_ε) such that $F_\varepsilon(u_\varepsilon) < C$,

(ii) a lower bound inequality of the sequence $(F_\varepsilon(u_\varepsilon))$,

(iii) an upper bound inequality of the sequence $(F_\varepsilon(u_\varepsilon))$.

Here we replace the steps (ii) and (iii) by

(ii′) an upper equality of the sequence $(F_\varepsilon(u_\varepsilon))$,

(iii′) a lower bound inequality of the sequence $(F_\varepsilon(u_\varepsilon))$ which essentially uses a subdiffenrential inequality.

2 An Alternative Strategy

It consists, under (1.3), (1.5), (1.6) and (1.7), in proving the following three propositions. In the sequel, the symbols $\to$, $\rightharpoonup$ and $\overset{*}{\rightharpoonup}$ stand for the strong convergence, the weak convergence and the weak star convergence, respectively. As usual, the letter C denotes various constants and for all $\xi = (\xi_1, \xi_2, \xi_3)$ in $\mathbb{R}^3$, $\hat{\xi}$ stands for (ξ_1, ξ_2).

Proposition 2.1 (compactness property)**.** *Let (u_ε) be a sequence such that* $\sup F_\varepsilon(u_\varepsilon)$ *is finite. Then (u_ε) is strongly relatively compact in $L^p(\Omega)$ and (v_ε), given by $v_\varepsilon := \frac{|\Omega|}{|T_\varepsilon|} 1_{T_\varepsilon} u_\varepsilon$, is bounded in $L^1(\Omega)$ and, up to a subsequence, (v_ε) weakly* converges in the space of bounded measures $\mathcal{M}_b(\Omega)$ to an element v of $L^p(\Omega)$.*

Proposition 2.2 (upper bound equality)**.** *For all (u, v) in $(L^p(\Omega))^2$, such that $\Phi(u, v) < +\infty$, there exists a sequence (u_ε) such that $u_\varepsilon \to u$ in $L^p(\Omega)$, $v_\varepsilon \overset{*}{\rightharpoonup} v$ in $\mathcal{M}_b(\Omega)$ and*

$$\lim_{\varepsilon\to 0} F_\varepsilon(u_\varepsilon) = \Phi(u, v).$$

Proposition 2.3 (lower bound inequality)**.** *For all u in $L^p(\Omega)$ and for all sequences (u_ε) such that $u_\varepsilon \to u$ in $L^p(\Omega)$, $v_\varepsilon \overset{*}{\rightharpoonup} v$ in $\mathcal{M}_b(\Omega)$, one has:*

$$\liminf_{\varepsilon\to 0} F_\varepsilon(u_\varepsilon) \geq \Phi(u, v).$$

The proofs of these propositions are presented in the following sections.

2.1 Proof of Proposition 2.1

Compactness property was already proved in [2].

2.2 Proof of Proposition 2.2

Our sole contribution is to prove that we can replace inequality by equality, for that we use the same approximation u'_ε of u as in [2]

$$u'_\varepsilon = (1 - \theta_\varepsilon)u + \theta_\varepsilon w_\varepsilon.$$

The function θ_ε is first defined on the closure of $\omega_\varepsilon := \cup_{i\in I_\varepsilon} Y^i_\varepsilon$ as a $(-\varepsilon/2, \varepsilon/2)^2$-periodic continuous function which satisfies $0 \leq \theta_\varepsilon \leq 1$, $\theta_\varepsilon = 1$ on $D_\varepsilon := \cup_{i\in I_\varepsilon} D^i_{r_\varepsilon}$, $\theta_\varepsilon = 0$ on $\overline{\omega}_\varepsilon \setminus \cup_{i\in I_\varepsilon} D^i_{R_\varepsilon}$, where $D^i_{R_\varepsilon}$ is the disk of $\mathbb{R}^2$ centered at $\hat{x}^i_\varepsilon$ of radius R_ε such that $r_\varepsilon \ll R_\varepsilon \ll \varepsilon$. Next θ_ε is assumed to vanish on $\overline{\omega} \setminus \omega_\varepsilon$ and

$$w_\varepsilon(\hat{x}, x_3) = \sum_{i\in I_\varepsilon} \left(\frac{1}{|D^i_{r_\varepsilon}|} \int_{D^i_{r_\varepsilon}} v(\hat{y}, x_3)\, d\hat{y} \right) 1_{Y^i_\varepsilon}(\hat{x}).$$

The approximation u'_ε does not satisfy the boundary condition on $\Gamma_0 \cap (\omega_0 \cup \omega_L)$ so that, as in [2], we introduce a sharper approximation

$$u^\#_\varepsilon := u\varphi_\varepsilon + u'_\varepsilon(1 - \varphi_\varepsilon).$$

Here φ_ε is a $C^\infty(\overline{\Omega})$ function which satisfies $\varphi_\varepsilon = 1$ on Γ_0, $\varphi_\varepsilon = 0$ on $\overline{\Omega}\backslash\Sigma_\varepsilon$, $|\nabla\varphi_\varepsilon| \le C/r_\varepsilon$ on $\overline{\Omega}$ where $\Sigma_\varepsilon := \{x \in \Omega : \mathrm{dist}(x, \Gamma_0) < r_\varepsilon\}$. We assume that u and v are Lipschitz on $\overline{\Omega}$ and there exists $L > 0$ such that

$$\left|\frac{\partial v}{\partial x_3}(\hat{x}', x_3) - \frac{\partial v}{\partial x_3}(\hat{x}'', x_3)\right| < L|\hat{x}' - \hat{x}''| \quad \forall\, (\hat{x}', x_3), (\hat{x}'', x_3) \in \overline{\Omega}. \tag{2.1}$$

Letting Ψ be any continuous function on $\overline{\Omega}$ such that $0 \le \Psi \le 1$, we introduce F_ε^Ψ, Φ^Ψ defined by similar formulae as the ones of F_ε and Φ but with $\Psi\, dx$ in place of dx. We will prove the lemma:

Lemma 2.4.

$$\lim_{\varepsilon\to 0} F_\varepsilon^\Psi(u_\varepsilon') = \Phi^\Psi(u, v).$$

If Lemma 2.4 is proved, then, by a classical approximation process, we can deduce

$$\lim_{\varepsilon\to 0} F_\varepsilon(u_\varepsilon') = \Phi(u, v). \tag{2.2}$$

Finally, we complete the proof of (2.2) for any (u, v) such that $\Phi(u, v) < +\infty$ by approximation and diagonalization arguments.

Proof of Lemma 2.4. We split $F_\varepsilon^\Psi(u_\varepsilon')$ in three parts

$$F_\varepsilon^\Psi(u_\varepsilon') = F_\varepsilon^{\Psi 1}(u_\varepsilon') + F_\varepsilon^{\Psi 2}(u_\varepsilon') + F_\varepsilon^{\Psi 3}(u_\varepsilon'). \tag{2.3}$$

First, we consider

$$F_\varepsilon^{\Psi 1}(u_\varepsilon') := \int_{\Omega\backslash B_\varepsilon \cup T_\varepsilon} \phi_p(\nabla u_\varepsilon')\Psi\, dx = \int_{\Omega\backslash B_\varepsilon \cup T_\varepsilon} \phi_p(\nabla u)\Psi\, dx,$$

where $B_\varepsilon := \cup_{i\in\varepsilon I_\varepsilon} D^i_{R_\varepsilon}\backslash\overline{D^i_{r_\varepsilon}} \times (0, L)$. Hence, the assumption $R_\varepsilon \ll \varepsilon$ yields $\lim_{\varepsilon\to 0}|B_\varepsilon \cup T_\varepsilon| = 0$ and, consequently,

$$\lim_{\varepsilon\to 0} F_\varepsilon^{\Psi 1}(u_\varepsilon') = \int_\Omega \phi_p(\nabla u)\Psi\, dx.$$

Next, we pay attention to

$$F_\varepsilon^{\Psi 2}(u_\varepsilon') := \int_{B_\varepsilon} \phi_p(\nabla u_\varepsilon')\Psi\, dx.$$

Writing

$$z_\varepsilon := (v - u)\widehat{\nabla}\theta_\varepsilon, \tag{2.4}$$

we obtain

$$\nabla u_\varepsilon' = z_\varepsilon + (w_\varepsilon - v)\nabla\theta_\varepsilon + (1 - \theta_\varepsilon)\nabla u + \theta_\varepsilon\nabla w_\varepsilon.$$

Let us show

$$\lim_{\varepsilon\to 0}\int_{B_\varepsilon} (\phi_p(\nabla u_\varepsilon') - \phi_p(z_\varepsilon))\Psi\, dx = 0. \tag{2.5}$$

The function ϕ_p, being convex and positively homogeneous of degree p, satisfies

$$\forall\, \xi, \eta \in \mathbb{R}^n,\ n = 1, 2, 3, \quad |\phi_p(\xi) - \phi_p(\eta)| \leq C|\xi - \eta|(|\xi|^{p-1} + |\eta|^{p-1}). \tag{2.6}$$

Therefore, Hölder inequality yields

$$\left|\int_{B_\varepsilon} (\phi_p(\nabla u'_\varepsilon) - \phi_p(z_\varepsilon))\Psi\, dx\right| \leq C \left(\int_{B_\varepsilon} |\nabla u'_\varepsilon - z_\varepsilon|^p\, dx\right)^{\frac{1}{p}} \left(\int_{B_\varepsilon} |\nabla u'_\varepsilon|^p\, dx + \int_{B_\varepsilon} |z_\varepsilon|^p\, dx\right)^{\frac{1}{p'}}.$$

The smoothness of (u, v) implies

$$\left.\begin{array}{l} u'_\varepsilon = u \text{ on } \Omega\backslash(B_\varepsilon \cup T_\varepsilon), \quad |u'_\varepsilon| \leq C \text{ on } \Omega, \quad |\nabla w_\varepsilon| \leq C \text{ on } B_\varepsilon, \\ u'_\varepsilon = w_\varepsilon \text{ on } T_\varepsilon, \quad |w_\varepsilon - v| \leq C R_\varepsilon \text{ on } B_\varepsilon, \end{array}\right\} \tag{2.7}$$

so that

$$\int_{B_\varepsilon} |\nabla u'_\varepsilon|^p\, dx + \int_{B_\varepsilon} |z_\varepsilon|^p\, dx \leq C\varepsilon^{-2} \int_{D(r_\varepsilon, R_\varepsilon)} \phi_p(\widehat{\nabla}\theta_\varepsilon)\, d\hat{x},$$

$$\int_{B_\varepsilon} |\nabla u'_\varepsilon - z_\varepsilon|^p\, dx \leq C R_\varepsilon^p \varepsilon^{-2} \int_{D(r_\varepsilon, R_\varepsilon)} \phi_p(\widehat{\nabla}\theta_\varepsilon)\, d\hat{x},$$

where $D(r_\varepsilon, R_\varepsilon) = \left\{\hat{x} \in \mathbb{R}^2 : r_\varepsilon < |\hat{x}| < R_\varepsilon\right\}$. Hence, if we choose θ_ε such that

$$\exists\, M > 0\ ; \qquad \varepsilon^{-2} \int_{D(r_\varepsilon, R_\varepsilon)} \phi_p(\widehat{\nabla}\theta_\varepsilon)\, d\hat{x}\ \leq M \qquad \forall\, \varepsilon > 0, \tag{2.8}$$

then (2.5) is true. We finally have

$$\begin{aligned} & \lim_{\varepsilon \to 0} F_\varepsilon^{\Psi 2}(u'_\varepsilon) \\ = & \lim_{\varepsilon \to 0} \int_{B_\varepsilon} \phi_p(z_\varepsilon)\Psi\, dx \\ = & \lim_{\varepsilon \to 0} \int_{B_\varepsilon} |v - u|^p \phi_p(\widehat{\nabla}\theta_\varepsilon)\Psi\, dx \\ = & \lim_{\varepsilon \to 0} \int_{D(r_\varepsilon, R_\varepsilon)} \phi_p(\widehat{\nabla}\theta_\varepsilon)\, d\hat{x} \int_0^L \sum_{i \in I_\varepsilon} |v - u|^p(\hat{y}^i_\varepsilon, x_3)\Psi(\hat{y}^i_\varepsilon, x_3)\, dx_3 \quad (\text{with } \hat{y}^i_\varepsilon \in Y^i_\varepsilon) \\ = & \lim_{\varepsilon \to 0} \varepsilon^{-2} \int_{D(r_\varepsilon, R_\varepsilon)} \phi_p(\widehat{\nabla}\theta_\varepsilon)\, d\hat{x} \int_0^L \sum_{i \in I_\varepsilon} |Y^i_\varepsilon||v - u|^p(\hat{y}^i_\varepsilon, x_3)\Psi(\hat{y}^i_\varepsilon, x_3)\, dx_3. \end{aligned}$$

Observe that $\lim_{\varepsilon \to 0} \int_0^L \sum_{i \in I_\varepsilon} |Y^i_\varepsilon||v - u|^p(\hat{y}^i_\varepsilon, x_3)\Psi(\hat{y}^i_\varepsilon, x_3)\, dx_3 = \int_\Omega |v - u|^p \Psi\, dx$. To get the lowest upper bound in Proposition 2.2, it is clear that θ_ε has to be the solution of the capacitary problem

$$(\mathcal{P}^{\text{cap}}_\varepsilon) \quad \min \left\{ \int_{D(r_\varepsilon, R_\varepsilon)} \phi_p(\widehat{\nabla}\varphi)\, d\hat{x} : \begin{array}{l} \varphi \in W^{1,p}(D(r_\varepsilon, R_\varepsilon)), \\ \varphi(\hat{x}) = 1 \text{ on } |\hat{x}| = r_\varepsilon, \\ \varphi(\hat{x}) = 0 \text{ on } |\hat{x}| = R_\varepsilon. \end{array} \right\}$$

As observed in [2], we have

$$\theta_\varepsilon = \begin{cases} \frac{\log R_\varepsilon - \log|\hat{x}|}{\log R_\varepsilon - \log r_\varepsilon} & \text{if } p=2, \\ \frac{R_\varepsilon^s - |\hat{x}|^s}{R_\varepsilon^s - r_\varepsilon^s} & \text{if } p \neq 2 \quad (s = \frac{p-2}{p-1}) \end{cases}$$

and

$$\int_{D(r_\varepsilon, R_\varepsilon)} \phi_p(\widehat{\nabla}\theta_\varepsilon)\, d\hat{x} = \frac{2\pi}{p}\Gamma_p(r_\varepsilon, R_\varepsilon),$$

where $\Gamma_p(r_\varepsilon, R_\varepsilon) := \begin{cases} \frac{1}{\log R_\varepsilon - \log r_\varepsilon} & \text{if } p=2, \\ (\frac{s}{R_\varepsilon^s - r_\varepsilon^s})^{p-1} & \text{if } p \neq 2 \quad (s = \frac{p-2}{p-1}). \end{cases}$

Note that

$$\lim_{\varepsilon\to 0} \varepsilon^{-2}\Gamma_p(r_\varepsilon, R_\varepsilon) = \gamma.$$

If $\gamma < +\infty$, then (2.8) is satisfied and

$$\lim_{\varepsilon\to 0} F_\varepsilon^{\Psi 2}(u'_\varepsilon) = \frac{2\pi\gamma}{p}\int_\Omega |v-u|^p \Psi\, dx.$$

When $\gamma = +\infty$, it suffices to prove that $\lim_{\varepsilon\to 0} F_\varepsilon^{\Psi 2}(u'_\varepsilon) = 0$. Due to (2.7), the result is a consequence of $F_\varepsilon^{\Psi 2}(u'_\varepsilon) \le C R_\varepsilon^p \varepsilon^{-2}\Gamma_p(r_\varepsilon, R_\varepsilon)$, which tends to zero.

Now, we consider the remaining part

$$F_\varepsilon^{\Psi 3}(u'_\varepsilon) := \int_{T_\varepsilon} \lambda_\varepsilon \phi_p(\nabla w_\varepsilon)\Psi\, dx.$$

Recalling the assumption (2.1) on v and using the local Lipschitz property (2.6), we deduce

$$\begin{aligned} \lim_{\varepsilon\to 0} F_\varepsilon^{\Psi 3}(u'_\varepsilon) &= \lim_{\varepsilon\to 0}\frac{1}{|T_\varepsilon|}\int_{T_\varepsilon} \lambda_\varepsilon \phi_p\left(\frac{\partial v}{\partial x_3}\right)\Psi\, dx \\ &= \frac{k\pi}{p}\int_\Omega \left|\frac{\partial v}{\partial x_3}\right|^p \Psi\, dx, \end{aligned}$$

as shown in [2]. ■

Now, we will prove the upper bound equality by using the sharper approximation $(u_\varepsilon^\#)$. We start with

$$F_\varepsilon(u_\varepsilon^\#) = \int_{\Sigma_\varepsilon} a_\varepsilon \phi_p(\nabla u_\varepsilon^\#)\, dx + \int_{\Omega\setminus\Sigma_\varepsilon} a_\varepsilon \phi_p(\nabla u'_\varepsilon)\, dx. \tag{2.9}$$

Conditions (2.7) imply $|u'_\varepsilon - u| \le C(r_\varepsilon 1_{T_\varepsilon} + R_\varepsilon 1_{B_\varepsilon})$. Hence

$$\int_{\Sigma_\varepsilon} a_\varepsilon \phi_p(\nabla u_\varepsilon^\#)\, dx \le C\left(|\Sigma_\varepsilon| + \int_{\Sigma_\varepsilon} a_\varepsilon(x)|\nabla u'_\varepsilon|^p\, dx + \lambda_\varepsilon |T_\varepsilon \cap \Sigma_\varepsilon| + \left(\frac{R_\varepsilon}{r_\varepsilon}\right)^p |\Sigma_\varepsilon|\right).$$

Lemma 2.4 implies that for every $\Psi \in C^0(\overline{\Omega}, [0,1])$, such that $\Psi = 1$ on a small neighborhood of $\Gamma_0 \cap (\omega_0 \cup \omega_L)$,

$$\begin{aligned}\limsup_{\varepsilon\to 0}\int_{\Sigma_\varepsilon} a_\varepsilon \phi_p(\nabla u_\varepsilon^{\#})\,dx &\leq \limsup_{\varepsilon\to 0}\int_{\Omega} a_\varepsilon \phi_p(\nabla u'_\varepsilon)\Psi\,dx\\ &= \int_\Omega \left(\phi_p(\nabla u) + \frac{k\pi}{p}\left|\frac{\partial v}{\partial x_3}\right|^p + \frac{2\pi\gamma}{p}|v-u|^p\right)\Psi\,dx.\end{aligned}$$

Thus, by letting Ψ tend to zero, we deduce

$$\lim_{\varepsilon\to 0}\int_{\Sigma_\varepsilon} a_\varepsilon \phi_p(\nabla u_\varepsilon^{\#})\,dx = \lim_{\varepsilon\to 0}\int_{\Sigma_\varepsilon} a_\varepsilon \phi_p(\nabla u'_\varepsilon)\,dx = 0,$$

and

$$\begin{aligned}\lim_{\varepsilon\to 0}\int_{\Omega\setminus\Sigma_\varepsilon} a_\varepsilon \phi_p(\nabla u'_\varepsilon)\,dx &= \lim_{\varepsilon\to 0}\left(\int_\Omega a_\varepsilon \phi_p(\nabla u'_\varepsilon)\,dx - \int_{\Sigma_\varepsilon} a_\varepsilon \phi_p(\nabla u'_\varepsilon)\,dx\right)\\ &= \int_\Omega \phi_p(\nabla u) + \frac{k\pi}{p}\left|\frac{\partial v}{\partial x_3}\right|^p + \frac{2\pi\gamma}{p}|v-u|^p\,dx,\end{aligned}$$

which proves the result for (u,v) smooth. We complete the proof by a standard approximation of (u,v) and a diagonalization argument [1].

2.3 Proof of Proposition 2.3

It is enough to consider $\liminf_\varepsilon F_\varepsilon(u_\varepsilon) < +\infty$. Due to the compactness property, (u,v) is in $(L^p(\Omega))^2$. We first consider the term $F^2_\varepsilon(u_\varepsilon)$. Let (u_η, v_η) be Lipschitz on $\overline{\Omega}$ such that $\lim_{\eta\to 0}\|u_\eta - u\|_{L^p(\Omega)} + \|v_\eta - v\|_{L^p(\Omega)} = 0$. We define $(v_\eta - u_\eta)_\varepsilon := \sum_{i\in I_\varepsilon}(v_\eta - u_\eta)(\hat{x}^i_\varepsilon, x_3)1_{Y^i_\varepsilon}$ and $z_{\eta\varepsilon} := (v_\eta - u_\eta)_\varepsilon \widehat{\nabla}\theta_\varepsilon$. Because of the local Lipschitz property (2.6) of ϕ_p and $(u,v) \in (L^p(\Omega))^2$, Hölder inequality implies

$$\lim_{\varepsilon\to 0}\int_{B_\varepsilon} \phi_p(z_{\eta\varepsilon}) - \phi_p(z_\varepsilon)\,dx = 0.$$

The proof of the upper bound equality allows us to write

$$\lim_{\varepsilon\to 0}\phi_p(z_{\eta\varepsilon}) = \frac{2\pi\gamma}{p}\int_\Omega |v_\eta - u_\eta|^p\,dx.$$

The convexity of ϕ_p and the fact that $\phi_p(\nabla u_\varepsilon) \geq \phi_p(\widehat{\nabla} u_\varepsilon)$ yield

$$\begin{aligned}\liminf_{\varepsilon\to 0} F^2_\varepsilon(u_\varepsilon) &\geq \liminf_{\varepsilon\to 0}\int_{B_\varepsilon}\phi_p(\widehat{\nabla} u_\varepsilon)\,dx\\ &\geq \liminf_{\varepsilon\to 0}\int_{B_\varepsilon}\phi_p(z_{\eta\varepsilon})\,dx\\ &\quad + \liminf_{\varepsilon\to 0}\int_{B_\varepsilon}\phi'_p(z_{\eta\varepsilon})\cdot(\widehat{\nabla} u_\varepsilon - z_{\eta\varepsilon})\,dx. \qquad (2.10)\end{aligned}$$

The very definition of ϕ_p implies

$$\begin{aligned} \phi'_p(\xi) &= |\xi|^{p-2}\xi && \forall \xi \in \mathbb{R}^n,\ n = 1,2,3, \\ \phi'_p(t\xi) &= \phi'_p(t)\phi'_p(\xi) && \forall (t,\xi) \in \mathbb{R}\times\mathbb{R}^n, n = 1,2,3, \\ \phi'_p(\xi)\cdot\xi &= p\phi_p(\xi) && \forall \xi \in \mathbb{R}^n,\ n = 1,2,3. \end{aligned}$$

Hence

$$\lim_{\varepsilon\to 0}\int_{B_\varepsilon} \phi'_p(z_{\eta\varepsilon})\cdot z_{\eta\varepsilon}\, dx = 2\pi\gamma\int_\Omega |v_\eta - u_\eta|^p\, dx. \tag{2.11}$$

For the other term of (2.10), we have

$$\int_{B_\varepsilon} \phi'_p(z_{\eta\varepsilon})\cdot\widehat{\nabla} u_\varepsilon\, dx = \sum_{i\in I_\varepsilon}\int_0^L \phi'_p(v_\eta - u_\eta)(\hat{x}^i_\varepsilon, x_3)\int_{D^i(r_\varepsilon,R_\varepsilon)} \phi'_p(\widehat{\nabla}\theta_\varepsilon)\cdot\widehat{\nabla} u_\varepsilon\, d\hat{x}\, dx_3,$$

where $D^i(r_\varepsilon,R_\varepsilon) = D^i_{R_\varepsilon}\setminus\overline{D^i_{r_\varepsilon}}$. Let ν be the outer normal on $\partial D^i(r_\varepsilon,R_\varepsilon)$, the very definition of θ_ε as a solution of $(\mathcal{P}^{\text{cap}}_\varepsilon)$ yields

$$\begin{aligned} \int_{D^i(r_\varepsilon,R_\varepsilon)} \phi'_p(\widehat{\nabla}\theta_\varepsilon)\cdot\widehat{\nabla} u_\varepsilon\, d\hat{x} &= \int_{\partial D^i(r_\varepsilon,R_\varepsilon)} (\phi'_p(\widehat{\nabla}\theta_\varepsilon)\cdot\nu) u_\varepsilon\, dl \\ &= \int_{\partial D^i_{R_\varepsilon}} (\phi'_p(\widehat{\nabla}\theta_\varepsilon)\cdot\nu) u_\varepsilon\, dl + \int_{\partial D^i_{r_\varepsilon}} (\phi'_p(\widehat{\nabla}\theta_\varepsilon)\cdot\nu) u_\varepsilon\, dl \\ &= -\tilde{u}^i_\varepsilon\int_{\partial D^i_{r_\varepsilon}} (\phi'_p(\widehat{\nabla}\theta_\varepsilon)\cdot\nu)\, dl + \tilde{v}^i_\varepsilon\int_{\partial D^i_{r_\varepsilon}} (\phi'_p(\widehat{\nabla}\theta_\varepsilon)\cdot\nu) u_\varepsilon\, dl \end{aligned}$$

where $\tilde{u}^i_\varepsilon := \frac{\int_{\partial D^i_{R_\varepsilon}} (\phi'_p(\widehat{\nabla}\theta_\varepsilon)\cdot\nu)u_\varepsilon\, dl}{\int_{\partial D^i_{r_\varepsilon}} (\phi'_p(\widehat{\nabla}\theta_\varepsilon)\cdot\nu)\, dl} = \frac{1}{2\pi R_\varepsilon}\int_{\partial D^i_{R_\varepsilon}} (\phi'_p(\widehat{\nabla}\theta_\varepsilon)\cdot\nu)u_\varepsilon\, dl$, $\tilde{u}_\varepsilon := \sum_{i\in I_\varepsilon}\tilde{u}^i_\varepsilon 1_{Y^i_\varepsilon}$, $\tilde{v}^i_\varepsilon := \frac{\int_{\partial D^i_{r_\varepsilon}} (\phi'_p(\widehat{\nabla}\theta_\varepsilon)\cdot\nu)v_\varepsilon\, dl}{\int_{\partial D^i_{r_\varepsilon}} (\phi'_p(\widehat{\nabla}\theta_\varepsilon)\cdot\nu)\, dl} = \frac{1}{2\pi r_\varepsilon}\int_{\partial D^i_{r_\varepsilon}} (\phi'_p(\widehat{\nabla}\theta_\varepsilon)\cdot\nu)v_\varepsilon\, dl$, $\tilde{v}_\varepsilon := \sum_{i\in I_\varepsilon}\tilde{v}^i_\varepsilon 1_{Y^i_\varepsilon}$. Thus,

$$\begin{aligned} \int_{D^i(r_\varepsilon,R_\varepsilon)} \phi'_p(\widehat{\nabla}\theta_\varepsilon)\cdot\widehat{\nabla} u_\varepsilon\, d\hat{x} &= (\tilde{v}^i_\varepsilon - \tilde{u}^i_\varepsilon)\int_{\partial D^i_{r_\varepsilon}} (\phi'_p(\widehat{\nabla}\theta_\varepsilon)\cdot\nu)\, dl \\ &= (\tilde{v}^i_\varepsilon - \tilde{u}^i_\varepsilon)\int_{D^i(r_\varepsilon,R_\varepsilon)} \phi'_p(\widehat{\nabla}\theta_\varepsilon)\cdot\widehat{\nabla}\theta_\varepsilon\, d\hat{x} \\ &= 2\pi\Gamma_p(r_\varepsilon,R_\varepsilon)(\tilde{v}^i_\varepsilon - \tilde{u}^i_\varepsilon), \end{aligned}$$

and

$$\int_{B_\varepsilon} \phi'_p(z_{\eta\varepsilon})\cdot\widehat{\nabla} u_\varepsilon\, dx = 2\pi\Gamma_p(r_\varepsilon,R_\varepsilon)\int_\Omega \phi'_p((v_\eta - u_\eta)_\varepsilon)(\tilde{v}_\varepsilon - \tilde{u}_\varepsilon)\, dx.$$

It was shown in [2] that $(\tilde{v}_\varepsilon - \tilde{u}_\varepsilon) \rightharpoonup (v-u)$ in $L^p(\Omega)$. On the other hand, $(v_\eta - u_\eta)$ being smooth and ϕ'_p being continuous from $L^p(\Omega)$ to $L^{p'}(\Omega)$, we have $\phi'_p((v_\eta - u_\eta)_\varepsilon) \to \phi'_p(v_\eta - u_\eta)$ in $L^{p'}(\Omega)$. Hence,

$$\lim_{\varepsilon\to 0}\int_{B_\varepsilon} \phi'_p(z_{\eta\varepsilon})\cdot\widehat{\nabla} u_\varepsilon\, dx = 2\pi\gamma\int_\Omega \phi'_p(v_\eta - u_\eta)(v-u)\, dx. \tag{2.12}$$

Therefore, (2.10), (2.11) and (2.12) imply

$$\begin{aligned}\liminf_{\varepsilon\to 0} F_\varepsilon^2(u_\varepsilon) \quad &\geq \quad \frac{2\pi\gamma}{p}\int_\Omega |v_\eta - u_\eta|^p\, dx \\ &+2\pi\gamma\left[\int_\Omega |v_\eta - u_\eta|^p\, dx - \int_\Omega \phi_p'(v_\eta - u_\eta)(v-u)\, dx\right].\end{aligned}$$

The expected lower bound for $F_\varepsilon^2(u_\varepsilon)$ is obtained by letting η tend to zero.

To complete the proof it suffices to use the arguments of [2] concerning the lower bounds for $F_\varepsilon^1(u_\varepsilon)$, $F_\varepsilon^3(u_\varepsilon)$ and the fact that v belongs to $L^p(\omega, W^{1,p}(0,L))$.

2.4 The Final Result

The following theorem, a convergence result for the minimizer of $(\mathcal{P}_\varepsilon)$, is a standard consequence of the previous three propositions.

Theorem 2.5. *Let the assumptions* (1.3) *and* (1.5) *hold with* $(k,\gamma) \in (0,+\infty)$. *Then the unique solution* $\overline{u}_\varepsilon$ *of* $(\mathcal{P}_\varepsilon)$ *converges weakly in* $W^{1,p}(\Omega)$ *to the unique solution* $\overline{u}$ *of the problem*

$$\min\left\{\min\left\{\Phi(u,v) - L(u) : v \in L^p(\Omega)\right\} : u \in L^p(\Omega)\right\},$$

where Φ *and* L *are defined by* (1.4) *and* (1.2) *respectively.*

Proof. The proof of this theorem is the same as that in [2]. ■

3 Conclusions and Remarks

The previous analysis can be easily extended to the case when ϕ_p is replaced by any strictly convex function which satisfies

$$\exists\, M > 0,\ \exists\, r \in (1,p)\ ; \qquad |W(\xi) - \phi_p(\xi)| \leq M|\xi|^r \qquad \forall\, \xi \in \mathbb{R}^3, \tag{3.1}$$

the density function associated with $\Phi(u,v)$ becomes

$$W(\nabla u) + 2\pi\gamma|v-u|^p + W\left(\frac{\partial v}{\partial x_3}\right).$$

Indeed, (3.1) and Hölder inequality imply

$$\left|\int_{B_\varepsilon} W(\nabla u_\varepsilon)\,dx - \int_{B_\varepsilon} \phi_p(\nabla u_\varepsilon)\,dx\right| \leq M|B_\varepsilon|^{1-\frac{r}{p}} \int_\Omega |\nabla u_\varepsilon|^p\,dx,$$

while our arguments and those of [2] to derive the upper bound and lower bound respectively are valid when ϕ_p is replaced by any convex function satisfying a growth condition like

$$\exists\, \alpha, \beta > 0\ ; \qquad \alpha(|\xi|^p - 1) \leq W(\xi) \leq \beta(1 + |\xi|^p) \qquad \forall\, \xi \in \mathbb{R}^3,$$

which is an obvious consequence of (3.1).

Eventually, the key arguments of our analysis are the identification of γ, θ_ε in terms of the solution of capacitary problems and the use of the p-positive homogeneity and convexity of ϕ_p and of the fact that $\phi_p(\xi) \geq \phi_p(\hat{\xi})$. Thus, it is easy to guess what could be $\Phi(u,v)$, when ϕ_p is replaced by any strictly convex function and when the cross sections of the fibers are smooth star-shaped domains of $\mathbb{R}^2$. We hope that our proposed strategy will be able to reduce and overcome the involved technical difficulties.

4 Acknowlagement

Somsak was supported by a research fund from Commission on Higher Education and the Thailand Research Fund (MRG5080422). Nuttawat was supported by a scholarship from The Ministry Staff Development Project of the Ministry of University Affairs.

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(Received 13 May 2008)

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