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Coefficient Functionals for Starlike Functions of Reciprocal Order

Virendra Kumar[1], **Sushil Kumar**[2] **and Nak Eun Cho**[3,*]

[1] *Department of Mathematics, Ramanujan College, University of Delhi, Kalkaji, New Delhi–110019, India*
e-mail : vktmaths@yahoo.in

[2] *Bharati Vidyapeeth's College of Engineering, Delhi–110063, India*
e-mail : sushilkumar16n@gmail.com

[3] *Department of Applied Mathematics, Pukyong National University, Busan–48513, Korea*
e-mail : necho@pknu.ac.kr

Abstract Several properties of the class $\mathcal{S}_r^*(\alpha)$ of starlike functions of reciprocal order $\alpha\,(0 \leq \alpha < 1)$ defined on the open unit disk have been studied in this paper. The paper begins with a sufficient condition for analytic functions to be in the class $\mathcal{S}_r^*(\alpha)$. Further, the sharp bounds on third order Hermitian-Toeplitz determinant, initial inverse coefficients and initial logarithmic coefficients for functions in the class $\mathcal{S}_r^*(\alpha)$ are derived.

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1. Introduction

Let the symbol $\mathcal{A}$ denote the class of normalised analytic functions $f(z) = \sum_{n=1}^{\infty} a_n z^n$; $(a_1 = 1)$ defined on $\mathbb{D} := \{z \in \mathbb{C} \colon |z| < 1\}$. The subclass of $\mathcal{A}$ consisting of one-to-one functions denoted by $\mathcal{S}$. A function $f \in \mathcal{S}$ is starlike of order α $(0 \leq \alpha < 1)$, denoted by $\mathcal{S}^*(\alpha)$, if $\Re(zf'(z)/f(z)) > \alpha$ for all $z \in \mathbb{D}$. Note that $\mathcal{S}^*(0) = \mathcal{S}^*$ is the class of starlike functions on $\mathbb{D}$. The function $f \in \mathcal{A}$ is starlike of reciprocal order α, denoted by $\mathcal{S}_r^*(\alpha)$, if

$$\Re\left\{\frac{f(z)}{zf'(z)}\right\} > \alpha, \quad z \in \mathbb{D}.$$

The function $g(z) = ze^{(1-\alpha)z}$ is a starlike function of reciprocal order $1/(2-\alpha)$ [1]. Note that every function $f \in \mathcal{S}_r^* := \mathcal{S}_r^*(0)$ is starlike and univalent. Ravichandran and Kumar [2] investigated the argument estimates for the analytic functions $f \in \mathcal{S}_r^*(\alpha)$. Frasin and Sabri [3] derived certain sufficient conditions for starlikeness of reciprocal order of analytic functions in $\mathbb{D}$. For more related results of some associated classes, see

*Corresponding author.

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[4–8]. Next, we are giving terminologies, concepts and literature which will be used in this paper.

For positive integers q and n, the Hermitian-Toeplitz determinant of order n associated with the coefficients of the function $f \in \mathcal{A}$, is given by

$$T_{q,n}(f) := \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ \overline{a}_{n+1} & a_n & \cdots & a_{n+q-2} \\ \vdots & \vdots & \vdots & \vdots \\ \overline{a}_{n+q-1} & \overline{a}_{n+q-2} & \cdots & a_n \end{vmatrix}.$$

The Hermitian-Toeplitz determinant $T_{q,1}(f)$ is rotationally invariant [9, 10]. Thus, the third order Hermitian-Toeplitz determinant $T_{3,1}(f)$ is given by

$$T_{3,1}(f) := 2\operatorname{Re}(a_2^2\,\bar{a_3}) - 2|a_2|^2 - |a_3|^2 + 1. \tag{1.1}$$

Cudna *et al.* [11] investigated the sharp lower and upper bounds for the second and third Hermitian-Toeplitz determinants for the classes of starlike and convex functions of order α. Jastrzębski *et al.* [12] determined the sharp estimates on the second and third order Hermitian-Toeplitz determinants for the close-to-star functions. Recently, the sharp upper and lower bounds on the Hermitian-Toeplitz determinant of the third order are computed for the classes of certain strongly starlike functions [13]. For recent updates on this type of problem one may refer to the papers [14–18].

In view of Koebe one quarter theorem, it is noted that the image of $\mathbb{D}$ under a univalent function contains a disk $\mathbb{D}_{1/4} := \{z \in \mathbb{C} \colon |z| < 1/4\}$. Thus for every univalent function f there exists inverse function f^{-1} such that $f^{-1}(f(z)) = z$ for $z \in \mathbb{D}$ and $f(f^{-1}(\omega)) = \omega$ for $|\omega| < r_0(f)$, where $r_0(f) \geq 1/4$. The Taylor series expansion of the function f^{-1} is given as

$$f^{-1}(\omega) = \omega + A_2\omega^2 + A_3\omega^3 + \cdots .$$

Löwner [19] determined the coefficient estimates for the inverse function $f \in \mathcal{S}$. The authors [20, 21] obtained bounds on the initial inverse coefficients for the classes of starlike and convex functions of order $\alpha\,(0 \leq \alpha < 1)$. In [22], the authors computed bounds on the inverse coefficients for Janowski starlike functions. For more details related to the inverse coefficient problem, see [23–26]. The logarithmic coefficients γ_n are associated with the function $f \in \mathcal{S}$ in the following way:

$$\log \frac{f(z)}{z} = 2\sum_{n=1}^{\infty} \gamma_n z^n, \ z \in \mathbb{D}. \tag{1.2}$$

For the Koebe function $k(z) = z/(1 - e^{i\theta}z)^2$, the n^{th} logarithmic coefficient is given by $\gamma_n = e^{in\theta}/n$ for each θ and for all $n \geq 1$. Using the concept of integral means, Duren and Leung [27] found the sharp bound on the n^{th} logarithmic coefficient of univalent functions. For details, see [28–32]. Elhosh [33] proved $\gamma_n \leq 1/n$ for close-to-convex functions. In 2018, the authors in the paper [34] determined the sharp estimates on γ_1, γ_2 and γ_3 for functions belonging to certain subclasses of close-to-convex functions. Recently, Adegani *et al.* [35] computed the sharp bounds on γ_n for the Ma-Minda starlike and convex functions.

Motivated by the stated research works, we first establish coefficient inequalities for functions f belonging to the class $\mathcal{S}_r^*(\alpha)$. The best possible lower and upper bounds on the third order Hermitian-Toeplitz determinant $T_{3,1}(f)$ and the bounds on the initial inverse coefficients A_2, A_3, A_4, A_5 are computed for such functions. In addition, the bounds on the initial logarithmic coefficients γ_1, γ_2, γ_3, γ_4 are also determined.

2. Preliminary Results

Let the class $\mathcal{P}$ consists of analytic functions of the form $p(z) = 1+p_1z+p_2z^2+p_3z^3+\cdots$ satisfying the inequality $\operatorname{Re} p(z) > 0$ for all $z \in \mathbb{D}$. There is a close relation between the class $\mathcal{P}$ and the class $\mathbf{B}$ of Schwarz function $w(z) = c_1z + c_2z^2 + c_3z^3 + \cdots$ and satisfying the condition $|w(z)| < 1$ for $z \in \mathbb{D}$. Consider the coefficient functional $\Psi(\mu,\nu) = |c_3 + \mu c_1c_2 + \nu c_1^3|$ associated with the coefficients of w for $w \in \mathbf{B}$ and μ, $\nu \in \mathbb{R}$. Let us assume that the symbols Ω_k's are defined as follows:

$$\begin{aligned}
\Omega_1 &:= \left\{(\mu,\nu) \in \mathbb{R}^2 : |\mu| \le 1/2,\ \ |\nu| \le 1\right\},\\
\Omega_2 &:= \left\{(\mu,\nu) \in \mathbb{R}^2 : \frac{1}{2} \le |\mu| \le 2,\ \ \frac{4}{27}(|\mu|+1)^3 - (|\mu|+1) \le \nu \le 1\right\},\\
\Omega_4 &:= \left\{(\mu,\nu) \in \mathbb{R}^2 : |\mu| \ge 1/2,\ \ \nu \le -\frac{2}{3}(|\mu|+1)\right\},\\
\Omega_5 &:= \left\{(\mu,\nu) \in \mathbb{R}^2 : |\mu| \le 2,\ \ \nu \ge 1\right\},\\
\Omega_6 &:= \left\{(\mu,\nu) \in \mathbb{R}^2 : 2 \le |\mu| \le 4,\ \ \nu \ge \frac{1}{12}(\mu^2+8)\right\},\\
\Omega_7 &:= \left\{(\mu,\nu) \in \mathbb{R}^2 : |\mu| \ge 4,\ \nu \ge \frac{2}{3}(|\mu|-1)\right\},\\
\Omega_8 &:= \left\{(\mu,\nu) \in \mathbb{R}^2 : \frac{1}{2} \le |\mu| \le 2,\ -\frac{2}{3}(|\mu|+1) \le \nu \le \frac{4}{27}(|\mu|+1)^3 - (|\mu|+1)\right\},\\
\Omega_9 &:= \left\{(\mu,\nu) \in \mathbb{R}^2 : |\mu| \ge 2,\ -\frac{2}{3}(|\mu|+1) \le \nu \le \frac{2|\mu|(|\mu|+1)}{\mu^2+2|\mu|+4}\right\},\\
\Omega_{10} &:= \left\{(\mu,\nu) \in \mathbb{R}^2 : 2 \le |\mu| \le 4,\ \frac{2|\mu|(|\mu|+1)}{\mu^2+2|\mu|+4} \le \nu \le \frac{1}{12}(\mu^2+8)\right\},\\
\Omega_{11} &:= \left\{(\mu,\nu) \in \mathbb{R}^2 : |\mu| \ge 4,\ \frac{2|\mu|(|\mu|+1)}{\mu^2+2|\mu|+4} \le \nu \le \frac{2|\mu|(|\mu|-1)}{\mu^2-2|\mu|+4}\right\},\\
\Omega_{12} &:= \left\{(\mu,\nu) \in \mathbb{R}^2 : |\mu| \ge 4,\ \frac{2|\mu|(|\mu|-1)}{\mu^2-2|\mu|+4} \le \nu \le \frac{2}{3}(|\mu|-1)\right\}.
\end{aligned}$$

Lemma 2.1. [36, Lemma 2, p. 128] *If $w \in \mathbf{B}$, then for any real numbers μ and ν, we have*

$$\Psi(\mu,\nu) \le \begin{cases}
1, & (\mu,\nu) \in \Omega_1 \cup \Omega_2 \cup \{(2,1)\};\\
|\nu|, & (\mu,\nu) \in \bigcup_{k=3}^{4} \Omega_k;\\
\frac{2}{3}(|\mu|+1)\left(\frac{|\mu|+1}{3(|\mu|+\nu+1)}\right)^{1/2}, & (\mu,\nu) \in \Omega_8 \cup \Omega_9;\\
\frac{1}{3}\nu\left(\frac{\mu^2-4}{\mu^2-4\nu}\right)\left(\frac{\mu^2-4}{3(\nu-1)}\right)^{1/2}, & (\mu,\nu) \in \Omega_{10} \cup \Omega_{11} \setminus \{(2,1)\};\\
\frac{2}{3}(|\mu|-1)\left(\frac{|\mu|-1}{3(|\mu|-\nu-1)}\right)^{1/2}, & (\mu,\nu) \in \Omega_{12}.
\end{cases}$$

The extremal functions, up to rotations, are of the form

$$w_1(z) = z^3,\ \ w_2(z) = z,\ \ w_3(z) = \frac{z(t_1 - z)}{1 - t_1 z},\ \ w_4(z) = \frac{z(t_2 + z)}{1 + t_2 z}$$

and $w_5(z) = c_1 z + c_2 z^2 + c_3 z^3 + \cdots$, *where the parameters* t_1, t_2 *and the coefficients* c_i *are given by*

$$t_1 = \left(\frac{|\mu| + 1}{3(|\mu| + \nu + 1)}\right)^{\frac{1}{2}},\ t_2 = \left(\frac{|\mu| - 1}{3(|\mu| - \nu - 1)}\right)^{\frac{1}{2}},\ c_1 = \left(\frac{2\nu(\mu^2+2) - 3\mu^2}{3(\nu - 1)(\mu^2 - 4\nu)}\right)^{\frac{1}{2}},$$

$$c_2 = (1 - c_1^2)e^{i\theta_0},\ c_3 = -c_1 c_2 e^{i\theta_0},$$

where

$$\theta_0 = \pm \arccos\left[\frac{\mu}{2}\left(\frac{\nu(\mu^2+8) - 2(\mu^2+2)}{2\nu(\mu^2+2) - 3\mu^2}\right)^{1/2}\right].$$

Lemma 2.2. [24, Lemma 3, p. 254] *Let* $p \in \mathcal{P}$. *Then*

$$2p_2 = p_1^2 + (4 - p_1^2)\xi$$

for some $\xi \in \overline{\mathbb{D}}$.

Lemma 2.3. [37, Lemma 2.3, p. 507] *Let* $p \in \mathcal{P}$. *Then for all* $n, m \in \mathbb{N}$,

$$|\mu p_n p_m - p_{m+n}| \le \begin{cases} 2, & 0 \le \mu \le 1; \\ 2|2\mu - 1|, & \text{elsewhere}. \end{cases}$$

If $0 < \mu < 1$, *then the inequality is sharp for the function* $p(z) = (1 + z^{m+n})/(1 - z^{m+n})$. *In the other cases, the inequality is sharp for the function* $\hat{p}_0(z) = (1 + z)/(1 - z)$.

Lemma 2.4. [38, Lemma 1] *Let* $p \in \mathcal{P}$. *Then, for any real number* μ,

$$|\mu p_3 - p_1^3| \le \begin{cases} 2|\mu - 4|, & \mu \le \dfrac{4}{3}; \\[2ex] 2\mu\sqrt{\dfrac{\mu}{\mu - 1}}, & \dfrac{4}{3} < \mu. \end{cases}$$

The result is sharp. The extremal function is given by

$$\tilde{p}(z) = \begin{cases} \hat{p}_0(z), & \mu \le \frac{4}{3}; \\ \frac{1 - z^2}{z^2 - 2\sqrt{\frac{\mu}{\mu - 1}}\, z + 1}, & \mu > \frac{4}{3}. \end{cases}$$

3. Sufficient Condition and Hermitian-Toeplitz Determinant

Throughout our discussion, we assume $0 \le \alpha < 1$. First, we establish a sufficient condition for the function $f \in \mathcal{A}$ to be in the class $\mathcal{S}_r^*(\alpha)$.

Theorem 3.1. *If* $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \mathcal{A}$ *satisfies the following inequality*

$$\sum_{n=2}^{\infty} ((n - 1) + |1 + (1 - 2\alpha)n|)|a_n| \le 2(1 - \alpha), \tag{3.1}$$

then $f \in \mathcal{S}_r^*(\alpha)$.

Proof. Since $\Re w > \alpha$ if and only if $|w-1| < |w+(1-2\alpha)|$. Therefore, $f \in \mathcal{S}_r^*(\alpha)$ if

$$\left|\frac{f(z)}{zf'(z)} - 1\right| < \left|\frac{f(z)}{zf'(z)} + 1 - 2\alpha\right|,$$

or equivalently

$$|f(z) - zf'(z)| - |f(z) + (1-2\alpha)zf'(z)| < 0.$$

Set $|F(z)| = |f(z) - zf'(z)| - |f(z) + (1-2\alpha)zf'(z)|$. However, by using the hypothesis, we have

$$\begin{aligned}|F(z)| &= \left|\sum_{n=2}^{\infty}(1-n)a_n z^n\right| - \left|2(1-\alpha)z + \sum_{n=2}^{\infty}(1+(1-2\alpha)n)a_n z^n\right| \\ &\leq \sum_{n=2}^{\infty}(n-1)|a_n| + \sum_{n=2}^{\infty}|1+(1-2\alpha)n||a_n| - 2(1-\alpha) \\ &= \sum_{n=2}^{\infty}(n-1+|1+(1-2\alpha)n|)|a_n| - 2(1-\alpha) \leq 0.\end{aligned}$$

The result follows. ∎

Corollary 3.2. *If $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \mathcal{A}$ satisfies $\sum_{n=2}^{\infty} n|a_n| \leq 1$, then $f \in \mathcal{S}_r^*$.*

This corollary follows by taking $\alpha = 0$ in the Theorem 3.1. However this could be seen as a trivial consequence of the fact that $\sum_{n=2}^{\infty} n|a_n| \leq 1$ is sufficient for $f \in \mathcal{S}_r^*$. More generally, we have following result:

Theorem 3.3. *If $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \mathcal{A}$ satisfies the inequality $\sum_{n=2}^{\infty} n|a_n| \leq 1$, then $f \in \mathcal{S}_r^*(\alpha)$ for $\alpha \leq \frac{1}{2}$.*

Proof. If $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \mathcal{A}$ satisfies the inequality $\sum_{n=2}^{\infty} n|a_n| \leq 1$, then

$$\left|\frac{zf'(z)}{f(z)} - 1\right| < 1.$$

Let $w = f(z)/zf'(z)$. Then

$$\left|\frac{1}{w} - 1\right| < 1 \quad \text{or} \quad |1-w| < |w| \quad \text{or} \quad \Re(w) > \frac{1}{2}.$$

Thus, we conclude that $f \in \mathcal{S}_r^*(1/2)$. ∎

Next result provides the best possible estimates on the third order Hermitian-Toeplitz determinants for the classes $\mathcal{S}_r^*(\alpha)$.

Theorem 3.4. *Let $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \mathcal{S}_r^*(\alpha)$. Then*

$$T_{3,1}(f) \leq \begin{cases} 16\alpha^4 - 64\alpha^3 + 87\alpha^2 - 46\alpha + 8, & 0 \leq \alpha \leq \frac{1}{4}; \\ 1, & \frac{1}{4} < \alpha < 1. \end{cases}$$

and

$$T_{3,1}(f) \geq \begin{cases} -(1-\alpha)^2, & 0 \leq \alpha < \frac{1}{2}; \\ 16\alpha^4 - 64\alpha^3 + 87\alpha^2 - 46\alpha + 8, & \frac{1}{2} \leq \alpha < 1. \end{cases}$$

The above estimates are best possible.

Proof. For some $p \in \mathcal{P}$, each function $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \mathcal{S}_r^*(\alpha)$ satisfies

$$\frac{f(z)}{zf'(z)} = \alpha + (1-\alpha)p(z). \tag{3.2}$$

On comparing the coefficients of like power terms in the expression (3.2) , we get

$$a_2 =(1-\alpha)p_1, \tag{3.3}$$

$$a_3 =\frac{1}{2}(\alpha-1)\left(2(\alpha-1)p_1^2+p_2\right), \tag{3.4}$$

$$a_4 =-\frac{1}{6}(\alpha-1)\left(6(\alpha-1)^2p_1^3+7(\alpha-1)p_1p_2-2p_3\right), \tag{3.5}$$

$$\begin{aligned} a_5 =&\frac{1}{24}(\alpha-1)(24(\alpha-1)^3p_1^4+46(\alpha-1)^2p_1^2p_2-20(\alpha-1)p_1p_3 \\ &+9(\alpha-1)p_2^2+6p_4). \end{aligned} \tag{3.6}$$

Since the classes $\mathcal{S}_r^*(\alpha)$ and $\mathcal{P}$ are invariant under rotation and $|p_1| \leq 2$, there is no loss in considering $0 \leq p_1 \leq 2$. Using (3.3), (3.4) in the expression (1.1) and then Lemma 2.2, we get

$$\begin{aligned} T_{3,1}(f) =&1+2\operatorname{Re} a_2^2a_3-2|a_2|^2-|a_3|^2 \\ =&\frac{1}{16}(1-\alpha)^2(4\alpha-5)(4\alpha-3)p_1^4-2(1-\alpha)^2p_1^2-\frac{1}{16}(1-\alpha)^2(4-p_1^2)^2|\zeta|^2 \\ &-\frac{1}{8}(1-\alpha)^2(4-p_1^2)p_1^2\operatorname{Re}\zeta+1 \\ =&\Psi(p_1^2,|\zeta|,\operatorname{Re}\zeta). \end{aligned} \tag{3.7}$$

If $p_1 = 0$, then we have

$$\Psi(0,|\zeta|,\operatorname{Re}\zeta) = 1-(1-\alpha)^2|\zeta|^2 \leq 1 \tag{3.8}$$

and if $p_1 = 2$, then

$$\Psi(4,|\zeta|,\operatorname{Re}\zeta) = 16\alpha^4-64\alpha^3+87\alpha^2-46\alpha+8. \tag{3.9}$$

We now procced to find the maximum of $\Psi(p_1^2,|\zeta|,\operatorname{Re}\zeta)$. From (3.7), with the settings $x := p_1^2 \in [0,4]$ and $y = |\zeta| \in [0,1]$, it is easy to see that

$$\begin{aligned} \Psi(p_1^2,|\zeta|,\operatorname{Re}\zeta) \leq \Psi(p_1^2,|\zeta|,-|\zeta|) \\ &= \frac{1}{16}(1-\alpha)^2(4\alpha-5)(4\alpha-3)x^2-2(1-\alpha)^2x \\ &\quad -\frac{1}{16}(1-\alpha)^2(4-x)^2y^2+\frac{1}{8}(1-\alpha)^2(4-x)xy+1 \\ &=: G(\alpha,x,y). \end{aligned}$$

(A): The function $G(\alpha,x,y)$ is defined on $\Omega := [0,4]\times[0,1]$. On the boundary of Ω, we have

$$G(\alpha,x,0) = \frac{1}{16}(4\alpha-5)(4\alpha-3)(1-\alpha)^2x^2-2(1-\alpha)^2x+1$$

and

$$G(\alpha,x,1) = -(\alpha-2)\alpha+\frac{1}{4}\left(4\alpha^2-8\alpha+3\right)(\alpha-1)^2x^2-(\alpha-1)^2x.$$

A1: To maximize $G(\alpha, x, 0)$, we proceed as follows. For $0 \le \alpha < 3/4$, we find that

$$G''(\alpha, x, 0) = \frac{1}{8}(\alpha - 1)^2(4\alpha - 5)(4\alpha - 3) < 0$$

holds for $0 \le \alpha < 3/4$ but for this range of α there is no critical point of $G(x, 0)$ and so G has no maximum in $(0, 4)$. For $\alpha = 3/4$, we have $G(3/4, x, 0) = 1 - x/8 \le 1$. In case $\alpha > 3/4$ the $G'(\alpha, x, 0) > 0$ which reveals that there is no maximum of G in $(0, 4)$.

A2: Now consider the function $G(\alpha, x, 1)$. For $0 \le \alpha < 1/2$, we find that

$$G''(\alpha, x, 1) = \frac{1}{2}(\alpha - 1)^2(2\alpha - 3)(2\alpha - 1) > 0$$

hence $G(\alpha, x, 1)$ has no maximum in $(0, x)$. Further computation reveals that $G'(\alpha, x, 1) < 0$ for $1/2 < \alpha < 1$ but the function $G(\alpha, x, 1) = 0$ has no critical point in $(0, 4)$ for this range of α. Finally, $G(1/2, x, 1) = (3 - x)/4 \le 3/4 < 1$.

A3: We now find the maximum of G inside the domain $(0, 4) \times (0, 1)$. For $\alpha = 3/4$, the function G takes the form

$$g(x, y) = -\frac{1}{256}(4 - x)^2 y^2 + \frac{1}{128}(4 - x)xy - \frac{x}{8} + 1.$$

It is a matter of simple calculation to verify that $g_y(x, y) = 0 = g_x(x, y)$ holds for $x = 16$ and $y = -4/3$. No critical point in $(0, 4) \times (0, 1)$. Further, in the case when $\alpha \ne 3/4$, we have find that

$$G_y(x, y) = \frac{1}{8}(1 - \alpha)^2(4 - x)x - \frac{1}{8}(1 - \alpha)^2(4 - x)^2 y = 0$$

and

$$G_x(x, y) = 2(\alpha - 1)^2\left((\alpha - 1)^2 x - 1\right) = 0$$

hold for

$$x = x_1 = \frac{1}{(\alpha - 1)^2}, \quad y = y_1 = \frac{1}{(2\alpha - 3)(2\alpha - 1)}.$$

It is verified that $(x_1, y_1) \in (0, 4) \times (0, 1)$ and $G(x_1, y_1) = 0$. From the above discussion in A1, A2, A3, in view of (3.8) and (3.9), we have

$$\begin{aligned} G(\alpha, x, y) &\le \max\left\{1; 16\alpha^4 - 64\alpha^3 + 87\alpha^2 - 46\alpha + 8\right\} \\ &= \begin{cases} 16\alpha^4 - 64\alpha^3 + 87\alpha^2 - 46\alpha + 8, & 0 \le \alpha \le 1/4, \\ 1, & 1/4 \le \alpha < 1. \end{cases} \end{aligned}$$

The sharpness of the upper bound follows for the function f_0 defined by

$$\frac{f_0(z)}{zf_0'(z)} = \begin{cases} \frac{1+(1-2\alpha)z^3}{1-z^3}, & 1/4 \le \alpha < 1; \\ \frac{1+(1-2\alpha)z}{1-z}, & 0 \le \alpha \le 1/4. \end{cases}$$

(B): Using (3.7) we can write

$$\begin{aligned} \Psi(p_1^2, |\zeta|, \operatorname{Re}\zeta) &\ge \Psi(p_1^2, |\zeta|, |\zeta|) \\ &\ge \Psi(p_1^2, 1, 1) \\ &=: h(x), \end{aligned}$$

where h is given by $h(x) = -(\alpha-2)\alpha + (\alpha-1)^4x^2 - 2(\alpha-1)^2x \ \ (0 \leq x \leq 4)$. Now at the end points

$$h(0) = (2-\alpha)\alpha \ \text{ and } \ h(4) = 16\alpha^4 - 64\alpha^3 + 87\alpha^2 - 46\alpha + 8.$$

It can be seen that $x = x_2 = 1/(\alpha-1)^2$ only root of

$$h'(x) = 2(\alpha-1)^4x - 2(\alpha-1)^2 = 0.$$

Further $h''(x_2) = 2(\alpha-1)^4 > 0$ and $h(x_2) = -(\alpha-1)^2$. Here $h(0) = (2-\alpha)\alpha > 0$ and $h(x_2) = -(\alpha-1)^2 < 0$. Hence

$$h(x) \geq \min\{h(0), h(4), h(x_2)\} = \begin{cases} h(x_2), & 0 \leq \alpha < 1/2; \\ h(4), & 1/2 \leq \alpha < 1. \end{cases}$$

Sharpness follows for the functions f_1 defined by

$$\frac{f_1(z)}{zf_1'(z)} = \begin{cases} (1-\alpha)\tilde{p}(z) + \alpha, & 0 \leq \alpha < 1/2; \\ \frac{1+(1-2\alpha)z}{1-z}, & 1/2 \leq \alpha < 1, \end{cases} \tag{3.10}$$

where

$$\tilde{p}(z) = \frac{1-z^2}{1 - \frac{1}{1-\alpha}z + z^2}$$

which completes the proof. ■

4. Inverse Coefficients

Next theorem gives the bounds on initial inverse coefficients of the starlike functions of reciprocal order α.

Theorem 4.1. *Let $f(z) = z + \sum_{n=2}^{\infty} a_nz^n \in \mathcal{S}_r^*(\alpha)$. Then*

$$\begin{aligned} |A_2| \leq& 2(1-\alpha), \\ |A_3| \leq& (1-\alpha)(5-4\alpha), \\ |A_4| \leq& \begin{cases} 1, & 0 \leq \alpha \leq \frac{11}{16}; \\ |12\alpha^2 - 32\alpha + 19|, & \frac{11}{16} \leq \alpha < 1, \end{cases} \\ |A_5| \leq& \begin{cases} -\frac{1}{6}(\alpha-1)\left(96\alpha^3 - 404\alpha^2 + 501\alpha - 196\right), & 0 \leq \alpha < \frac{15}{29}; \\ \frac{1}{6}(\alpha-1)\left(96\alpha^3 - 288\alpha^2 + 325\alpha - 136\right), & \frac{15}{29} \leq \alpha < 1. \end{cases} \end{aligned}$$

All bounds are sharp except for $|A_5|$ in the case $15/29 \leq \alpha < 1$. The extremal function f_1 is given by (3.10).

Proof. Since $f^{-1}(\omega) = \omega + A_2\omega^2 + A_3\omega^3 + A_4\omega^4 + \cdots$ in some neighbourhood of origin, so we have $f(f^{-1}(\omega)) = \omega$. That is,

$$\begin{aligned} \omega &= f^{-1}(\omega) + a_2(f^{-1}(\omega))^2 + a_3(f^{-1}(\omega))^3 + \cdots \\ &= \omega + A_2\omega^2 + A_3\omega^3 + A_4\omega^4 + \cdots + a_2(\omega + A_2\omega^2 + A_3\omega^3 + A_4\omega^4 + \cdots)^2 \\ &\quad + a_3(\omega + A_2\omega^2 + A_3\omega^3 + A_4\omega^4 + \cdots)^3. \end{aligned}$$

A simple calculation gives the following relations:

$$A_2 = -a_2, \tag{4.1}$$
$$A_3 = 2a_2^2 - a_3, \tag{4.2}$$
$$A_4 = -5a_2^3 + 5a_2a_3 - a_4, \tag{4.3}$$
$$A_5 = 14a_2^4 - 21a_2^2a_3 + 6a_2a_4 + 3a_3^2 - a_5. \tag{4.4}$$

By using (3.3), (3.4), (3.5), (3.6) in (4.1), (4.2), (4.3), (4.4), we get

$$A_2 = -(1-\alpha)p_1$$
$$A_3 = -\frac{1}{2}(\alpha-1)\left(p_2 - 2(\alpha-1){p_1}^2\right) \tag{4.5}$$
$$A_4 = \frac{1}{3}(\alpha-1)\left(3(\alpha-1)^2{p_1}^3 - 4(\alpha-1)p_1p_2 - p_3\right) \tag{4.6}$$
$$A_5 = \frac{1}{24}(\alpha-1)(24(\alpha-1)^3{p_1}^4 - 58(\alpha-1)^2{p_1}^2p_2 - 28(\alpha-1)p_1p_3 + 9(\alpha-1){p_2}^2 - 6p_4). \tag{4.7}$$

By using of the fact $|p_n| \le 2$, the desired bound on A_2 can be readily obtained. In view of Lemma 2.3 and the expression (4.5), we get

$$|A_3| = \left|\frac{1}{2}(1-\alpha)\left(p_2 + 2(1-\alpha)p_1^2\right)\right| \le (1-\alpha)(5-4\alpha).$$

Since $p \in \mathcal{P}$ if and only if $w(z) = (p(z)-1)/(p(z)+1) \in \mathcal{B}$, we have following relations

$$p_1 = 2c_1,\ p_2 = 2c_2 + 2c_1^2,\ p_3 = 2c_3 + 4c_1c_2 + 2c_1^3. \tag{4.8}$$

In view of (4.8) and (4.6), we have

$$\begin{aligned} A_4 =& \frac{2}{3}(\alpha-1)\left(\left(12\alpha^2 - 32\alpha + 19\right){c_1}^3 + (6-8\alpha)c_1c_2 - c_3\right) \\ =& \frac{2}{3}(\alpha-1)(c_3 + \mu c_1c_2 + \nu {c_1}^3), \end{aligned} \tag{4.9}$$

where $\mu = -(6-8\alpha)$ and $\nu = -\left(12\alpha^2 - 32\alpha + 19\right)$. It is noted that $\mu \ge 0$ for $3/4 < \alpha < 1$. Now we verify the following three cases to use the Lemma 2.1:

(1) $|\mu| \le \frac{1}{2}$ and $|\nu| \le 1$ is equivalent to

$$-\frac{1}{2} \le 8\alpha - 6 \le \frac{1}{2} \text{ and } -1 \le -12\alpha^2 + 32\alpha - 19 \le 1$$

which holds for $\frac{13}{16} \le \alpha < 1$. This gives $(\mu,\nu) \in \Omega_1$ when $\frac{13}{16} \le \alpha < 1$.

(2) $|\mu| \le 2$ and $\nu \le -1$ is equivalent to

$$-\frac{1}{2} \le 8\alpha - 6 \le \frac{1}{2} \text{ and } -12\alpha^2 + 32\alpha - 19 \le -1$$

which holds for

$$\frac{11}{16} \le \alpha \le \frac{1}{6}\left(8 - \sqrt{10}\right).$$

So for $\frac{11}{16} \le \alpha \le \frac{1}{6}\left(8 - \sqrt{10}\right)$, we have $(\mu,\nu) \in \Omega_3$.

(3) $|\mu| \geq 1/2$ and $\nu \leq \frac{2}{3}(|\mu|+1)$ equivalently can be written as $\mu \geq 1/2, \nu \leq \frac{2}{3}(\mu+1)$ in case of $\mu \geq 0$ or $\mu \leq -1/2, \nu \leq \frac{2}{3}(-\mu+1)$ in case of $\mu < 0$. Now we see that $\mu \geq 1/2, \nu \leq \frac{2}{3}(\mu+1)$ holds for $\frac{13}{16} \leq \alpha < 1$. Further, computation shows that $\mu \leq -1/2, \nu \leq \frac{2}{3}(-\mu+1)$ holds for $0 \leq \alpha \leq \frac{11}{16}$.

Using triangle inequality in expression (4.7), we get

$$\frac{24}{1-\alpha}|A_5| \leq \left|24(1-\alpha)^3{p_1}^4 + 9(1-\alpha){p_2}^2 + 6p_4\right| + \left|58(\alpha-1)^2{p_1}^2p_2 - 28(1-\alpha)p_1p_3\right|. \tag{4.10}$$

Consider

$$\left|58(\alpha-1)^2{p_1}^2p_2 - 28(1-\alpha)p_1p_3\right| = 56(1-\alpha)|p_1|\left|\frac{29(1-\alpha)}{14}p_1p_2 - p_3\right|.$$

By applying Lemma 2.3, we have

$$56(1-\alpha)|p_1|\left|\frac{29(1-\alpha)}{14}p_1p_2 - p_3\right| \leq 56(1-\alpha)\times\begin{cases} 2, & \frac{15}{29} \leq \alpha < 1; \\ \frac{2(22-29\alpha)}{7}, & 0 \leq \alpha < \frac{15}{29}. \end{cases} \tag{4.11}$$

Therefore, in view of (4.10) and (4.11), for $0 \leq \alpha < \frac{15}{29}$, we have

$$\frac{24}{1-\alpha}|A_5| \leq \left|24(1-\alpha)^3\text{p1}^4\right| + \left|9(1-\alpha)\text{p2}^2\right| + |6\text{p4}| + \frac{112(1-\alpha)(22-29\alpha)}{7}$$

and for $\frac{15}{29} \leq \alpha < 1$, we have

$$\frac{24}{1-\alpha}|A_5| \leq \left|24(1-\alpha)^3\text{p1}^4\right| + \left|9(1-\alpha)\text{p2}^2\right| + |6\text{p4}| + 112(1-\alpha).$$

Using the fact $|p_i| \leq 2$, we have

$$|A_5| \leq -\frac{1}{6}(\alpha-1)\left(96\alpha^3 - 404\alpha^2 + 501\alpha - 196\right) \quad \text{for } 0 \leq \alpha < \frac{15}{29}$$

and

$$|A_5| \leq \frac{1}{6}(\alpha-1)\left(96\alpha^3 - 288\alpha^2 + 325\alpha - 136\right) \quad \text{for } \frac{15}{29} \leq \alpha < 1.$$

This completes the proof. ■

5. Logarithmic Coefficients

In this section, we determine the estimates on initial logarithmic coefficients for the starlike functions of reciprocal order α.

Theorem 5.1. *Let* $\alpha_0 = \frac{1}{6}\left(-\sqrt[3]{44-3\sqrt{177}} - \frac{7}{\sqrt[3]{44-3\sqrt{177}}} + 7\right) \approx 0.170516$ *and* $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \mathcal{S}_r^*(\alpha)$*. Then*

$$|\gamma_1| \leq 1-\alpha,$$

$$|\gamma_2| \leq \frac{1-\alpha}{2},$$

$$|\gamma_3| \leq \begin{cases} \frac{(1-\alpha)(7-4\alpha)}{9}\sqrt{\frac{7-4\alpha}{6-3\alpha^2}}, & \alpha_0 \leq \alpha < 1; \\ \frac{(\alpha-2)(1-\alpha)\left(4\alpha^2-4\alpha-1\right)}{9\sqrt{3}(1-\alpha)}\sqrt{\frac{2-\alpha}{\alpha}}, & 0 \leq \alpha \leq \alpha_0, \end{cases}$$

$$|\gamma_4| \leq \begin{cases} \frac{(1-\alpha)(1+2(1+2\sqrt{2})(1-\alpha))}{4}, & 2/3 \leq \alpha < 1; \\ \frac{(1-\alpha)(1+2(5+2\sqrt{2}-6\alpha)(1-\alpha))}{4}, & 0 \leq \alpha < 2/3. \end{cases}$$

All estimates are sharp except on γ_4*.*

Proof. In view of series expansion (1.2), it follows that

$$\gamma_1 = \frac{a_2}{2}, \tag{5.1}$$

$$\gamma_2 = \frac{1}{2}\left(a_3 - \frac{a_2^2}{2}\right), \tag{5.2}$$

$$\gamma_3 = (a_4 - a_2 a_3 + \frac{1}{3}a_2^3), \tag{5.3}$$

$$\gamma_4 = \frac{1}{2}\left(-\frac{a_2^4}{4} + a_2^2 a_3 - a_2 a_4 - \frac{a_3^2}{2} + a_5\right). \tag{5.4}$$

By using (3.3), (3.4), (3.5), (3.6) in (5.1), (5.2), (5.3), (5.4), we get

$$\begin{aligned} \gamma_1 &= \frac{(1-\alpha)p_1}{2}, \\ \gamma_2 &= \frac{1}{4}(1-\alpha)\left((1-\alpha)p_1^2 - p_2\right), \\ \gamma_3 &= \frac{1}{6}(\alpha-1)\left(-(\alpha-1)^2 p_1^3 - 2(\alpha-1)p_1 p_2 + p_3\right), \\ \gamma_4 &= \frac{1}{8}(\alpha-1)\left((\alpha-1)^3 p_1^4 + 3(\alpha-1)^2 p_1^2 p_2 - 2(\alpha-1)p_1 p_3 + (\alpha-1)p_2^2 + p_4\right). \end{aligned} \tag{5.5}$$

By making use of the fact $|p_n| \leq 2$ and Lemma 2.3, we get the required bound on γ_1 and γ_2. In view of (4.8), γ_3 is written in terms of Schwarz coefficients as

$$\gamma_3 = -\frac{1}{3}(\alpha-1)\left(\left(4\alpha^2-4\alpha-1\right)c_1^3 + 2(2\alpha-3)c_1 c_2 - c_3\right)$$

so that

$$|\gamma_3| = \frac{1-\alpha}{3}\left|c_3 + \mu c_1 c_2 + \nu c_1^3\right|, \tag{5.6}$$

where $\mu = 2(3-2\alpha)$ and $\nu = 1+4\alpha-4\alpha^2$. It is noted that $\mu = 2(3-2\alpha) \geq 0$ for $0 \leq \alpha < 1$.

(i) Let $\frac{1}{2} \leq \alpha < 1$. Now consider $2 \leq \mu \leq 4$ and $\nu \geq (\mu^2+8)/12$. This is equivalent to

$$2 \leq 2(3-2\alpha) \leq 4 \text{ and } -4\alpha^2 + 4\alpha + 1 \geq \frac{1}{12}\left(4(3-2\alpha)^2 + 8\right),$$

or equivalently

$$\frac{1}{2} \leq \alpha \leq 1 \text{ and } 3\alpha \geq 2\alpha^2 + 1$$

that is true for $\frac{1}{2} \leq \alpha < 1$ and hence $(\mu, \nu) \in \Omega_6$.

(ii) Let

$$\alpha_0 = \frac{1}{6}\left(-\sqrt[3]{44 - 3\sqrt{177}} - \frac{7}{\sqrt[3]{44 - 3\sqrt{177}}} + 7 \right) \approx 0.170516$$

be smallest positive root of $2x^3 - 7x^2 + 7x - 1 = 0$. Now assume that $0 \leq \alpha \leq \alpha_0$. It is easy to see that $\mu \geq 2$ for all $0 \leq \alpha < 1$. Now the condition

$$-\frac{2}{3}(\mu + 1) \leq \nu \leq \frac{2\mu(\mu + 1)}{\mu^2 + 2\mu + 4}$$

is equivalent to

$$\frac{2}{3}(4\alpha - 7) \leq -4\alpha^2 + 4\alpha + 1 \leq \frac{(2\alpha - 3)(4\alpha - 7)}{4\alpha^2 - 14\alpha + 13}.$$

Further computation reveals that this holds for all $\alpha \in [0, \alpha_0]$. Thus $(\mu, \nu) \in \Omega_9$.

(iii) Let $\alpha_0 \leq \alpha \leq 1/2$. It can be verified that $\mu \geq 4$ for $\alpha \leq 1/2$. Now the conditions

$$\frac{2\mu(\mu + 1)}{\mu^2 + 2\mu + 4} \leq \nu \leq \frac{2\mu(\mu - 1)}{\mu^2 - 2\mu + 4}$$

are equivalent to

$$\frac{(2\alpha - 3)(4\alpha - 7)}{4\alpha^2 - 14\alpha + 13} \leq -4\alpha^2 + 4\alpha + 1 \leq \frac{(2\alpha - 3)(4\alpha - 5)}{4\alpha^2 - 10\alpha + 7}$$

which holds for $\alpha_0 \leq \alpha \leq 1/2$. Thus $(\mu, \nu) \in \Omega_{11}$.

By using Lemma 2.1, the above discussion reveals that

$$\begin{aligned} \left|c_3 + \mu c_1 c_2 + \nu c_1^3\right| &\leq \begin{cases} \frac{2}{3}(|\mu| + 1)\left(\frac{|\mu|+1}{3(|\mu|+\nu+1)}\right)^{1/2}, & \alpha_0 \leq \alpha < 1; \\ \frac{1}{3}\nu\left(\frac{\mu^2-4}{\mu^2-4\nu}\right)\left(\frac{\mu^2-4}{3(\nu-1)}\right)^{1/2}, & 0 \leq \alpha \leq \alpha_0, \end{cases} \\ &= \begin{cases} \frac{1}{3}(7 - 4\alpha)\sqrt{\frac{7-4\alpha}{6-3\alpha^2}}, & \alpha_0 \leq \alpha < 1; \\ \frac{(\alpha-2)(4\alpha^2-4\alpha-1)}{3\sqrt{3}(1-\alpha)}\sqrt{\frac{2-\alpha}{\alpha}}, & 0 \leq \alpha \leq \alpha_0. \end{cases} \end{aligned} \tag{5.7}$$

From expression (5.6) and (5.7), we get the desired estimate on γ_3. In view of (5.5), we have

$$|\gamma_4| \leq \frac{1}{8}(1-\alpha)\left(\left|(\alpha - 1)^3 p_1^4 - 2(\alpha - 1)p_1 p_3\right| + \left|3(\alpha - 1)^2 p_1^2 p_2 + (\alpha - 1)p_2^2\right| + |p_4|\right). \tag{5.8}$$

Using Lemma 2.3 and Lemma 2.4, we have

$$\left|3(\alpha - 1)^2 {p_1}^2 p_2 + (\alpha - 1){p_2}^2\right| \leq \begin{cases} 4(1 - \alpha), & 2/3 \leq \alpha < 1; \\ 4(1 - \alpha)(5 - 6\alpha), & 0 \leq \alpha < 2/3, \end{cases} \tag{5.9}$$

and

$$\left|(\alpha - 1)^3 p_1^4 - 2(\alpha - 1)p_1 p_3\right| \leq 8\sqrt{2}(1 - \alpha). \tag{5.10}$$

From (5.8), (5.9) and (5.10), we get the required bound on γ_4. ∎

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