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Fixed Point Theorems for $F-$Contraction in Generalized Asymmetric Metric Spaces

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Abstract Recently, a new type of mapping called $F-$contraction was introduced in the literature as a generalization of the concepts of contractive mappings. This present article extends the new notion in generalized asymmetric metric spaces and establishing the existence and uniqueness of fixed point for them. Non-trivial examples are further provided to support the hypotheses of our results.

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1. Introduction

Metric fixed point theory has its roots in methods from the late 19th century, when successive approximations were used to establish the existence and uniqueness of solutions to equations, and especially differential equations. This approach is particularly associated with the work of Picard, although it was Stefan Banach who in 1922 in [1] developed the ideas involved in an abstract setting.

Banach's contraction principle is a fundamental result in fixed point theory. Due to its importance, several authors have obtained many interesting extensions and generalizations see [2–6].

Many generalizations of the concept of metric spaces are defined and some fixed point theorems were proved in these spaces. In particular, asymmetric metric spaces were introduced by Wilson [7] as a generalization of metric spaces. many mathematicians worked on this interesting space. For more, the reader can refer to [8, 9].

A. Branciari in [10] initiated the notions of a generalized metric space as a generalization of a metric space, where the triangular inequality of metric spaces replaced by

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$d(x, y) \leq d(x, u) + d(u, v) + d(v, y)$ (quadrilateral Inequality). Various fixed point results were established on such spaces, see [11–16] and references therein.

Combining conditions used for definitions of asymmetric metric and generalized metric spaces, Piri et al [17] announced the notions of generalized asymmetric metric space and establish nice results of fixed point on such space.

In this paper, inspired by the work done in [18, 19], we introduce the notion of $F-$contraction and establish some new fixed point theorems for mappings in the setting of complete generalized asymmetric metric spaces. Moreover, an illustrative examples is presented to support the obtained results.

2. Preliminaries

In the following, we recollect some definitions which will be useful in our main results.

Definition 2.1. [10]. Let X be a non-empty set and $d : X \times X \to \mathbb{R}^+$ be a mapping such that for all $x, y \in X$ and for all distinct points $u, v \in X$, each of them different from x and y, on has

(i) $d(x, y) = 0$ if and only if $x = y$;
(ii) $d(x, y) = d(y, x)$ for all distinct points $x, y \in X$;
(iii) $d(x, y) \leq d(x, u) + d(u, v) + d(v, y)$. (quadrilateral Inequality)

Then (X, d) is called a generalized metric space.

Definition 2.2. [17]. Let X be a non-empty set and $d : X \times X \to \mathbb{R}^+$ be a mapping such that for all $x, y \in X$ and for all distinct points $u, v \in X$, each of them different from x and y, on has

(i) $d(x, y) = 0$ if and only if $x = y$;
(ii) $d(x, y) \leq d(x, u) + d(u, v) + d(v, y)$. (quadrilateral Inequality) Then (X, d) is called a generalized asymmetric metric space.

Example 2.3. Consider $X = \{0, 1, 2, 3\}$ Let $d : X \times X \to R^+$ be mapping defined by
(i) $d(x, y) = 0$ if and only $x = y$;
(ii) $d(0, 1) = d(1, 0) = d(2, 1) = d(2, 0) = d(3, 0) = d(2, 3) = d(3, 1) = 1$;
(iii) $d(1, 2) = d(0, 2) = 2$;
(iv) $d(0, 3) = 3$, $d(3, 2) = 4$;
(v) $d(1, 3) = 2$.

Clearly, (X, d) is not asymmetric metric space. Indeed, $d(1, 3) = 4 > d(1, 2) + d(2, 3) = 3$.
But is a complete generalized asymmetric metric space.

Definition 2.4. [17]. Let (X, d) is a generalized asymmetric metric space and $\{x_n\}_{n\in\mathbb{N}}$ be a sequence in X, and $x \in X$. Then

(i) We say that $\{x_n\}_{n\in\mathbb{N}}$ forward (backward) converges to x if and only if

$$\lim_{n\to+\infty} d(x, x_n) = \lim_{n\to+\infty} d(x_n, x) = 0.$$

(ii) We say that $\{x_n\}_{n\in\mathbb{N}}$ forward (backward) Cauchy if

$$\lim_{n,m\to+\infty} d(x_n, x_m) = \lim_{n,m\to+\infty} d(x_m, x_n) = 0.$$

Lemma 2.5. [17]. *Let (X, d) be an generalized asymmetric metric space and $\{x_n\}_n$ be a forward (or backward) Cauchy sequence with pairwise disjoint elements in X. If $\{x_n\}_n$ forward converges to $x \in X$ and backward converges to $y \in X$, then $x = y$.*

Definition 2.6. [17]. Let (X, d) be a generalized asymmetric metric space. X is said to be forward (backward) complete if every forward (backward) Cauchy sequence $\{x_n\}_{n\in\mathbb{N}}$ in X forward (backward) converges to $x \in X$.

Definition 2.7. [17]. Let (X, d) be a generalized asymmetric metric space. X is said to be complete if X is forward and backward complete.

The following definition was given by Samet et al in [20].

Definition 2.8. [20]. Let $T : X \to X$ be function and $\alpha : X \times X \to [0, +\infty[$ be a function. We say that T is an α-admissible mapping if

$$x, y \in X, \alpha(x, y) \geq 1 \Rightarrow \alpha(Tx, Ty) \geq 1.$$

The following definition was given by Salimi et al in [21].

Definition 2.9. [21]. Let T be a self-mapping on X and $\alpha, \eta : X \times X \to [0, +\infty[$ be two functions. We say that T is an α-admissible mapping with respect to η if

$$x, y \in X, \alpha(x, y) \geq \eta(x, y) \Rightarrow \alpha(Tx, Ty) \geq \eta(Tx, Ty).$$

Definition 2.10. [22]. Let (X, d) be a metric space. Let $\alpha, \eta : X \times X \to [0, +\infty[$ and $T : X \to X$ be functions. We say T is an α -$\eta-$continuous mapping on (X, d), if for given $x \in X$ and sequence $\{x_n\}_{n\in\mathbb{N}}$ with $x_n \to x$, $\alpha(x_n, x_{n+1}) \geq \eta(x_n, x_{n+1})$ for all $n \in N \Rightarrow Tx_n \to Tx$.

Definition 2.11. [5]. Let $\mathbb{F}$ be the family of all functions $F\colon \mathbb{R}^+ \to \mathbb{R}$ such that
(i) F is strictly increasing;
(ii) For each sequence $\{x_n\}_{n\in\mathbb{N}}$ of positive numbers

$$\lim_{n\to 0} x_n = 0, \quad \text{if and only if} \quad \lim_{n\to\infty} F(x_n) = -\infty;$$

(iii) There exists $k \in]0, 1[$ such that $\lim_{x\to 0} x^k F(x) = 0$.

Recently Piri et al. [4], replaced the condition (iii) in [5] by the condition : F is continuous.

Definition 2.12. [4]. Let $\Im$ be the family of all functions $F\colon \mathbb{R}^+ \to \mathbb{R}$ such that
(i) F is strictly increasing;
(ii) For each sequence $\{x_n\}_{n\in\mathbb{N}}$ of positive numbers

$$\lim_{n\to 0} x_n = 0, \quad \text{if and only if} \quad \lim_{n\to\infty} F(x_n) = -\infty;$$

(iii) F is continuous.

3. Main Result

The following Definition is similar to Definition 2.10 in the framework of generalized asymmetric metric space.

Definition 3.1. Let (X, d) be a generalized asymmetric metric space. Let $\alpha, \eta : X \times X \to [0, +\infty[$ and $T : X \to X$ be functions. We say T is an α -$\eta-$continuous mapping on (X, d), if for given $x \in X$ and sequence $\{x_n\}_{n\in\mathbb{N}}$ with $x_n \to x$, $\alpha(x_n, x_{n+1}) \geq \eta(x_n, x_{n+1})$ for all $n \in N \Rightarrow Tx_n \to Tx$.

Example 3.2. Consider $X = \{0, 1, 2, 3\}$. Let $d : X \times X \to R^+$, be a mapping defined by

(i) $d(x,y)=0 \Leftrightarrow x=y \; for \; all \; x,y\in X$;
(ii) $d(0,1)=d(1,0)=d(2,1)=d(2,0)=d(3,0)=d(2,3)=d(3,1)=1$;
(iii) $d(1,2)=d(0,2)=2$;
(iv) $d(0,3)=d(3,2)=3$, $d(1,3)=4$. Clearly, (X,d) is a complete generalized asymmetric metric space.

Let $T : X\to X$, be given by
$T(0)=T(1)=0$, $T(2)=1$, $T(3)=2$.
Define, $\alpha,\eta\colon X \times X\to [0,+\infty[$ by

$$\alpha(x,y)=\frac{x+y}{\max\{x,y\}+1}, \text{ and } \eta(x,y)=\frac{|x-y|}{\max\{x,y\}+1},$$

so,

$$\frac{|x-y|}{\max\{x,y\}+1}\leq\frac{x+y}{\max\{x,y\}+1}\forall x,y\in X.$$

so, we have

$$\eta(x,y)\leq\alpha(x,y)\quad \forall x,y\in X.$$

Since $T(x)\in X\ \forall x\in X$, imply

$$\eta(Tx,Ty)\leq\alpha(Tx,Ty) \;\; for \; all \; x,y\in X.$$

Therefore, T is not continuous, but T is α-η -continuous on (X,d).

Definition 3.3. Let (X,d) be a generalized asymmetric metric space. A mapping T : $X\times X$ is said to be an α -η-contraction of type (A) on (X,d), if there exist $F\in\mathbb{F}$, α, η: $X\times X\to[0,+\infty[$ and $\tau>0$ such that

$$\begin{aligned}\max\{d(Tx,Ty),d(Ty,Tx)\} > 0 \Rightarrow & F\left[\max\{d(Tx,Ty),d(Ty,Tx)\}\right]+\tau \\ & \leq F\left[\max\{d(x,y),d(y,x)\}\right].\end{aligned} \tag{3.1}$$

Theorem 3.4. *Let (X,d) be a complete generalized asymmetric metric space and let $T:X\to X$ be a mapping satisfying the following assertions:*
(i) T is α-admissible mapping with respect to η;
(ii) T is α-η -F-contraction of type (A);
(iii) there exists $x_0\in X$ such that $\alpha(x_0,Tx_0)\geq\eta(x_0,Tx_0)$ and $\alpha(x_0,T^2x_0)\geq\eta(x_0,T^2x_0)$;
(iv) T is α-η-continuous.
Then T has a fixed point. Moreover, if $\alpha(x,y)\geq\eta(x,x)$ for all $x,y\in Fix(T)$, then T has a unique fixed point

Proof. First step. We shall prove that

$$\lim_{n\to\infty} d(x_n,x_{n+1})=0 \text{ and } \lim_{n\to\infty} d(x_{n+1},x_n)=0.$$

Let $x_0\in X$ such that $\alpha(x_0,Tx_0)\geq\eta(x_0,Tx_0)$ and $\alpha\left(x_0,T^2x_0\right)\geq\eta\left(x_0,T^2x_0\right)$.
For such x_0, we define the sequence $\{x_n\}_{n\in\mathbb{N}}$ by $x_n=T^nx_0=Tx_{n-1}$.
Since, T is α-admissible mapping with respect to η we have

$$\alpha(x_0,Tx_0)=\alpha(x_0,x_1)\geq\eta(x_0,Tx_0)=\eta(x_0,x_1)$$

and

$$\alpha\left(x_0,T^2x_0\right)=\alpha(x_0,x_2)\geq\eta\left(x_0,T^2x_0\right)=\eta(x_0,x_2).$$

By continuing this process, we have

$$\eta(x_n, Tx_n) = \eta(x_n, x_{n+1}) \quad \leq \alpha(x_n, x_{n+1}), \tag{3.2}$$

and

$$\eta(x_n, Tx_{n+1}) = \eta(x_n, x_{n+2}) \quad \leq \alpha(x_n, x_{n+2}). \tag{3.3}$$

for all $n \in N$. If there exist $n_0 \in N$ such that $d(x_{n_0}, x_{n_0+1}) = 0$ or $d\left(x_{n_{0+1}}, x_{n_0}\right) = 0$, then x_{n_0} is fixed point of T.
So, we assume that $d(x_n, Tx_n) > 0$ and $d(Tx_n, x_n) > 0$ for all $n \in N$.
Therefore,

$$\max[d(x_n, x_{n+1}), d(x_{n+1}, x_n)] > 0.$$

So from assumption of the theorem, we get,

$$F[\max\{(d(x_n, x_{n+1})), d(x_{n+1}, x_n)\}] + \tau \leq F[\max\{d(x_{n-1}, x_n), d(x_n, x_{n-1})\}],$$

which implies,

$$\begin{aligned} &F[\max\{d(x_n, x_{n+1}), d(x_{n+1}, x_n)\}] \leq F[\max\{d(x_{n-1}, x_n), d(x_n, x_{n-1})\}] - \tau \\ &\leq F[\max\{d(x_{n-2}, x_{n-1}), d(x_{n-1}, x_{n-2})\}] - 2\tau \\ &\leq ... \leq F[\max\{d(x_0, x_1), d(x_1, x_0)\}] - n\tau \end{aligned}$$

Therefore,

$$F[\max\{(d(x_n, x_{n+1})), d(x_{n+1}, x_n)\}] \leq F[\max\{d(x_0, x_1), d(x_1, x_0)\}] - n\tau. \tag{3.4}$$

Letting n $\rightarrow +\infty$ in (3.4), we obtain

$$\lim_{n\to\infty} F[\max\{(d(x_n, x_{n+1})), d(x_{n+1}, x_n)\}] = -\infty.$$

From (F_2), we obtain

$$\lim_{n\to\infty} \max\{(d(x_n, x_{n+1})), d(x_{n+1}, x_n)\} = 0.$$

Therefore,

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = 0 \text{ and } \lim_{n\to\infty} d(x_{n+1}, x_n) = 0. \tag{3.5}$$

Second step. We shall prove that

$$\lim_{n\to\infty} d(x_n, x_{n+2}) = 0 \text{ and } \lim_{n\to\infty} d(x_{n+2}, x_n) = 0.$$

We assume that $x_n \neq x_m$ for every $n, m \in \mathbb{N}$. Indeed, suppose that $x_n = x_m$ for some $n = m + k$ with $k > 0$, so we have $Tx_n = Tx_m$. By (3.1) implies that

$$\begin{aligned} &F[\max\{d(x_m, x_{m+1}), d(x_{m+1}, x_m)\}] = F[\max\{d(x_n, x_{n+1}), d(x_{n+1}, x_n)\}] \\ &\leq F[\max\{d(x_{n-1}, x_n), d(x_n, x_{n-1})\}] - \tau. \\ &< F[\max\{d(x_{n-1}, x_n), d(x_n, x_{n-1})\}]. \end{aligned}$$

Since F is strictly increasing, so

$$\max\{d(x_n, x_{n+1}), d(x_{n+1}, x_n)\} < \max\{d(x_{n-1}, x_n), d(x_n, x_{n-1})\}.$$

Continuing this process, we have

$$\max\{d(x_m, x_{m+1}), d(x_{m+1}, x_m)\} < \max\{d(x_m, x_{m+1}), d(x_{m+1}, x_m)\}.$$

which is a contradiction. Therefore,

$$\max\left\{d\left(x_m,x_n\right),d\left(x_n,x_m\right)\right\}>0$$

for every $n,m\in\mathbb{N}$, $n\neq m$
Substituting $x=x_n$ and $y=x_{n+2}$. Then

$$max\left\{d\left(x_n,x_{n+2}\right),d\left(x_{n+2,}x_n\right)\right\}>0.$$

Now, applying (3.1) with $x=x_{n-1}$ and $y=x_{n+1}$, we get

$$F\left[\max\left\{d\left(x_n,x_{n+2}\right),d\left(x_{n+2},x_n\right)\right\}\right]+\tau\leq F\left[\max\left\{d\left(x_{n-1},x_{n+1}\right),d\left(x_{n+1},x_{n-1}\right)\right\}\right].$$

So, $F\left(\max\left\{d\left(x_n,x_{n+2}\right),d\left(x_{n+2},x_n\right)\right\}\right)\leq F\left[\max\left\{d\left(x_{n-1},x_{n+1}\right),d\left(x_{n+1},x_{n-1}\right)\right\}\right]-\tau$

$$\leq F\left[\max\left\{d\left(x_{n-2},x_n\right),d\left(x_n,x_{n-2}\right)\right\}\right]-2\tau...\leq F\left[\max\left\{d\left(x_0,x_2\right),d\left(x_2,x_0\right)\right\}\right]-n\tau.$$

Therefore,

$$F\left[\max\left\{d\left(x_n,x_{n+2}\right),d\left(x_{n+2},x_n\right)\right\}\right]\leq F\left[\max\left\{d\left(x_0,x_2\right),d\left(x_2,x_0\right)\right\}\right]-n\tau. \quad (3.6)$$

Letting n $\to+\infty$ in (3.6), we obtain

$$\lim_{n\to\infty}F\left[\max\left\{\left(d\left(x_n,x_{n+2}\right)\right),d\left(x_{n+2},x_n\right)\right\}\right]=-\infty$$

and from $\left(F_2\right)$, we obtain

$$\lim_{n\to\infty}\max\left\{\left(d\left(x_n,x_{n+2}\right)\right),d\left(x_{n+2},x_n\right)\right\}=0.$$

Therefore,

$$\lim_{n\to\infty}d\left(x_n,x_{n+2}\right)=0 \text{ and } \lim_{n\to\infty}d\left(x_{n+2},x_n\right)=0. \quad (3.7)$$

Third step. We shall prove that, $\{x_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence in (X,d).
Now, from $\left(F_3\right)$, there exists $k\in\,]0,1[$ such that

$$\lim_{n\to\infty}\left[\max\left\{\left(d\left(x_n,x_{n+1}\right)\right),d\left(x_{n+1},x_n\right)\right\}\right]^kF\left(\max\left\{\left(d\left(x_n,x_{n+1}\right)\right),d\left(x_{n+1},x_n\right)\right\}\right)=0. \quad (3.8)$$

Since

$$F\left[\max\left\{d\left(x_n,x_{n+1}\right),d\left(x_{n+1},x_n\right)\right\}\right]\leq F\left[\max\left\{d\left(x_0,x_1\right),d\left(x_1,x_0\right)\right\}\right]-n\tau,$$

we have

$$\begin{aligned}&\left[\max\left\{\left(d\left(x_n,x_{n+1}\right)\right),d\left(x_{n+1},x_n\right)\right\}\right]^kF\left[\max\left\{d\left(d\left(x_n,x_{n+1}\right)\right),d\left(x_{n+1},x_n\right)\right\}\right]\\&\leq\left[\max\left\{\left(d\left(x_n,x_{n+1}\right)\right),d\left(x_{n+1},x_n\right)\right\}\right]^kF\left[\max\left\{d\left(x_0,x_1\right),d\left(x_1,x_0\right)\right\}\right]\\&-n\tau\left[\max\left\{\left(d\left(x_n,x_{n+1}\right)\right),d\left(x_{n+1},x_n\right)\right\}\right]^k\end{aligned}$$

Therefore,

$$\begin{aligned}&\left[\max\left\{\left(d\left(x_n,x_{n+1}\right)\right),d\left(x_{n+1},x_n\right)\right\}\right]^kF\left[\max\left\{d\left(x_n,x_{n+1}\right),d\left(x_{n+1},x_n\right)\right\}\right]\\&-\left[\max\left\{\left(d\left(x_n,x_{n+1}\right)\right),d\left(x_{n+1},x_n\right)\right\}\right]^kF\left[\max\left\{d\left(x_0,x_1\right),d\left(x_1,x_0\right)\right\}\right]\\&\leq-n\tau\left[\max\left\{\left(d\left(x_n,x_{n+1}\right)\right),d\left(x_{n+1},x_n\right)\right\}\right]^k\\&\leq 0.\end{aligned} \quad (3.9)$$

Letting n $\to+\infty$ in (3.9) and from (3.8), we obtain

$$\lim_{n\to\infty}n\left[\max\left\{\left(d\left(x_n,x_{n+1}\right)\right),d\left(x_{n+1},x_n\right)\right\}\right]^k=0.$$

From the definition of the limit, there exists $n_0 \in \mathbb{N}$ such that

$$[\max\{(d(x_n, x_{n+1})), d(x_{n+1}, x_n)\}] \leq \frac{1}{n^{\frac{1}{k}}}, \forall n \geq n_0.$$

Which implies

$$(d(x_n, x_{n+1})) \leq \frac{1}{n^{\frac{1}{k}}} \text{ and } d(x_{n+1}, x_n) \leq \frac{1}{n^{\frac{1}{k}}} \forall n \geq n_0. \tag{3.10}$$

On the other hand from (F_3) there exists $k \in]0,1[$ such that

$$\lim_{n\to\infty}[\max\{(d(x_n, x_{n+2})), d(x_{n+2}, x_n)\}]^k F(\max\{(d(x_n, x_{n+2})), d(x_{n+2}, x_n)\}) = 0. \tag{3.11}$$

Since

$$F[\max\{d(x_n, x_{n+2}), d(x_{n+2}, x_n)\}] \leq F[\max\{d(x_0, x_2), d(x_2, x_0)\}] - n\tau,$$

Therefore,

$$\begin{aligned}&[\max\{(d(x_n, x_{n+2})), d(x_{n+2}, x_n)\}]^k F[\max\{(d(x_n, x_{n+2})), d(x_{n+2}, x_n)\}]\\ &\leq [\max\{(d(x_n, x_{n+2})), d(x_{n+2}, x_n)\}]^k F[\max\{d(x_0, x_2), d(x_2, x_0)\}]\\ &- n\tau[\max\{(d(x_n, x_{n+2})), d(x_{n+2}, x_n)\}]^k.\end{aligned}$$

So, we have

$$\begin{aligned}&[\max\{d(x_n, x_{n+2}), d(x_{n+2}, x_n)\}]^k F[\max\{(d(x_n, x_{n+2})), d(x_{n+2}, x_n)\}]\\ &- [\max\{(d(x_n, x_{n+2})), d(x_{n+2}, x_n)\}]^k F[\max\{d(x_0, x_2), d(x_2, x_0)\}]\\ &\leq -n\tau[\max\{(d(x_n, x_{n+2})), d(x_{n+2}, x_n)\}]^k \leq 0.\end{aligned} \tag{3.12}$$

Letting n $\to +\infty$ in (3.12) and from (11), we obtain

$$\lim_{x\to\infty} n[\max\{(d(x_n, x_{n+2})), d(x_{n+2}, x_n)\}]^k = 0.$$

From the definition of the limit, there exists $n_1 \in \mathbb{N}$ such that

$$[\max\{(d(x_n, x_{n+2})), d(x_{n+2}, x_n)\}] \leq \frac{1}{n^{\frac{1}{k}}} \text{ } for \text{ } all \text{ } n \geq n_1.$$

So,

$$d(x_n, x_{n+2}) \leq \frac{1}{n^{\frac{1}{k}}} \text{ and } d(x_{n+2}, x_n) \leq \frac{1}{n^{\frac{1}{k}}} \text{ } for \text{ } all \text{ } n \geq n_1. \tag{3.13}$$

Next, we show that $\{x_n\}_{n\in\mathbb{N}}$ is a Forward Cauchy sequence, i.e,

$$\lim_{n\to\infty} d(x_n, x_{n+k}) = 0 \text{ } for \text{ } all \text{ } k \in \mathbb{N}.$$

The cases $k = 1$ and $k = 2$, are proved, respectively by (3.5) and (3.7).
Now, we take $k \geq 3$. It is sufficient to examine two cases:
Case (I): Suppose that $k = 2m + 1$ where $m \geq 1$. By using the quadrilateral inequality together we have

$$d(x_n, x_{n+k}) = d(x_n, x_{n+2m+1}) \leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \dots\dots + d(x_{n+2m}, x_{n+2m+1}).$$

Using (3.10), we have

$$d(x_n, x_{n+k}) \leq \frac{1}{n^{\frac{1}{k}}} + \frac{1}{(n+1)^{\frac{1}{k}}} + ... + \frac{1}{(n+2m)^{\frac{1}{k}}}$$
$$\leq \sum_{i=n}^{i=n+2m} \frac{1}{i^{\frac{1}{k}}} \leq \sum_{i=n}^{\infty} \frac{1}{i^{\frac{1}{k}}} < \infty. \quad (3.14)$$

Case (II): Suppose that $k = 2m$ where $m \geq 1$. Again, by applying the quadrilateral inequality we have

$$d(x_n, x_{n+k}) = d(x_n, x_{n+2m}) \leq d(x_n, x_{n+2}) + d(x_{n+2}, x_{n+3}) + ... + d(x_{n+2m-1}, x_{n+2m}).$$

Using (3.10) and (3.13), we have

$$d(x_n, x_{n+k}) \leq d(x_n, x_{n+2}) + \frac{1}{(n+2)^{\frac{1}{k}}} + ... + \frac{1}{(n+2m-1)^{\frac{1}{k}}}$$
$$\leq \frac{1}{n^{\frac{1}{k}}} + \sum_{i=n+2}^{i=n+2m-1} \frac{1}{i^{\frac{1}{k}}} \leq \sum_{i=n}^{\infty} \frac{1}{i^{\frac{1}{k}}} < \infty. \quad (3.15)$$

By combining the expressions (3.14) and (3.15), we obtain

$$d(x_n, x_{n+k}) \leq \sum_{i=n}^{\infty} \frac{1}{i^{\frac{1}{k}}} < \infty. \quad (3.16)$$

Letting n $\to +\infty$ in (3.16), we obtain

$$\lim_{n\to\infty} d(x_n, x_{n+k}) = 0.$$

Consequently, $\{x_n\}_{n\in\mathbb{N}}$ is forward Cauchy sequence in X. Secondly we show $\{x_n\}_{n\in\mathbb{N}}$ is a Backward Cauchy sequence, i.e.

$$\lim_{x\to\infty} d(x_{n+k}, x_n) = 0, \ for\ all\ k \in \mathbb{N}.$$

The cases $k = 1$ and $k = 2$, are proved, respectively, by (3.5) and (3.7) i.e.,

$$\lim_{x\to\infty} d(x_{n+1}, x_n) = \lim_{x\to\infty} d(x_{n+2}, x_n) = 0.$$

Now, we take $k \geq 3$. It is sufficient to examine two cases:
Case (I): Suppose that $k = 2m + 1$ where $m \geq 1$. Then, by using the quadrilateral inequality we have

$$d(x_{n+k}, x_n) = d(x_{n+2m+1}, x_n)$$
$$\leq d(x_{n+2m+1}, x_{n+2m}) + d(x_{n+2m}, x_{n+2m-1}) + ... + d(x_{n+1}, x_n).$$

From (3.10) we have

$$d(x_n, x_{n+k}) \leq \frac{1}{n^{\frac{1}{k}}} + \frac{1}{(n+1)^{\frac{1}{k}}} + ... + \frac{1}{(n+2m)^{\frac{1}{k}}}$$
$$\leq \sum_{i=n}^{i=n+2m} \frac{1}{i^{\frac{1}{k}}} \leq \sum_{i=n}^{\infty} \frac{1}{i^{\frac{1}{k}}} < \infty. \quad (17)$$

Case (II): Suppose that $k=2m$ where $m\geq 1$. Again, by applying the quadrilateral inequality we have

$$\begin{aligned}d\left(x_{n+k},x_n\right)&=d\left(x_{n+2m},x_n\right)\\&\leq d\left(x_{n+2m},x_{n+2m-2}\right)+d\left(x_{n+2m-2},x_{n+2m-3}\right)+......+d\left(x_{n+1},x_n\right).\end{aligned}$$

Using (3.10) and (3.13), we obtain

$$\begin{aligned}d\left(x_{n+k},x_n\right)&\leq d\left(x_{n+2m},x_{n+2m-2}\right)+\frac{1}{(n+2m-3)^{\frac{1}{k}}}+....+\frac{1}{(n+1)^{\frac{1}{k}}}+\frac{1}{n^{\frac{1}{k}}}\\&\leq\frac{1}{(n+2m-2)^{\frac{1}{k}}}+\sum_{i=n}^{i=n+2m-3}\frac{1}{i^{\frac{1}{k}}}\leq\sum_{i=n}^{\infty}\frac{1}{i^{\frac{1}{k}}}<\infty.\end{aligned}\tag{3.17}$$

For two cases, we obtain

$$d\left(x_{n+k},x_n\right)\leq\sum_{i=n}^{\infty}\frac{1}{i^{\frac{1}{k}}}<\infty\ \forall k\geq 3.\tag{3.18}$$

Letting $n\to\infty$ in (3.18), we obtain

$$\lim_{n\to\infty}d\left(x_{n+k},x_n\right)=0.$$

Consequently, $\{x_n\}_{n\in\mathbb{N}}$ is a Backward Cauchy sequence in (X,d). We deduce that $\{x_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence in complete generalized asymmetric metric space (X,d). By completeness of (X,d), there exists z, $w\in X$ such that

$$\lim_{n\to\infty}d\left(x_n,z\right)=0\text{ and }\lim_{n\to\infty}d\left(w,x_n\right)=0.$$

By Lemma (2.5), we get $z=w$.
On the other hand

$$\eta\left(x_n,x_{n+1}\right)\leq\alpha\left(x_n,x_{n+1}\right).$$

Since T is $\alpha-\eta$-continuous, we have $x_{n+1}=Tx_n\to Tz$ as $n\to\infty$.
In this stage, we show that, $d\left(z,Tz\right)=0$ or $d\left(Tz,z\right)=0$. Observe that

$$\lim_{n\to\infty}d\left(Tx_n,Tz\right)=\lim_{n\to\infty}d\left(Tz,Tx_n\right)=0.\tag{3.19}$$

From the quadrilateral inequality we get,

$$d\left(Tx_n,Tz\right)\leq d\left(Tx_n,x_n\right)+d\left(x_n,z\right)+d\left(z,Tz\right),\tag{3.20}$$

and

$$d\left(z,Tz\right)\leq d\left(z,x_n\right)+d\left(x_n,Tx_n\right)+d\left(Tx_n,Tz\right).\tag{3.21}$$

By letting $n\to\infty$ in (3.20) and (3.21), we obtain

$$d\left(z,Tz\right)\leq\lim_{n\to\infty}d\left(Tx_n,Tz\right)\leq d\left(z,Tz\right).$$

Therefore,

$$\lim_{n\to\infty}d\left(Tx_n,Tz\right)=d\left(z,Tz\right).\tag{3.22}$$

Again by quadrilateral inequality, we get

$$d\left(Tz,Tx_n\right)\leq d\left(Tz,z\right)+d\left(z,x_n\right)+d\left(x_n,Tx_n\right),\tag{3.23}$$

and

$$d\left(Tz,z\right)\leq d\left(Tz,Tx_{n}\right)+d\left(Tx_{n},x_{n}\right)+d\left(x_{n},z\right). \tag{3.24}$$

By letting $n\to\infty$ in inequality (3.23) and (3.24), we obtain

$$d\left(Tz,z\right)\leq\lim_{n\to\infty}d\left(Tz,Tx_{n}\right)\leq d\left(Tz,z\right). \tag{3.25}$$

Therefore

$$\lim_{n\to\infty}d\left(Tz,Tx_{n}\right)=d\left(Tz,z\right). \tag{3.26}$$

From (3.19) , (3.22) and (3.26) we prove that $Tz=z$.
Uniqueness.
Let $u,v\in Fix(T)$ where $u\neq v$. Substituting $x=u$ and $y=v$ in (3.1), we obtain

$$\max\left[d\left(Tv,Tu\right),d\left(Tu,Tv\right)\right]=\max\left[d\left(v,u\right),d\left(u,v\right)\right]>0.$$

So from the assumption of theorem, we have

$$\alpha(u,v)\geq\eta(v,v),$$

and

$$\begin{aligned}F\left[\max\{d\left(Tv,Tu\right),d\left(Tu,Tv\right)\}\right]-\tau&=F\left[\max\{d\left(v,u\right),d\left(u,v\right)\}\right]-\tau\\&<\max\left[d\left(v,u\right),d\left(u,v\right)\right].\end{aligned}$$

Which is a contradiction. Hence, $u=v$. Therefore, T has a unique fixed point. ■

If in Theorem 3.4 we take $\alpha(x,y)=\eta(x,y)=1$ for all $x,y\in X$, then we deduce the Corollary.

Corollary 3.5. *Let (X,d) be a complete generalized asymmetric metric space. Let $T:X\to X$ be a self-mapping satisfying If for $x,y\in X$ with $\max\left[d\left(Tx,Ty\right),d\left(Ty,Tx\right)\right]>0$ we have*

$$F\left[\max\left\{d\left(Tx,Ty\right),d\left(Ty,Tx\right)\right\}\right]+\tau\leq F\left[\max\left\{d\left(x,y\right),d\left(y,x\right)\right\}\right]$$

where $\tau>0$ and $F\in\digamma$. Then T has a unique fixed point.

Example 3.6. Consider $X=\{0,1,2,3\}$. Let $d:X\times X\to\mathbf{R}^{+}$, be a mapping defined by

(i) $d(x,y)=d(y,x)=0\Leftrightarrow x=y\ \forall x,y\in X$;

(ii) $d(0,1)=d(1,0)=d(2,1)=d(2,0)=d(3,0)=d(2,3)=d(3,1)=1$;

(iii) $d(1,2)=d(0,2)=2$;

(vi) $d(0,3)=d(3,2)=3,d(1,3)=4$.

Clearly, (X,d) is not asymmetric metric spaces, from

$$d(1,3)=4>d(1,2)+d(2,3)=3.$$

But is a complete generalized asymmetric metric space.
Let $T : X \to X$, be given by

$$\begin{cases} T(0) = T(1) = 0 \\ T(2) = 1 \\ T(3) = 2. \end{cases}$$

$T(0) = T(1) = 0$, $T(2) = 1$, $T(3) = 2$.
Define, $\alpha, \eta \colon X \times X \to [0, +\infty[$ by

$$\alpha(x, y) = \frac{x+y}{\max\{x, y\}+1}, \text{ and} \eta(x, y) = \frac{|x-y|}{\max\{x, y\}+1},$$

so,

$$\frac{|x-y|}{\max\{x, y\}+1} \leq \frac{x+y}{\max\{x, y\}+1} \forall x, y \in X,$$

then,

$$\eta(x, y) \leq \alpha(x, y) \forall\, x, y \in X,$$

and $T(x) \in X$ $\forall x \in X$, imply

$$\eta(Tx, Ty) \leq \alpha(Tx, Ty) \forall\, x, y \in X.$$

Hence, T is an α -admissible mapping with respect to η.
On the other hand, $\alpha(3, T3) = \frac{5}{4} \geq \frac{1}{4} = \eta(3, T3)$. and $\alpha\left(3, T^2 3\right) = 1 \geq \frac{1}{2} = \eta\left(3, T^2 3\right)$,
Clearly, T is $\alpha - \eta$ continuous.
If

$$lim_{n\to\infty} x_n = x \text{ and } \eta(x_n, Tx_n) \leq \alpha(x_n, Tx_n),$$

then,

$$lim_{n\to\infty} Tx_n = Tx, \quad \forall x_n \in X.$$

On the other hand

$$\begin{cases} \eta(0, T(0)) = \eta(0, 0) = 0 \leq \alpha(0, x) \forall x \in X; \\ \eta(1, T(1)) = \eta(1, 0) = \frac{1}{2} \leq \alpha(1, x) \forall x \in X; \\ \eta(2, T(2)) = \eta(2, 1) = \frac{1}{3} \leq \alpha(2, x) \ \forall x \in X; \\ \eta(3, T(3)) = \eta(3, 2) = \frac{1}{4} \leq \alpha(3, x) \ \forall x \in X. \end{cases}$$

Then

$$\eta(x, Tx) \leq \alpha(x, y), \ \forall x, y \in X.$$

Suppose $F(x) = \ln(x)$, and $\tau = \ln\left(\frac{3}{2}\right)$. Obviously, $F \in \digamma$.
First observe that $\max\{d(Tx, Ty), d(Ty, Tx)\} > 0 \Leftrightarrow \{x = 0, y = 2\}$, $\{x = 1, y = 2\}$, $\{x = 0, y = 3\}$, $\{x = 1, y = 3\}$ or $\{x = 2, y = 3\}$.
For $x = 0,\ y = 2$, we have

$$\begin{aligned} \ln(\max\{d(T(0), T(2)), d(T(2), T(0))\}) + \tau &= \ln(\max\{d(0, 1), d(1, 0)\}) + \tau = \ln\frac{3}{2} \\ &\leq \ln(\max\{d(0, 2), d(2, 0)\}) = \ln 2. \end{aligned}$$

For $x=1,\ y=2$, we have

$$\begin{aligned}\ln\left(\max\left\{d(T(1),T(2)),d(T(2),T(1)\right\}\right)+\tau &= \ln\left(\max d(0,1),d(1,0)\right)+\frac{1}{2}=\ln\frac{3}{2}\\ &\leq \ln\left(\max\left\{d(1,2),d(2,1)\right\}\right)=\ln 2.\end{aligned}$$

For $x=0,\ y=3$, we have

$$\begin{aligned}\ln\left(\max\left\{(T(0),T(3)),d(T(3),T(0))\right\}\right)+\tau &= \ln\left(\max d(0,2),d(2,0)\right)\\ &= \ln\frac{3}{2}+\ln 2=\ln 3\\ &\leq \ln\left(\max\left\{d(0,3),d(3,0)\right\}\right)=\ln 3.\end{aligned}$$

For $x=1,\ y=3$, we have

$$\begin{aligned}\ln\left(\max\left\{(T(1),T(3)),d(T(3),T(1))\right\}\right)+\tau &= \ln\left(\max d(0,2),d(2,0)\right)\\ &= \ln\frac{3}{2}+\ln 2=\ln 3\\ &\leq \ln\left(\max\left\{d(1,3),d(3,1)\right\}\right)=\ln 4.\end{aligned}$$

For $x=2,\ y=3$, we have

$$\begin{aligned}\ln\left(\max\left\{(T(2),T(3)),d(T(3),T(2))\right\}\right)+\tau &= \ln\left(\max d(1,2),d(2,1)\right)\\ &= \ln\frac{3}{2}+\ln 2=\ln 3\\ &\leq \ln\left(\max\left\{d(2,3),d(3,2)\right\}\right)=\ln 3.\end{aligned}$$

Therefore,

$$\begin{aligned}\max\left[d\left(Tx,Ty\right),d\left(Ty,Tx\right)\right]>0 \Rightarrow &F\left[\max\left\{d\left(Tx,Ty\right),d\left(Ty,Tx\right)\right\}\right]+\tau\\ &\leq F\left[\max\left\{d\left(x,y\right),d\left(y,x\right)\right\}\right].\end{aligned}$$

Hence T has a unique fixed point i.e, $T(0)=0$.

Theorem 3.7. *Let (X,d) be a complete generalized asymmetric metric space and let $T:X\to X$ be a self-mapping satisfying the following assertions:*
(i) T is α-admissible mapping with respect to η;
(ii) T is α-η -F-contraction of type (A);
(iii) there exists $x_0\in X$ such that $\alpha(x_0,Tx_0)\geq\eta(x_0,Tx_0)$ and $\alpha\left(x_0,T^2x_0\right)\geq\eta\left(x_0,T^2x_0\right)$;
(iv) If x_n is a sequence in X such that $\alpha(x_n,x_{n+1})\geq\eta(x_n,x_{n+1})$ with $x_n\to x$ as $n\to\infty$, then either $\eta(Tx_n,T^2x_n)\leq\alpha(Tx_n,x)$ or $\eta(T^2x_n,T^3x_n)\leq\alpha(T^2x_n,x)$ holds for all $n\in N$. Then T has a fixed point. Moreover, T has a unique fixed point when $\alpha(x,y)\geq\eta(x,x)$ for all $x,y\in Fix(T)$.

Proof. Let $x_0\in X$ such that

$$\alpha\left(x_0,Tx_0\right)\geq\eta\left(x_0,Tx_0\right)\text{and }\alpha\left(x_0,T^2x_0\right)\geq\eta\left(x_0,T^2x_0\right).$$

As in the proof of Theorem (3.4), we can conclude that

$$\eta\left(x_n,x_{n+1}\right)\leq\alpha\left(x_n,x_{n+1}\right),$$

and

$$\eta\left(x_n,x_{n+2}\right)\ \leq\alpha\left(x_n,x_{n+2}\right).$$

Therefore,

$$\lim_{n\to\infty} x_n = x \quad for\ all\ n \in \mathbb{N}.$$

From (iv), we have

$$\eta(x_{n+1}, x_{n+2}) \ \leq \alpha(x_{n+1}, x) \text{ or } \eta(x_{n+2}, x_{n+3}) \ \leq \alpha(x_{n+2}, x) \ ,$$

holds for all $n \in \mathbb{N}$. Equivalently, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that

$$\eta(x_{n_k}, Tx_{n_k}) = \eta\left(x_{n_k}, x_{n_{k+1}}\right) \leq \alpha(x_{n_k}, x).$$

On the other hand from (ii) we deduce that

$$F[\max\{d(Tx_{n_k}, Tx), d(Tx, Tx_{n_k})\}] + \tau \leq F[\max\{d(x_{n_k}, x), d(x, x_{n_k})\}].$$

Since F is strictly increasing, so

$$\max\{d(Tx_{n_k}, Tx), d(Tx, Tx_{n_k})\} < \max\{d(x_{n_k}, x), d(x, x_{n_k})\}. \tag{3.27}$$

By taking limit as $k \to \infty$ in (3.27), we get

$$\lim_{k\to\infty}[\max\{d(Tx_{n_k}, Tx), d(Tx, Tx_{n_k})\}] \leq \lim_{k\to\infty}[\max\{d(x_{n_k}, x), d(x, x_{n_k})\}].$$

Therefore,

$$\lim_{k\to\infty}[\max\{d(Tx_{n_k}, Tx), d(Tx, Tx_{n_k})\}] = 0.$$

so, we have

$$\lim_{k\to\infty} d(Tx_{n_k}, Tx) = \lim_{k\to\infty} d(Tx, Tx_{n_k}) = 0$$

Then $\lim_{k\to\infty} Tx_{n_k} = Tx$, therefore $\lim_{n\to\infty} Tx_n = Tx$. As in proof of Theorem (3.4) we can conclude that $Tz = z$.
Uniqueness: follow similarly as in Theorem (3.4). ■

Definition 3.8. Let (X, d) be a generalized asymmetric metric space. A mapping T : $X \times X$ is said to be an α -η-contraction of type (B) on (X, d), if there exist $F \in \Im$, α, η: $X \times X \to [0, +\infty[$ and $\tau > 0$ such that

$$\begin{aligned}\max\{d(Tx,Ty), d(Ty,Tx)\} \ > 0 \Rightarrow & F[\max\{d(Tx,Ty), d(Ty,Tx)\}] + \tau \\ & \leq F[\max\{d(x,y), d(y,x)\}].\end{aligned} \tag{3.28}$$

Theorem 3.9. *Let (X, d) be a complete generalized asymmetric metric space and let $T : X \to X$ be a self-mapping satisfying the following assertions:*
(i) T is α-admissible mapping with respect to η;
(ii) T is α-η -F-contraction of type (B);
(iii) there exists $x_0 \in X$ such that$\alpha(x_0, Tx_0) \geq \eta(x_0, Tx_0)$ and $\alpha\left(x_0, T^2x_0\right) \geq \eta\left(x_0, T^2x_0\right)$;
(iv) T is α-η-continuous.
Then T has a fixed point. Moreover, T has a unique fixed point when $\alpha(x, y) \geq \eta(x, x)$ for all $n \in \mathbb{N}$ $x, y \in Fix(T)$.

Proof. First step. Let $x_0 \in X$ such that

$$\alpha\left(x_0, Tx_0\right) \geq \eta\left(x_0, Tx_0\right) \text{and } \alpha\left(x_0, T^2x_0\right) \geq \eta\left(x_0, T^2x_0\right).$$

As in the proof of Theorem (3.4) we can conclude that

$$\lim_{n\to\infty} d\left(x_n, x_{n+1}\right) = \lim_{n\to\infty} d\left(x_{n+1}, x_n\right) \quad = 0, \tag{3.29}$$

and

$$\lim_{n\to\infty} d\left(x_n, x_{n+2}\right) = \lim_{n\to\infty} d\left(x_{n+2}, x_n\right) \quad = 0. \tag{3.30}$$

Second step. Next we show that $\{x_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence.i.e.

$$\lim_{n,m\to\infty} d\left(x_m, x_n\right) = \lim_{n,m\to\infty} d\left(x_n, x_m\right) = 0$$

Now, we define a function D from $X \times X \to \mathbb{R}^+$ as follows

$$D\left(x, y\right) = \max(d(x, y), d(y, x)) \quad for\ all\ x, y \in X.$$

Now, we claim that, $\lim_{n,m\to\infty} D\left(x_m, x_n\right) = 0$. Arguing by contradiction. we assume that there exists $\varepsilon > 0$ we can find and sequences $(m\left(k\right))_k$ and $(n\left(k\right))_k$ of positive integers such that
for all positive integers, $n(k) > m(k) > k$,

$$D(x_{m(k)}, x_{n(k)}) \geq \varepsilon$$

and

$$D(x_{m(k)}, x_{n(k)-1}) < \varepsilon.$$

So, we have

$$\begin{aligned}\varepsilon &\leq D\left(x_{m_{(k)}}, x_{n_{(k)}}\right)\\ &\leq D\left(x_{m_{(k)}}, x_{n(k)-1}\right) + D\left(x_{n(k)-1}, x_{n(k)+1}\right) + D\left(x_{n(k)+1}, x_{n_{(k)}}\right)\\ &< \varepsilon + D\left(x_{n(k)-1}, x_{n(k)+1}\right) + D\left(x_{n(k)+1}.x_{n_{(k)}}\right)\end{aligned}$$

So,

$$\lim_{k\to\infty} D(x_{m(k)}, x_{n(k)}) = \varepsilon. \tag{3.31}$$

Again by the quadrilateral inequality, for all $n \in \mathbb{N}$, we have the following two inequalities

$$D\left(x_{m_{(k)+1}}, x_{n_{(k)+1}}\right) \leq D\left(x_{m_{(k)+1}}, x_{m(k)}\right) + D\left(x_{m(k)}, x_{n_{(k)}}\right) + D\left(x_{n(k)}, x_{n_{(k)+1}}\right), \tag{3.32}$$

and

$$D\left(x_{m_{(k)}}, x_{n_{(k)}}\right) \leq D\left(x_{m_{(k)}}, x_{m(k)+1}\right) + D\left(x_{m(k)+1}, x_{n_{(k)+1}}\right) + D\left(x_{n(k)+1}, x_{n_{(k)}}\right). \tag{3.33}$$

Letting $k \to +\infty$ in (3.32) and (3.33), we obtain

$$\lim_{k\to\infty} D\left(x_{m_{(k)+1}}, x_{n_{(k)+1}}\right) = \varepsilon. \tag{3.34}$$

From (34) there exists $n_0 \in \mathbb{N}$ such that

$$D\left(x_{m_{(k)+1}}, x_{n_{(k)+1}}\right) \geq \frac{\varepsilon}{2}\ for\ all\ n \geq n_0. \tag{3.35}$$

Therefore,

$$\max\left\{d\left(x_{m_{(k)+1}}, x_{n_{(k)+1}}\right), d\left(x_{n_{(k)+1}}, x_{m_{(k)+1}}\right)\right\} \geq \frac{\varepsilon}{2} \ \forall n \geq n_0 .$$

So,

$$\max\left\{d\left(Tx_{m_{(k)}}, Tx_{n_{(k)}}\right), d\left(x_{n_{(k)}}, Tx_{m_{(k)}}\right)\right\} \geq \frac{\varepsilon}{2} \ for\ all\ n \geq n_0. \tag{3.36}$$

On the other hand, T is α-η -F-contraction, we obtain

$$\tau + F\left(D\left(x_{m_{(k)+1}}, x_{n_{(k)+1}}\right)\right) \leq F\left(D\left(x_{m_{(k)}}, x_{n_{(k)}}\right)\right). \tag{3.37}$$

Letting $k \to +\infty$ in (3.37), and from (iii) of Definition 2.12 we obtain

$$\tau + F\left(\frac{\varepsilon}{2}\right) \leq F\left(\frac{\varepsilon}{2}\right) \Rightarrow \tau \leq 0.$$

Which is contradiction, it follows that

$$\lim_{n,m\to\infty} D\left(x_m, x_n\right) = 0. \tag{3.38}$$

therefore,

$$\lim_{n,m\to\infty} d\left(x_m, x_n\right) = \lim_{n,m\to\infty} d\left(x_n, x_m\right) = 0. \tag{3.39}$$

Hence $\{x_n\}_{n\in\mathbb{N}}$ is Forward and Backward Cauchy sequence in X. As in proof of Theorem (3.4), we can proved that T has is unique fixed point $z \in X$. ■

Example 3.10. Consider $X = \{0, 1, 2, 3\}$. Let $d : X \times X \to \mathbf{R}^+$, be a mapping defined by

(i) $d(x, y) = 0 \Leftrightarrow x = y \quad \forall x, y \in X$;

(ii) $d(0, 1) = d(1, 0) = d(2, 1) = d(2, 0) = d(3, 0) = d(2, 3) = d(3, 1) = 1$;

(iii) $d(1, 2) = d(0, 2) = 2$;

(iv) $d(0, 3) = d(3, 2) = 3, d(1, 3) = 4$. Clearly, (X, d) is not asymmetric metric spaces, from

$$d(1, 3) = 4 > d(1, 2) + d(2, 3) = 3.$$

But is a complete generalized asymmetric space. Let $T : X \to X$, be given by

$$\begin{cases} T(0) = T(1) = 0 \\ T(2) = 1 \\ T(3) = 2. \end{cases}$$

Define, α, η: $X \times X \to [0, +\infty[$ by

$$\alpha\left(x, y\right) = \frac{x + y}{\max\left\{x, y\right\} + 1}, \text{ and } \eta\left(x, y\right) = \frac{|x - y|}{\max\left\{x, y\right\} + 1}.$$

As in the proof of Example (3.12) , we can proved that T satisfying (i), (iii) and (iv). Define $F :]0, +\infty[\to \mathbb{R}$ by

$$F\left(x\right) = \frac{-1}{x} + \frac{x}{4} \ and\ \tau = \frac{5}{12}.$$

Obviously, $F \in \Gamma$. First observe that $\max\{d(Tx,Ty), d(Ty,Tx)\} > 0 \Leftrightarrow \{x=0, y=2\}$, $\{x=1, y=2\}$, $\{x=0, y=3\}$, $\{x=1, y=3\}$ or $\{x=2, y=3\}$.
For $x=0, y=2$, we have

$$\begin{aligned} F\left(\max\{d(T(0),T(2)), d(T(2),T(0))\}\right) + \tau &= F\max\{d(0,1), d(1,0))\} + \tau \\ &= -\frac{3}{4} + \frac{5}{12} = \frac{-1}{12} \\ &\leq F\left(\max\{d(0,2), d(2,0)\}\right) \\ &= F(2) = 0. \end{aligned}$$

For $x=1, y=2$, we have

$$\begin{aligned} F\left(\max\{d(T(1),T(2)), d(T(2),T(1)\}\right) + \tau &= F\left(\max d(0,1), d(1,0)\right) \\ &= \frac{5}{12} - \frac{3}{4} = \frac{-1}{12} \\ &\leq F\left(\max\{d(1,2), d(2,1)\}\right) \\ &= F(2) = 0. \end{aligned}$$

For $x=0, y=3$, we have

$$\begin{aligned} F\left(\max\{d(T(0),T(3)), d(T(3),T(0))\}\right) + \tau &= F\left(\max d(0,2), d(2,0)\right) \\ &= \frac{5}{12} + F(2) = \frac{5}{12} \\ &\leq F\left(\max\{d(0,3), d(3,0)\}\right) \\ &= F(3) = \frac{5}{12}. \end{aligned}$$

For $x=1, y=3$, we have

$$\begin{aligned} F\left(\max\{(T(1),T(3)), d(T(3),T(1))\}\right) + \tau &= F\left(\max d(0,2), d(2,0)\right) \\ &= \frac{5}{12} + F(2) = \frac{5}{12} \\ &\leq F\left(\max\{d(1,3), d(3,1)\}\right) \\ &= F(4) = \frac{3}{4}. \end{aligned}$$

For $x=2, y=3$, we have

$$\begin{aligned} F\left(\max\{(T(2),T(3)), d(T(3),T(2))\}\right) + \tau &= F\left(\max d(1,2), d(2,1)\right) \\ &= \frac{5}{12} + F(2) = \frac{5}{12} \\ &\leq F\left(\max\{d(2,3), d(3,2)\}\right) \\ &= F(3) = \frac{5}{12}. \end{aligned}$$

Therefore, T is α-η -F-contraction of type (B).
Hence T satisfies in assumption of Theorem and is the unique fixed point of T, $z=0$.
If in Theorem (3.9) we take $\alpha(x,y) = \eta(x,y) = 1$ for all $x, y \in X$, then we deduce the Corollary.

Corollary 3.11. *Let (X,d) be a complete generalized asymmetric space and let $T : X \to X$ be a self-mapping such that for $x, y \in X$, $\max[d(Tx,Ty), d(Ty,Tx)] > 0$ we have*

$$F[\max\{d(Tx,Ty), d(Ty,Tx)\}] + \tau \leq F[\max\{d(x,y), d(y,x)\}]$$

where $\tau > 0$ and $F \in \Gamma$. Then T has a unique fixed point.

Theorem 3.12. *Let (X,d) be a complete g. a. m. s and let $T : X \to X$ be a self-mapping satisfying the following assertions*
(i) T is α-admissible mapping with respect to η;
(ii) T is α-η -F-contraction of type (B);
(iii) there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq \eta(x_0, Tx_0)$ and $\alpha(x_0, T^2x_0) \geq \eta(x_0, T^2x_0)$;
(iv) if x_n is a sequence in X such that $\alpha(x_n, x_{n+1}) \geq \eta(x_n, x_{n+1})$ with $x_n \to x$ as $n \to \infty$, then
either $\eta(Tx_n, T^2x_n) \leq \alpha(Tx_n, x)$ or $\eta(T^2x_n, T^3x_n) \leq \alpha(T^2x_n, x)$ holds for all $n \in N$.
Then T has a fixed point. Moreover, T has a unique fixed point when $\alpha(x,y) \geq \eta(x,x)$ for all $x, y \in Fix(T)$.

Proof. Let $x_0 \in X$ such that

$$\alpha(x_0, Tx_0) \geq \eta(x_0, Tx_0) \text{ and } \alpha(x_0, T^2x_0) \geq \eta(x_0, T^2x_0).$$

As in the proof of Theorem (3.10) we can conclude that

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = \lim_{n\to\infty} d(x_{n+1}, x_n) = 0, \tag{3.40}$$

and

$$\lim_{n\to\infty} d(x_n, x_{n+2}) = \lim_{n\to\infty} d(x_{n+2}, x_n) = 0. \tag{3.41}$$

As in the proof of Theorem (3.10) we can conclude that $\{x_n\}_{n\in\mathbb{N}}$ is Cauchy sequence in X.
So, from (iv), and either

$$\eta(x_{n+1}, x_{n+2}) \leq \alpha(x_{n+1}, x) \text{ or } \eta(x_{n+2}, x_{n+3}) \leq \alpha(x_{n+2}, x) \ \forall n \in \mathbb{N},$$

holds for all $n \in N$. Equivalently, there exists a subsequence x_{n_k} of x_n such that

$$\lim_{k\to\infty} [\max\{d(Tx_{n_k}, Tx), d(Tx, Tx_{n_k})\}] = 0.$$

As in the proof of Theorem (3.10) we can conclude that $x = Tx$.
Uniqueness follow similarly as in Theorem (3.10). ∎

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