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Refinement of the Jensen Inequality for Convex Functions of Several Variables with Applications

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Abstract In this manuscript we give a refinement of Jensen's inequality for convex functions of several variables associated to certain tuples. As applications we deduce refinements of Beck's inequality. At the end of the manuscript further generalization has been presented for n finite certain sequences.

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1. Introduction

The Jensen inequality for convex function is one of the most important inequality and no one can ignore the importance of this inequality in almost every field of science. Due to the great importance of this inequality, there is an extensive literature devoted to the Jensen inequality concerning different refinements, counterparts, generalizations in different manners and their applications [1–27]. Jensen's inequality has been given for different classes of functions such as $(\alpha, m)-$convex, $\alpha(x)-$convex, $Q-$class convex, $m-$convex, strongly convex and coordinate convex functions etc [28–32].

The following Jensen inequality for convex functions of several variables has been given in [33].

Theorem 1.1. *Let $I_1, I_2, \ldots, I_m$ be intervals in $\mathbb{R}$ and $y_j^i \in I_i, \zeta_j \in \mathbb{R}^+$ for $i = 1, 2, \ldots, m$, $j = 1, 2, \ldots, n$. If $\psi : I_1 \times I_2 \times \cdots \times I_m \to \mathbb{R}$ is a convex function, then*

$$\psi\left(\frac{\sum_{j=1}^n \zeta_j y_j^1}{\sum_{j=1}^n \zeta_j}, \frac{\sum_{j=1}^n \zeta_j y_j^2}{\sum_{j=1}^n \zeta_j}, \ldots, \frac{\sum_{j=1}^n \zeta_j y_j^m}{\sum_{j=1}^n \zeta_j}\right) \leq \frac{1}{\sum_{j=1}^n \zeta_j}\sum_{j=1}^n \zeta_j \psi(y_j^1, y_j^2, \ldots, y_j^m). \tag{1.1}$$

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Let $q : I \to \mathbb{R}$ be a continuous and strictly monotone function defined on the interval I, then the quasiarithmetic (q-mean) of vector $\mathbf{z} = (z_1, z_2, \ldots, z_n) \in I^n$ with positive weights $\boldsymbol{\zeta} = (\zeta_1, \zeta_2, \ldots, \zeta_n)$ is defined by

$$q(\mathbf{z}; \boldsymbol{\zeta}) = q^{-1}\left(\frac{1}{\sum_{j=1}^{n} \zeta_j} \sum_{j=1}^{n} \zeta_j q(z_j)\right). \tag{1.2}$$

Similarly, integral quasiarithmetic mean can be defined as:
Let $q : I \to \mathbb{R}$ be a continuous and strictly monotone function defined on the interval I, $g : [a, b] \to I$ and $p : [a, b] \to \mathbb{R}^+$ be integrable functions, then the quasiarithmetic mean of the function g with weight p is defined by

$$q(g; p) = q^{-1}\left(\frac{1}{\int_a^b p(\omega)d\omega} \int_a^b p(\omega)q(g(\omega))d\omega\right).$$

The following weighted version of Beck's inequality has been given in [34].

Theorem 1.2. *Let $q_i : I_i \to \mathbb{R}$ $(i = 1, 2, \ldots, m)$ be strictly monotone and $P : I_P \to \mathbb{R}$ be continuous and strictly increasing functions whose domains are intervals in $\mathbb{R}$, and $\psi : I_1 \times I_2 \times \cdots \times I_m \to I_P$ be a continuous function. Let $\mathbf{y}^i = (y_1^i, y_2^i, \ldots, y_n^i) \in I_1 \times I_2 \times \cdots \times I_n$, $i = 1, 2, \ldots, m$ and $\boldsymbol{\zeta} = (\zeta_1, \zeta_2, \ldots, \zeta_n)$ be a nonnegative n-tuple such that $\sum_{j=1}^{n} \zeta_j = 1$, then*

$$\psi\left(q_1(\mathbf{y}^1; \boldsymbol{\zeta}), q_2(\mathbf{y}^2; \boldsymbol{\zeta}), \ldots, q_m(\mathbf{y}^m; \boldsymbol{\zeta})\right) \geq P^{-1}\left(\sum_{j=1}^{n} \zeta_j P(\psi(y_j^1, y_j^2, \ldots, y_j^m))\right). \tag{1.3}$$

holds for all possible $\mathbf{y}^i (i = 1, 2, \ldots, n)$ and $\boldsymbol{\zeta}$, if and only if the function $\mathfrak{D}$ defined on $q_1(I_1) \times q_2(I_2) \times \cdots q_m(I_m)$ by

$$\mathfrak{D}(z_1, z_2, \ldots, z_n) = P(\psi(q_1^{-1}(z_1), q_2^{-1}(z_2), \ldots, q_m^{-1}(z_m))$$

is concave.

The inequality in (1.3) is reversed for all possible $\mathbf{y}^i (i = 1, 2, \ldots, m)$ and $\boldsymbol{\zeta}$, if and only if $\mathfrak{D}$ is convex.

Beck's original result (see [35, p. 249], [36, p. 300], [37], [38, p. 194]) was Theorem 1.2 for the case $m = 2$ which is stated as:

Theorem 1.3. *Let $K : I_K \to \mathbb{R}$, $L : I_L \to \mathbb{R}$ be strictly monotone and $N : I_N \to \mathbb{R}$ be continuous and strictly increasing functions whose domains are intervals in $\mathbb{R}$, and let $\psi : I_K \times I_L \to I_N$ be a continuous function. Let $\mathbf{a} = (a_1, a_2, \ldots, a_n) \in I_K^n$, $\mathbf{b} = (b_1, b_2, \ldots, b_n) \in I_L^n$ and $\boldsymbol{\zeta} = (\zeta_1, \zeta_2, \ldots, \zeta_n)$ be a nonnegative n-tuple such that $\sum_{j=1}^{n} \zeta_j = 1$. Then the following inequality holds*

$$\psi(K(\mathbf{a}; \boldsymbol{\zeta}), L(\mathbf{b}; \boldsymbol{\zeta})) \geq M(\psi(\mathbf{a}, \mathbf{b}); \boldsymbol{\zeta}), \tag{1.4}$$

$$\textit{where } \psi(\mathbf{a}, \mathbf{b}) = (\psi(a_1, b_1), \psi(a_2, b_2), \ldots, \psi(a_n, b_n)),$$

if and only if the function $H(s, t) = M(\psi(K^{-1}(s), L^{-1}(t)))$, is concave.

The inequality (1.4) holds in reverse direction if and only if H is convex.

Corollary 1.4. [38, p. 194] *If* $\psi(x,y)=x+y$ *and* $H(s,t)=M(K^{-1}(s)+L^{-1}(t))$, *and if* $E:=\frac{K'}{K''}, F:=\frac{L'}{L''}, G:=\frac{N'}{N''}$, *where all* K',L',N',K'',L'',N'' *are all positive, then* (1.4) *holds for all possible tuples* **a** *and* **b** *if and only if*

$$E(x)+F(y)\leq G(x+y). \tag{1.5}$$

Corollary 1.5. [38, p. 194] *Let* $\psi(x,y)=xy$ *and* $H(s,t)=M(K^{-1}(s)L^{-1}(t))$. *If* $A(x):=\frac{K'(x)}{K'(x)+xK''(x)}$, $B(x):=\frac{L'(x)}{L'(x)+xL''(x)}$, $C(x):=\frac{M'(x)}{M'(x)+xN''(x)}$, *and if the functions* K',L',N',A,B,C *are all positive, then* (1.4) *holds for all possible tuples* **a** *and* **b** *if and only if*

$$A(x)+B(y)\leq C(xy). \tag{1.6}$$

This paper is organized as: First of all we prove a refinement of discrete Jensen's inequality for convex functions of several variables connected to two certain tuples. As a application, we deduce refinement of the weighted generalized version of Beck's inequality. In particular, we discuss refinements of Beck's inequality. Also, we present integral version of the related results. At the end of the paper, we establish further generalization of Jensen's inequality related to n certain tuples.

2. Main Results

We begin to present refinement of Jensen's inequality for convex functions of several variables.

Theorem 2.1. *Let* $I_1,I_2,\ldots,I_m$ *be intervals in* $\mathbb{R}$, $\psi:I_1\times I_2\times\cdots\times I_m\to\mathbb{R}$ *be a convex function and* $y_j^i\in I_i,\zeta_j,\eta_j,\theta_j\in\mathbb{R}^+$ *for* $i=1,2,\ldots,m, j=1,2,\ldots,n$. *If* $\eta_j+\theta_j=1$ *for all* $j\in\{1,2,\ldots,n\}$, *then*

$$\begin{aligned}
\psi\left(\frac{\sum_{j=1}^n\zeta_jy_j^1}{\sum_{j=1}^n\zeta_j},\frac{\sum_{j=1}^n\zeta_jy_j^2}{\sum_{j=1}^n\zeta_j},\ldots,\frac{\sum_{j=1}^n\zeta_jy_j^m}{\sum_{j=1}^n\zeta_j}\right)
&\leq \frac{\sum_{j=1}^n\zeta_j\eta_j}{\sum_{j=1}^n\zeta_j}\psi\left(\frac{\sum_{j=1}^n\zeta_j\eta_jy_j^1}{\sum_{j=1}^n\zeta_j\eta_j},\ldots,\frac{\sum_{j=1}^n\zeta_j\eta_jy_j^m}{\sum_{j=1}^n\zeta_j\eta_j}\right)\\
&\quad+\frac{\sum_{j=1}^n\zeta_j\theta_j}{\sum_{j=1}^n\zeta_j}\psi\left(\frac{\sum_{j=1}^n\zeta_j\theta_jy_j^1}{\sum_{j=1}^n\zeta_j\theta_j},\ldots,\frac{\sum_{j=1}^n\zeta_j\theta_jy_j^m}{\sum_{j=1}^n\zeta_j\theta_j}\right)\\
&\leq \frac{1}{\sum_{j=1}^n\zeta_j}\sum_{j=1}^n\zeta_j\psi(y_j^1,y_j^2,\ldots,y_j^m).
\end{aligned}\tag{2.1}$$

If the function ψ *is concave then the reverse inequalities hold in* (2.1).

Proof. Since $\eta_j + \theta_j = 1$ for all $j \in \{1, 2, \ldots, n\}$, therefore we have

$$
\begin{aligned}
&\psi\left(\frac{\sum_{j=1}^n \zeta_j y_j^1}{\sum_{j=1}^n \zeta_j}, \frac{\sum_{j=1}^n \zeta_j y_j^2}{\sum_{j=1}^n \zeta_j}, \ldots, \frac{\sum_{j=1}^n \zeta_j y_j^m}{\sum_{j=1}^n \zeta_j}\right) \\
&\qquad = \psi\left(\frac{\sum_{j=1}^n \zeta_j \eta_j y_j^1}{\sum_{j=1}^n \zeta_j} + \frac{\sum_{j=1}^n \zeta_j \theta_j y_j^1}{\sum_{j=1}^n \zeta_j}, \ldots, \frac{\sum_{j=1}^n \zeta_j \eta_j y_j^m}{\sum_{j=1}^n \zeta_j} + \frac{\sum_{j=1}^n \zeta_j \theta_j y_j^m}{\sum_{j=1}^n \zeta_j}\right) \\
&\qquad = \psi\left(\frac{\sum_{j=1}^n \zeta_j \eta_j}{\sum_{j=1}^n \zeta_j} \frac{\sum_{j=1}^n \zeta_j \eta_j y_j^1}{\sum_{j=1}^n \zeta_j \eta_j} + \frac{\sum_{j=1}^n \zeta_j \theta_j}{\sum_{j=1}^n \zeta_j} \frac{\sum_{j=1}^n \zeta_j \theta_j y_j^1}{\sum_{j=1}^n \zeta_j \theta_j}, \ldots, \right. \\
&\qquad\qquad \left. \frac{\sum_{j=1}^n \zeta_j \eta_j}{\sum_{j=1}^n \zeta_j} \frac{\sum_{j=1}^n \zeta_j \eta_j y_j^m}{\sum_{j=1}^n \zeta_j \eta_j} + \frac{\sum_{j=1}^n \zeta_j \theta_j}{\sum_{j=1}^n \zeta_j} \frac{\sum_{j=1}^n \zeta_j \theta_j y_j^m}{\sum_{j=1}^n \zeta_j \theta_j}\right) \\
&\qquad \leq \frac{\sum_{j=1}^n \zeta_j \eta_j}{\sum_{j=1}^n \zeta_j} \psi\left(\frac{\sum_{j=1}^n \zeta_j \eta_j y_j^1}{\sum_{j=1}^n \zeta_j \eta_j}, \ldots, \frac{\sum_{j=1}^n \zeta_j \eta_j y_j^m}{\sum_{j=1}^n \zeta_j \eta_j}\right) \\
&\qquad\quad + \frac{\sum_{j=1}^n \zeta_j \theta_j}{\sum_{j=1}^n \zeta_j} \psi\left(\frac{\sum_{j=1}^n \zeta_j \theta_j y_j^1}{\sum_{j=1}^n \zeta_j \theta_j}, \ldots, \frac{\sum_{j=1}^n \zeta_j \theta_j y_j^m}{\sum_{j=1}^n \zeta_j \theta_j}\right) \\
&\qquad \leq \frac{1}{\sum_{j=1}^n \zeta_j} \sum_{j=1}^n \zeta_j \psi(y_j^1, y_j^2, \ldots, y_j^m).
\end{aligned}
$$

The first inequality has been obtained by using (1.1) for the case $n = 2$, while the second inequality has been obtained by using (1.1) on both the terms. ∎

The integral version of the above theorem can be stated as:

Theorem 2.2. *Let $I_1, I_2, \ldots, I_m$ be intervals in $\mathbb{R}$, $\psi : I_1 \times I_2 \times \cdots \times I_m \to \mathbb{R}$ be a convex function. Let $u, v, p, g_i : [a, b] \to \mathbb{R}$ be integrable functions such that $g_i(\omega) \in I_i, u(\omega), v(\omega), p(\omega) \in \mathbb{R}^+$ for all $\omega \in [a, b], i = 1, 2, \ldots, m$, and $v(\omega) + u(\omega) = 1$, $P = \int_a^b p(\omega)d\omega$. Then*

$$
\begin{aligned}
&\psi\left(\frac{1}{P}\int_a^b p(\omega)g_1(\omega)d\omega, \ldots, \frac{1}{P}\int_a^b p(\omega)g_m(\omega)d\omega\right) \\
&\leq \frac{1}{P}\int_a^b u(\omega)p(\omega)d\omega\psi\left(\frac{\int_a^b p(\omega)u(\omega)g_1(\omega)d\omega}{\int_a^b p(\omega)u(\omega)d\omega}, \ldots, \frac{\int_a^b p(\omega)u(\omega)g_m(\omega)d\omega}{\int_a^b p(\omega)u(\omega)d\omega}\right) \\
&+ \frac{1}{P}\int_a^b u(\omega)p(\omega)d\omega\psi\left(\frac{\int_a^b p(\omega)v(\omega)g_1(\omega)d\omega}{\int_a^b p(\omega)v(\omega)d\omega}, \ldots, \frac{\int_a^b p(\omega)v(\omega)g_m(\omega)d\omega}{\int_a^b p(\omega)v(\omega)d\omega}\right) \\
&\quad \leq \frac{1}{P}\int_a^b p(\omega)\psi(g_1(\omega), g_2(\omega), \ldots, g_m(\omega))d\omega.
\end{aligned}
\tag{2.2}
$$

If the function ψ is concave then the reverse inequalities hold in (2.2).

Remark 2.3. Analogously, related refinement can be given for Jensen's inequality (2.8) as given in [33].

In the following theorem we present refinement of the inequality (1.3).

Theorem 2.4. *Let $q_i : I_i \to \mathbb{R}$ $(i = 1, 2, \ldots, m)$ be strictly monotone and $P : I_P \to \mathbb{R}$ be continuous and strictly increasing functions whose domains are intervals in $\mathbb{R}$, and $\psi : I_1 \times I_2 \times \cdots \times I_m \to I_P$ be a continuous function. Let $\mathbf{y}^i = (y_1^i, y_2^i, \ldots, y_n^i) \in I_1 \times I_2 \times \cdots \times I_n$, $i = 1, 2, \ldots, m$. If $\zeta_j, \eta_j, \theta_j \in \mathbb{R}^+$ such that $\eta_j + \theta_j = 1$ for $j = 1, 2, \ldots, n$, then*

$$\begin{aligned}
\psi\Big(q_1(\mathbf{y}^1;\boldsymbol{\zeta})&\;, q_2(\mathbf{y}^2;\boldsymbol{\zeta}), \ldots, q_m(\mathbf{y}^m;\boldsymbol{\zeta})\Big) \\
&\geq P^{-1}\left[\frac{\sum_{j=1}^n \zeta_j\eta_j}{\sum_{j=1}^n \zeta_j} P\Big(\psi\Big(q_1(\mathbf{y}^1;\boldsymbol{\zeta}.\boldsymbol{\eta}), \cdots, q_m(\mathbf{y}^m;\boldsymbol{\zeta}.\boldsymbol{\eta})\Big)\Big)\right. \\
&\qquad \left.+\frac{\sum_{j=1}^n \zeta_j\theta_j}{\sum_{j=1}^n \zeta_j} P\Big(\psi\Big(q_1(\mathbf{y}^1;\boldsymbol{\zeta}.\boldsymbol{\theta}), \cdots, q_m(\mathbf{y}^m;\boldsymbol{\zeta}.\boldsymbol{\theta})\Big)\Big)\right] \\
&\geq P^{-1}\left(\frac{1}{\sum_{j=1}^n \zeta_j}\sum_{j=1}^n \zeta_j P(\psi(y_j^1, y_j^2, \ldots, y_j^m))\right).
\end{aligned} \tag{2.3}$$

if and only if the function $\mathfrak{D}$ defined on $q_1(I_1) \times q_2(I_2) \times \cdots q_m(I_m)$ by

$$\mathfrak{D}(z_1, z_2, \ldots, z_m) = P(\psi(q_1^{-1}(z_1), q_2^{-1}(z_2), \ldots, q_m^{-1}(z_m)) \tag{2.4}$$

is concave.

The inequalities in (2.3) hold in reverse direction for all possible $\mathbf{y}^i (i = 1, 2, \ldots, n)$ and $\boldsymbol{\zeta}$, if and only if $\mathfrak{D}$ is convex.

Proof. Replace ψ by $\mathfrak{D}$ and y_j^i by $q_j(y_j^i)$ $(i = 1, 2, \ldots, m, j = 1, 2, \ldots, n)$ and then applying the increasing function P^{-1} in the reverse inequality in (2.1), we obtain (2.3). ■

The integral version of the above theorem can be stated as:

Theorem 2.5. *Let $q_i : I_i \to \mathbb{R}$ $(i = 1, 2, \ldots, m)$ be strictly monotone and $T : I_T \to \mathbb{R}$ be continuous and strictly increasing functions whose domains are intervals in $\mathbb{R}$, and $\psi : I_1 \times I_2 \times \cdots \times I_m \to I_T$ be a continuous function. Let $u, v, p, g_i : [a, b] \to \mathbb{R}$ be integrable functions such that $g_i(\omega) \in I_i, u(\omega), v(\omega), p(\omega) \in \mathbb{R}^+$ for all $\omega \in [a, b], i = 1, 2, \ldots, m$, and $v(\omega) + u(\omega) = 1$, $P = \int_a^b p(\omega)d\omega$. Then*

$$\begin{aligned}
\psi\Big(q_1(g_1;p)&\;, q_2(g_2;p), \ldots, q_m(g_m;p)\Big) \\
&\geq T^{-1}\left[\frac{1}{P}\int_a^b u(\omega)p(\omega)d\omega T\Big(\psi\Big(q_1(g_1;p.u), \cdots, q_m(g_m;p.u)\Big)\Big)\right. \\
&\qquad \left.+\frac{1}{P}\int_a^b v(\omega)p(\omega)d\omega T\Big(\psi\Big(q_1(g_1;p.v), \cdots, q_m(g_m;p.v)\Big)\Big)\right] \\
&\geq T^{-1}\left(\frac{1}{P}\int_a^b p(\omega)T(\psi(g_1(\omega), g_2(\omega), \ldots, g_m(\omega))d\omega\right).
\end{aligned} \tag{2.5}$$

if and only if the function $\mathfrak{D}$ defined on $q_1(I_1) \times q_2(I_2) \times \cdots q_m(I_m)$ by

$$\mathfrak{D}(z_1, z_2, \ldots, z_m) = T(\psi(q_1^{-1}(z_1), q_2^{-1}(z_2), \ldots, q_m^{-1}(z_m))$$

is concave.

The inequalities in (2.5) hold in reverse direction for all possible $\mathbf{y}^i (i = 1, 2, \dots, n)$ *and* $\boldsymbol{\zeta}$*, if and only if* $\mathfrak{D}$ *is convex.*

As a consequence of the above theorem for the case $m = 2$, the following refinement of Beck's inequality holds:

Corollary 2.6. *Let all the assumptions of Theorem 1.3 hold. If* $\eta_j, \theta_j \in \mathbb{R}^+$ *such that* $\eta_j + \theta_j = 1$ *for* $j = 1, 2, \dots, n$*, then*

$$\psi(K(\mathbf{a};\boldsymbol{\zeta}), L(\mathbf{b};\boldsymbol{\zeta})) \geq M^{-1}\left[\frac{\sum_{j=1}^n \zeta_j\eta_j}{\sum_{j=1}^n \zeta_j} M\left(\psi\left(K(\mathbf{a};\boldsymbol{\zeta}.\boldsymbol{\eta}), L(\mathbf{b};\boldsymbol{\zeta}.\boldsymbol{\eta})\right)\right)\right.$$
$$\left.+\frac{\sum_{j=1}^n \zeta_j\theta_j}{\sum_{j=1}^n \zeta_j} M\left(\psi\left(K(\mathbf{a};\boldsymbol{\zeta}.\boldsymbol{\theta}), (L(\mathbf{a};\boldsymbol{\zeta}.\boldsymbol{\theta})\right)\right)\right] \geq M(\psi(\mathbf{a},\mathbf{b});\boldsymbol{\zeta}). \tag{2.6}$$

if and only if the function $H(s,t) = M(\psi(K^{-1}(s), L^{-1}(t)))$*, is concave.*

The inequality (2.6) holds in reverse direction if and only if H *is convex.*

Corollary 2.7. *Let* K, L, M *be twice continuously differentiable and strictly monotone functions such that* $K', L', M', K'', L'', M''$ *are all positive. If* $\eta_j, \theta_j \in \mathbb{R}^+$ *such that* $\eta_j + \theta_j = 1$ *for* $j = 1, 2, \dots, n$*, then*

$$K(\mathbf{a};\boldsymbol{\zeta}) + L(\mathbf{b};\boldsymbol{\zeta}) \geq M^{-1}\left[\frac{\sum_{j=1}^n \zeta_j\eta_j}{\sum_{j=1}^n \zeta_j} M\left(K(\mathbf{a};\boldsymbol{\zeta}.\boldsymbol{\eta}) + L(\mathbf{b};\boldsymbol{\zeta}.\boldsymbol{\eta})\right)\right.$$
$$\left.+\frac{\sum_{j=1}^n \zeta_j\theta_j}{\sum_{j=1}^n \zeta_j} M\left(K(\mathbf{a};\boldsymbol{\zeta}.\boldsymbol{\theta}) + L(\mathbf{a};\boldsymbol{\zeta}.\boldsymbol{\theta}))\right)\right] \geq M(\mathbf{a}+\mathbf{b};\boldsymbol{\zeta}). \tag{2.7}$$

holds for all possible tuples $\mathbf{a}, \mathbf{b}$ *and positive tuple* $\boldsymbol{\zeta}$ *if and only if*

$$E(x) + F(y) \leq G(x+y), \tag{2.8}$$

where $E := \frac{K'}{K''}, F := \frac{L'}{L''}, G := \frac{M'}{M''}$.

Proof. Let $H(s,t) = M(K^{-1}(s) + L^{-1}(t))$. We prove that the function H is concave. Since H is twice continuously differentiable function, therefore for the concavity of H, we show that

$$a_1^2\frac{\partial^2 H}{\partial s^2} + 2a_1a_2\frac{\partial^2 H}{\partial s\partial t} + a_2^2\frac{\partial^2 H}{\partial t^2} \leq 0, \text{ for all } a_1, a_2 \in \mathbb{R}. \tag{2.9}$$

But by taking partial derivatives of H of order 2 and using (2.8), we obtain (2.9).

Finally, using H and $\psi(x,y) = x + y$ in (2.6), we deduce (2.7). ∎

In the following corollary we present refinement of the inequality given in Corollary 1.5. The idea of the proof is similar to the proof of Corollary 1.5.

Corollary 2.8. *Let* K, L, M *be twice continuously differentiable and strictly monotone functions and let* $A(x) := \frac{K'(x)}{K'(x)+xK''(x)}$, $B(x) := \frac{L'(x)}{L'(x)+xL''(x)}, C(x) := \frac{M'(x)}{M'(x)+xN''(x)}$.

Also, assume that the functions K', L', N', A, B, C are all positive. If $\eta_j, \theta_j \in \mathbb{R}^+$ such that $\eta_j + \theta_j = 1$ for $j = 1, 2, \dots, n$, then

$$K(\mathbf{a};\boldsymbol{\zeta})L(\mathbf{b};\boldsymbol{\zeta}) \geq M^{-1}\left[\frac{\sum_{j=1}^{n}\zeta_j\eta_j}{\sum_{j=1}^{n}\zeta_j}M\Big(K(\mathbf{a};\boldsymbol{\zeta}.\boldsymbol{\eta})L(\mathbf{b};\boldsymbol{\zeta}.\boldsymbol{\eta})\Big) + \frac{\sum_{j=1}^{n}\zeta_j\theta_j}{\sum_{j=1}^{n}\zeta_j}M\Big(K(\mathbf{a};\boldsymbol{\zeta}.\boldsymbol{\theta})(L(\mathbf{a};\boldsymbol{\zeta}.\boldsymbol{\theta}))\Big)\right] \geq M(\mathbf{a}.\mathbf{b};\boldsymbol{\zeta}). \tag{2.10}$$

holds for all possible tuples $\mathbf{a}, \mathbf{b}$ and positive tuple $\boldsymbol{\zeta}$ if and only if

$$A(x) + B(y) \leq C(xy). \tag{2.11}$$

We give a refinement of the Minkowski's inequality.

Corollary 2.9. *Let I be an interval in $\mathbb{R}$, $\boldsymbol{y}_j = (y_j^1, y_j^2, \dots, y_j^m) \in I^m$, $\zeta_j, \eta_j, \theta_j \in \mathbb{R}^+(j = 1, 2, \dots, n)$ such that $\eta_j + \theta_j = 1$ for all $j \in \{1, 2, \dots, n\}$, and $M : I \to \mathbb{R}$ be a continuous and strictly monotone function. Consider the quasi-arithmetic mean function $M_n : I^n \to \mathbb{R}$ defined by*

$$M_n(\boldsymbol{y};\boldsymbol{\zeta}) = M^{-1}\left(\frac{1}{\sum_{j=1}^{n}\zeta_j}\sum_{j=1}^{n}\zeta_j M(y_j)\right)$$

is convex, then

$$M_m\left(\frac{1}{n}\sum_{j=1}^{n}\boldsymbol{y}_j;\boldsymbol{\zeta}\right) \leq \frac{1}{n}\sum_{j=1}^{n}\eta_j M_m\left(\boldsymbol{y}_\eta;\boldsymbol{\zeta}\right) + \frac{1}{n}\sum_{j=1}^{n}\theta_j M_m\left(\boldsymbol{y}_\theta;\boldsymbol{\zeta}\right) \leq \frac{1}{n}\sum_{j=1}^{n} M_m(\boldsymbol{y}_j;\boldsymbol{\zeta}), \tag{2.12}$$

where $\boldsymbol{y}_\eta = \left(\frac{\sum_{j=1}^{n}\eta_j y_j^1}{\sum_{j=1}^{n}\eta_j}, \dots, \frac{\sum_{j=1}^{n}\eta_j y_j^m}{\sum_{j=1}^{n}\eta_j}\right)$, $\boldsymbol{y}_\theta = \left(\frac{\sum_{j=1}^{n}\theta_j y_j^1}{\sum_{j=1}^{n}\theta_j}, \dots, \frac{\sum_{j=1}^{n}\theta_j y_j^m}{\sum_{j=1}^{n}\theta_j}\right)$.

Proof. The proof follows by using Theorem 2.1 for $\zeta_j = 1$ and then taking the function $M_m(;\boldsymbol{\zeta})$ instead of ψ. ■

Remark 2.10. Analogously as above we can give the integral version of Corollaries 2.6-2.9.

3. Further Generalization

In the following theorem, we present further refinement of the Jensen inequality related to tn sequences.

Theorem 3.1. *Let $I_1, I_2, \dots, I_m$ be intervals in $\mathbb{R}$, $\psi : I_1 \times I_2 \times \cdots \times I_m \to \mathbb{R}$ be a convex function and $y_j^i \in I_i$, $\zeta_j, \theta_j^l \in \mathbb{R}^+$ $(i = 1, 2, \dots, m, j = 1, 2, \dots, n, l = 1, 2, \dots, t)$ such that $\sum_{l=1}^{t}\theta_j^l = 1$ for each $j \in \{1, 2, \dots, n\}$, $\bar{\zeta} = \sum_{j=1}^{n}\zeta_j$. Assume that L_1 and L_2 are non* *empty disjoint subsets of* $\{1, 2, ..., m\}$ *such that* $L_1 \cup L_2 = \{1, 2, ..., t\}$. *Then*

$$\psi\left(\frac{\sum_{j=1}^{n}\zeta_j y_j^1}{\bar{\zeta}}, \frac{\sum_{j=1}^{n}\zeta_j y_j^2}{\bar{\zeta}}, \dots, \frac{\sum_{j=1}^{n}\zeta_j y_j^m}{\bar{\zeta}}\right)$$
$$\leq \frac{\sum_{j=1}^{n}\sum_{l\in L_1}\zeta_j\theta_j^l}{\bar{\zeta}}\psi\left(\frac{\sum_{j=1}^{n}\sum_{l\in L_1}\zeta_j\theta_j^l y_j^1}{\sum_{j=1}^{n}\sum_{l\in L_1}\zeta_j\theta_j^l}, \dots, \frac{\sum_{j=1}^{n}\sum_{l\in L_1}\zeta_j\theta_j^l y_j^m}{\sum_{j=1}^{n}\sum_{l\in L_1}\zeta_j\theta_j^l}\right)$$
$$+\frac{\sum_{j=1}^{n}\sum_{l\in L_2}\zeta_j\theta_j^l}{\bar{\zeta}}\psi\left(\frac{\sum_{j=1}^{n}\sum_{l\in L_2}\zeta_j\theta_j^l y_j^1}{\sum_{j=1}^{n}\sum_{l\in L_2}\zeta_j\theta_j^l}, \dots, \frac{\sum_{j=1}^{n}\sum_{l\in L_2}\zeta_j\theta_j^l y_j^m}{\sum_{j=1}^{n}\sum_{l\in L_2}\zeta_j\theta_j^l}\right)$$
$$\leq \frac{1}{\bar{\zeta}}\sum_{j=1}^{n}\zeta_j\psi(y_j^1, y_j^2, \dots, y_j^m). \tag{3.1}$$

If the function ψ *is concave then the reverse inequalities hold in* (3.1).

Proof. Since $\sum_{l=1}^{t}\theta_j^l = 1$ for each $j \in \{1, 2, \dots, n\}$, therefore we may write

$$\frac{1}{\bar{\zeta}}\sum_{j=1}^{n}\zeta_j\psi(y_j^1, y_j^2, \dots, y_j^m) = \frac{1}{\bar{\zeta}}\sum_{j=1}^{n}\sum_{l\in L_1}\theta_j^l\zeta_j\psi(y_j^1, y_j^2, \dots, y_j^m)$$
$$+\frac{1}{\bar{\zeta}}\sum_{j=1}^{n}\sum_{l\in L_2}\theta_j^l\zeta_j\psi(y_j^1, y_j^2, \dots, y_j^m). \tag{3.2}$$

Applying Jensen's inequality (1.1) on both terms on the right hand side of (3.2) we obtain

$$\frac{1}{\bar{\zeta}}\sum_{j=1}^{n}\zeta_j\psi(y_j^1, y_j^2, \dots, y_j^m)$$
$$\geq \frac{\sum_{j=1}^{n}\sum_{l\in L_1}\zeta_j\theta_j^l}{\bar{\zeta}}\psi\left(\frac{\sum_{j=1}^{n}\sum_{l\in L_1}\zeta_j\theta_j^l y_j^1}{\sum_{j=1}^{n}\sum_{l\in L_1}\zeta_j\theta_j^l}, \dots, \frac{\sum_{j=1}^{n}\sum_{l\in L_1}\zeta_j\theta_j^l y_j^m}{\sum_{j=1}^{n}\sum_{l\in L_1}\zeta_j\theta_j^l}\right)$$
$$+\frac{\sum_{j=1}^{n}\sum_{l\in L_2}\zeta_j\theta_j^l}{\bar{\zeta}}\psi\left(\frac{\sum_{j=1}^{n}\sum_{l\in L_2}\zeta_j\theta_j^l y_j^1}{\sum_{j=1}^{n}\sum_{l\in L_2}\zeta_j\theta_j^l}, \dots, \frac{\sum_{j=1}^{n}\sum_{l\in L_2}\zeta_j\theta_j^l y_j^m}{\sum_{j=1}^{n}\sum_{l\in L_2}\zeta_j\theta_j^l}\right)$$
$$\geq \psi\left[\frac{\sum_{j=1}^{n}\sum_{l\in L_1}\zeta_j\theta_j^l}{\bar{\zeta}}\left(\frac{\sum_{j=1}^{n}\sum_{l\in L_1}\zeta_j\theta_j^l y_j^1}{\sum_{j=1}^{n}\sum_{l\in L_1}\zeta_j\theta_j^l}, \dots, \frac{\sum_{j=1}^{n}\sum_{l\in L_1}\zeta_j\theta_j^l y_j^m}{\sum_{j=1}^{n}\sum_{l\in L_1}\zeta_j\theta_j^l}\right)\right.$$
$$\left.+\frac{\sum_{j=1}^{n}\sum_{l\in L_2}\zeta_j\theta_j^l}{\bar{\zeta}}\left(\frac{\sum_{j=1}^{n}\sum_{l\in L_2}\zeta_j\theta_j^l y_j^1}{\sum_{j=1}^{n}\sum_{l\in L_2}\zeta_j\theta_j^l}, \dots, \frac{\sum_{j=1}^{n}\sum_{l\in L_2}\zeta_j\theta_j^l y_j^m}{\sum_{j=1}^{n}\sum_{l\in L_2}\zeta_j\theta_j^l}\right)\right]$$

(By the convexity of ψ)

$$= \psi\left(\frac{\sum_{j=1}^{n}\zeta_j y_j^1}{\bar{\zeta}}, \frac{\sum_{j=1}^{n}\zeta_j y_j^2}{\bar{\zeta}}, \dots, \frac{\sum_{j=1}^{n}\zeta_j y_j^m}{\bar{\zeta}}\right). \tag{3.3}$$

■

The integral version of the above theorem can be states as:

Theorem 3.2. *Let* $I_1, I_2, \dots, I_m$ *be intervals in* $\mathbb{R}$, $\psi : I_1 \times I_2 \times \cdots \times I_m \to \mathbb{R}$ *be a convex function. Let* $p, g_i, u_l \in L[a, b]$ *such that* $g_i(\omega) \in I_i, p(\omega), u_l(\omega) \in \mathbb{R}^+$ *for all* $\omega \in [a, b]$

$(i = 1, 2, ..., n, l = 1, 2, ..., t)$ and $\sum_{l=1}^{t} u_l(\omega) = 1$, $P = \int_a^b p(\omega)d\omega$. Assume that L_1 and L_2 are non empty disjoint subsets of $\{1, 2, ..., t\}$ such that $L_1 \cup L_2 = \{1, 2, ..., t\}$. Then

$$\begin{aligned}
&\psi\left(\frac{1}{P}\int_a^b p(\omega)g_1(\omega)d\omega, \dots, \frac{1}{P}\int_a^b p(\omega)g_m(\omega)d\omega\right) \\
&\le \frac{1}{P}\int_a^b \sum_{l\in L_1} u_l(\omega)p(\omega)d\omega\psi\left(\frac{\int_a^b \sum_{l\in L_1} u_l(\omega)p(\omega)g_1(\omega)d\omega}{\int_a^b \sum_{l\in L_1} u_l(\omega)p(\omega)d\omega}, \dots, \frac{\int_a^b \sum_{l\in L_1} u_l(\omega)p(\omega)g_n(\omega)d\omega}{\int_a^b \sum_{l\in L_1} u_l(\omega)p(\omega)d\omega}\right) \\
&+ \frac{1}{P}\int_a^b \sum_{l\in L_2} u_l(\omega)p(\omega)d\omega\psi\left(\frac{\int_a^b \sum_{l\in L_2} u_l(\omega)p(\omega)g_1(\omega)d\omega}{\int_a^b \sum_{l\in L_2} u_l(\omega)p(\omega)d\omega}, \dots, \frac{\int_a^b \sum_{l\in L_1} u_l(\omega)p(\omega)g_n(\omega)d\omega}{\int_a^b \sum_{l\in L_1} u_l(\omega)p(\omega)d\omega}\right) \\
&\le \frac{1}{P}\int_a^b p(\omega)\psi(g_1(\omega), g_2(\omega), \dots, g_m(\omega))d\omega. \qquad (3.4)
\end{aligned}$$

If the function ψ is concave then the reverse inequalities hold in (3.4).

Remark 3.3. All the results presented in this paper may also be generalized using Theorem 3.1.

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