Thai Journal of Mathematics
Volume 20 Number 3 (2022)
Pages 1119–1136

http://thaijmath.in.cmu.ac.th

ISSN 1686-0209

Completion of C^*-algebra-valued Metric Spaces

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Abstract The concept of a C^*-algebra-valued metric space was introduced in 2014. It is a generalization of a metric space replacing the set of real numbers by a C^*-algebra. In this paper, we show that C^*-algebra-valued metric spaces are cone metric spaces in some point of view which is useful to extend results of the cone case to C^*-algebra-valued metric spaces. Then the completion theorem of C^*-algebra-valued metric spaces is obtained. Moreover, the completion theorem of C^*-algebra-valued normed spaces is verified and the connection with Hilbert C^*-modules, generalized inner product spaces, is also provided.

MSC: 54E50; 46B99; 46L07

Keywords: C^*-algebra-valued metric space; cone metric space; normed space; inner product space; completion

Submission date: 29.11.2021 / Acceptance date: 20.05.2022

1. Introduction

A metric space is one of attractive objects in mathematics which plays an important role in various branches of mathematics. It is a nonempty set X together with a distance function $d : X \times X \to \mathbb{R}$, which is often called a metric on X. Plenty of research papers study various kinds of spaces generalized from the definition of a metric space in different directions. Some authors remove or change initial properties of a metric space while others change the values of the distance function to be in generalized sets of real or complex numbers, such as, a Banach space or a C^*-algebra which can be seen in [1] and [2], respectively.

The concept of a C^*-algebra-valued metric space was first introduced in 2014 by Z. Ma and others. For this space the distance function was replaced by a function valued in a C^*-algebra $\mathbb{A}$. If we consider the set of all positive elements $\mathbb{A}_+$ of $\mathbb{A}$ as a cone of $\mathbb{A}$. A C^*-algebra-valued metric space is, in fact, a cone metric space which was introduced in 2004 by L. G. Huang and others, see more details about a cone metric space in [1]. Recently, there are many authors whose study area related to C^*-algebra-valued metric (like) spaces especially in mathematical analysis, see [3–9] for examples.

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The main purpose of this research is to study the completion for C^*-algebra-valued metric spaces and a C^*-algebra-valued normed spaces. We verify some facts and use them to extend the results from others in [10]. Then we discuss relationships between C^*-algebra-valued metric spaces and Hilbert C^*-modules, generalized inner product spaces whose scalar fields are replaced by some C^*-algebras.

The rest of the paper is organized as follows. In section 2 we derive the important definitions and theorems used to obtain our results. In section 3 We discuss on C^*-algebra-valued metric and normed spaces and the relation to cone metric spaces. In section 4 the connection to Hilbert C^*-modules is provided.

2. Preliminaries

This section provides a brief review of basic knowledge used in this research which can be found in [1, 2, 10–12]. We start with the concept of C^*-algebras and some necessary related properties. A C^*-algebra $\mathbb{A}$ is a *-algebra with a complete submultiplicative norm $\|\cdot\|_{\mathbb{A}}$ such that $\|a^*a\|_{\mathbb{A}} = \|a\|_{\mathbb{A}}^2$ for every $a \in \mathbb{A}$. If $\mathbb{A}$ admits a unit I ($aI = Ia = a$ for every $a \in \mathbb{A}$) such that $\|I\|_{\mathbb{A}} = 1$, we call $\mathbb{A}$ a *unital C^*-algebra*. It is known that not all C^*-algebras are unital. However, we can embed them as C^*-subalgebras in another unital C^*-algebras which are called the unitizations of C^*-algebras. We denote by $\widetilde{\mathbb{A}}$ the unitization of $\mathbb{A}$.

We say that $a \in \mathbb{A}$ is *invertible* if there is $b \in \mathbb{A}$ such that $ab = I = ba$. We denote by $\mathrm{Inv}(\mathbb{A})$ the set of all invertible elements of $\mathbb{A}$. The *spectrum* of a is the set

$$\sigma(a) = \sigma_{\mathbb{A}}(a) = \{\lambda \in \mathbb{C} : \lambda I - a \notin \mathrm{Inv}(A)\}.$$

If $\mathbb{A}$ is nonunital, we define $\sigma_{\mathbb{A}}(a) = \sigma_{\widetilde{\mathbb{A}}}(a)$. Let $\mathbb{A}_h = \{a\mathbb{A} : a = a^*\}$, the set of all *hermitian* elements of $\mathbb{A}$. An element $a \in \mathbb{A}_h$ with $\sigma(a) \subseteq [0, +\infty)$ is called *positive* and the set of all positive elements of $\mathbb{A}$ is denoted by $\mathbb{A}_+$. Now $\mathbb{A}_h$ becomes a partially ordered set by defining $a \leq b$ to mean $b - a \in \mathbb{A}_+$. It is obvious that $0_{\mathbb{A}} \leq a$ precisely for $a \in \mathbb{A}_+$ where $0_{\mathbb{A}}$ is the zero in $\mathbb{A}$. Thus, we may write $0_{\mathbb{A}} \leq a$ to indicate that a is positive.

Proposition 2.1. *Let $\mathbb{A}$ be a C^*-algebra. Then for each $x \in \mathbb{A}$ there is a unique pair of hermitian elements $a, b \in \mathbb{A}$ such that $x = a + bi$. More precisely, $a = \frac{1}{2}(x + x^*)$ and $b = \frac{1}{2i}(x - x^*)$.*

Theorem 2.2. *Let a be a positive element of a C^*-algebra $\mathbb{A}$. Then there is a unique $b \in \mathbb{A}_+$ such that $b^2 = a$.*

By the previous theorem we can define the square root of the positive element a to be the element b, we denote it by $a^{1/2}$. A brief review of some necessary properties for positive elements of a C^*-algebra is provided below, see more details in [11].

Proposition 2.3. *The sum of two positive elements in a C^*-algebra are positive.*

Theorem 2.4. *Let $\mathbb{A}$ be a C^*-algebra. The the following properties are satisfied.*

(1) *Suppose that $\mathbb{A}$ is unital and $a \in \mathbb{A}$ is hermitian. If $\|a - \alpha I\|_{\mathbb{A}} \leq \alpha$ for some $\alpha \in \mathbb{R}$, then a is positive. In the reverse direction, for every $\alpha \in \mathbb{R}$, if $\|a\|_{\mathbb{A}} \leq \alpha$ and a is positive, then $\|a - \alpha I\|_{\mathbb{A}} \leq \alpha$.*
(2) *For every $a, b, c \in \mathbb{A}_h$, $a \leq b$ implies $a + c \leq b + c$.*
(3) *For every real numbers $\alpha, \beta \geq 0$ and every $a, b \in \mathbb{A}_+$, $\alpha a + \beta b \in \mathbb{A}_+$.*

(4) $A_+ = \{a^*a : a \in \mathbb{A}\}$.
(5) *If* $a, b \in A_h$ *and* $c \in A$, *then* $a \leq b$ *implies* $c^*ac \leq c^*bc$.
(6) *If* $0_\mathbb{A} \leq a \leq b$, *then* $\|a\|_\mathbb{A} \leq \|b\|_\mathbb{A}$.

Proposition 2.5. *Let* $\gamma = \alpha + \beta i \in \mathbb{C}$ *and* $a \in \mathbb{A}_+$. *Then* $((\alpha^2+\beta^2)a)^{1/2} = |\gamma|a^{1/2}$.

Proof. It is obvious that $|\gamma|a^{1/2}$ is positive. Consider

$$(|\gamma|a^{1/2})^2 = |\gamma|^2(a^{1/2})^2 = (\alpha^2+\beta^2)a.$$

By Theorem 2.2 , we have $((\alpha^2+\beta^2)a)^{1/2} = |\gamma|a^{1/2}$. ■

Theorem 2.6. *Let* $a, b \in \mathbb{A}_+$. *Then* $a \leq b$ *implies* $a^{1/2} \leq b^{1/2}$.

Proposition 2.7. $\mathbb{A}_+$ *is closed in a* C^**-algebra* $\mathbb{A}$.

Proof. Let $\{x_n\}$ be a sequence in $\mathbb{A}_+$ converging to $x \in \mathbb{A}$. We first examine for the case that $\mathbb{A}$ is unital. Since $\mathbb{A}_h$ is closed in $\mathbb{A}$ and $\mathbb{A}_+ \subseteq \mathbb{A}_h$, we have $x \in \mathbb{A}_h$. Since $\{x_n\}$ is convergent, it is certainly bounded. Then there is a positive real number α such that $\|x_n\|_\mathbb{A} \leq \alpha$ for every $n \in \mathbb{N}$. We know that x_n is positive for every $n \in \mathbb{N}$. Thus, Theorem 2.4 implies that $\|x_n - \alpha I\|_\mathbb{A} \leq \alpha$ for every $n \in \mathbb{N}$. Consider

$$\|x - \alpha I\|_\mathbb{A} \leq \|x_n - x\|_\mathbb{A} + \|x_n - \alpha I\|_\mathbb{A} \leq \|x_n - x\|_\mathbb{A} + \alpha.$$

This implies that $\|x - \alpha I\|_\mathbb{A} \leq \alpha$. Since x is hermitian, again by Theorem 2.4 we have $x \in \mathbb{A}_+$. Therefore, $\mathbb{A}_+$ is closed in $\mathbb{A}$.

In case of non-unital C^*-algebra, we work on the unitization $\widetilde{\mathbb{A}}$. Now $\{(x_n, 0)\}$ is a sequence in $\widetilde{\mathbb{A}}_+$ converging to $(x, 0) \in \widetilde{\mathbb{A}}$. Now we apply the first case and obtain $(x, 0) \in \widetilde{\mathbb{A}}_+$, so $x \in \mathbb{A}_+$. Therefore $\mathbb{A}_+$ is closed in $\mathbb{A}$. ■

Next, we provide the definitions of a C^*-algebra-valued metric space, convergent sequences and Cauchy sequences in the space which are our main study.

Definition 2.8. Let X be a nonempty set and $d : X \times X \to \mathbb{A}$ be a function satisfying the following properties:

(C1) $d(x, y) \geq 0_\mathbb{A}$,
(C2) $d(x, y) = 0_\mathbb{A}$ if and only if $x = y$,
(C3) $d(x, y) = d(y, x)$,
(C4) $d(x, y) \leq d(x, z) + d(z, y)$,

for every $x, y, z \in X$. We call the function d a C^*-algebra-valued metric and call the triple $(X, \mathbb{A}, d)$ a C^*-algebra-valued metric space.

The C^*-algebra $\mathbb{A}$ in the above definition need not be unital, so our C^*-algebra-valued metric space is a generalization of that in [2]. We know that every C^*-algebra $\mathbb{A}$ can be embedded in $\widetilde{\mathbb{A}}$. Thus we can consider a C^*-algebra-valued metric space $(X, \mathbb{A}, d)$ as a C^*-algebra-valued metric space $(X, \widetilde{\mathbb{A}}, d)$ and work on $\widetilde{\mathbb{A}}$ if necessary.

The following definitions provides the conditions of convergent and Cauchy sequences in a C^*-algebra-valued metric space which are defined in [2, Definition 2.2]. We change some inequality in the definitions to correspond them with other familiar definitions that we use frequently.

Definition 2.9. Let $(X, \mathbb{A}, d)$ be a C^*-algebra-valued metric space. A sequence $\{x_n\}$ in X is said to *converge* to an element $x \in X$ (with respect to $\mathbb{A}$) if and only if for every $\varepsilon > 0$ there is a positive integer N such that for every integer $n \geq N$ we have $\|d(x_n, x)\|_{\mathbb{A}} < \varepsilon$. In this case we write $\lim_{n\to\infty} x_n = x$, and say that the sequence $\{x_n\}$ is *convergent*.

A sequence $\{x_n\}$ in X is said to be *Cauchy* (with respect to $\mathbb{A}$) if and only if for every $\varepsilon > 0$ there is a positive integer N such that for every integer $n, m \geq N$ we have $\|d(x_n, x_m)\|_{\mathbb{A}} < \varepsilon$.

We say that a C^*-algebra-valued metric space $(X, \mathbb{A}, d)$ is *complete* if every Cauchy sequence is convergent.

Next, we discuss cone metric spaces which closely related to C^*-algebra-valued metric spaces. We start with a cone of a real Banach space which was introduced in [1]. The definition is different from [12] which allows a cone to be trivial.

Definition 2.10. Let $(\mathbb{E}, \|\cdot\|_{\mathbb{E}})$ be a real Banach space. A nonempty closed subset P of $\mathbb{E}$ is called a cone if and only if it satisfies the following properties:

(P1) $P \neq \{0\}$,
(P2) For every real numbers $\alpha, \beta \geq 0$ and every $a, b \in P$, $\alpha a + \beta b \in P$,
(P3) If $x \in P$ and $-x \in P$, then $x = 0$.

Now we can define a partial order $\leq$ on $\mathbb{E}$ with respect to a cone P by $x \leq y$ to mean $y - x \in P$. We write $x < y$ to indicate that $x \leq y$ and $x \neq y$, and write $x \ll y$ if $y - x \in \text{Int}(P)$.

A cone P is said to be *normal* if and only if there exists a positive real number α such that for every $x, y \in \mathbb{E}$, $0 \leq x \leq y$ implies $\|x\|_{\mathbb{E}} \leq \alpha \|y\|_{\mathbb{E}}$. The following proposition is a consequence of Theorem 2.4. $\mathbb{A}_+$ is a cone in the sense of the preceding definition.

Proposition 2.11. *$\mathbb{A}_+$ is a normal cone of a unital C^*-algebra $\mathbb{A}$.*

Proof. We show that A_+ satisfies all conditions in Definition 2.10. We see that $A_+ \neq \{0\}$ since $I \in \mathbb{A}_+$. The condition P2 is a property of A_+ and the condition P3 is obtained by considering the spectrums of elements of $\mathbb{A}$ directly. Since $\mathbb{A}_+$ is closed by Proposition 2.7, A_+ is a cone of $\mathbb{A}$. Normality is obvious by the sixth item of Theorem 2.4. ∎

Definition 2.12. Let X be a nonempty set and $d : X \times X \to \mathbb{E}$ be a function satisfying the following properties:

(M1) $d(x, y) \geq 0_{\mathbb{E}}$,
(M2) $d(x, y) = 0_{\mathbb{E}}$ if and only if $x = y$,
(M3) $d(x, y) = d(y, x)$,
(M4) $d(x, y) \leq d(x, z) + d(z, y)$,

for every $x, y, z \in X$. We call the function d a *cone metric* and call the pair (X, d) a *cone metric space.*

Consider a unital C^*-algebra $\mathbb{A}$. If the scalar filed is restricted to the set of real numbers, $\mathbb{A}$ becomes a real Banach space. Thus, a C^*-algebra-valued metric space becomes a cone metric space.

Definition 2.13. Let (X, d) be a cone metric space. A sequence $\{x_n\}$ in X is said to *converge* to $x \in X$ (with respect to $\mathbb{E}$) if and only if for every $c \in \mathbb{E}$ with $c \gg 0$ there is a positive integer N such that for every integer $n \geq N$ we have $d(x_n, x) \ll c$. In this case we write $\lim_{n\to\infty} x_n = x$, and say that the sequence $\{x_n\}$ is *convergent.*

A sequence $\{x_n\}$ in X is said to be *Cauchy* (with respect to $\mathbb{E}$) if and only if for every $c \in \mathbb{E}$ with $c \gg 0$ there is a positive integer N such that for every integer $n, m \geq N$ we have $d(x_n, x_m) \ll c$.

We say that a cone metric space (X, d) is complete if every Cauchy sequence is convergent.

Lemma 2.14. *Let (X, d) be a cone metric space together with a normal cone. A sequence $\{x_n\}$ converges to $x \in X$ if and only if $\lim_{n\to\infty} d(x_n, x) = 0$. A sequence $\{x_n\}$ is Cauchy if and only if $\lim_{n,m\to\infty} d(x_n, x_m) = 0$.*

Lemma 2.15. *Let (X, d) be a cone metric space together with a normal cone and $x, y \in X$. Assume that sequences $\{x_n\}$ and $\{y_n\}$ converge to x and y, respectively. Then $\lim_{n\to\infty} d(x_n, y_n) = d(x, y)$.*

Definition 2.16. Let X be a vector space over the real field and $\|\cdot\|_X : X \to \mathbb{E}$ be a function. A pair $(X, \|\cdot\|_X)$ is called a *cone normed space* if $\|\cdot\|_X$ satisfies the following properties:

(1) $\|x\|_X = 0_{\mathbb{E}}$ if and only if $x = 0_X$,
(2) $\|\alpha x\|_X = |\alpha|\|x\|_X$,
(3) $\|x + y\|_X \leq \|x\|_X + \|y\|_X$,

for every $x, y \in X$ and every scalar α.

It is clear that each cone normed space is a cone metric space with the cone metric given by $d(x, y) = \|x-y\|_X$. Complete cone normed spaces are called *cone Banach spaces.*

Theorem 2.17. *Let (X, d) be a cone metric space over a normal cone. Then there is a complete cone metric space (X_c, d_c) which has a dense subspace W isometric with X. The space X_c is unique except for isometries.*

Theorem 2.18. *Let $(X, \|\cdot\|)$ be a cone normed space over a normal cone. Then there is a cone Banach space $(X_c, \|\cdot\|_c)$ which has a dense subspace W isometric with X. The space X_c is unique except for isometries.*

The two results above are completion theorems obtained in [10]. We apply the them to obtain our results. The isometry mentioned in that research is a mapping $T : X \to Y$ between cone metric spaces preserving distances, that is,

$$d_X(x, y) = d_Y(Tx, Ty),$$

for every $x, y \in X$, where d_X and d_Y are cone metrics on X and Y, respectively. Properties of the mapping T are different from those of the ordinary version only the values of d_X and d_Y which are not real numbers. For the second theorem the author applied the first one to obtain a bijective isometry from a cone normed space onto a dense metric subspace of the cone metric completion. By the setting that provided algebraic operations and a norm for the metric completion, the isometry was actually a linear operator. Thus it is an isomorphism between cone normed spaces, a vector space isomorphism which preserves cone norms. Since isomorphisms of cone normed spaces are always isometry, we have another version of the completion theorem for normed spaces.

Theorem 2.19. *Let $(X, \|\cdot\|)$ be a cone normed space over a normal cone. Then there is a cone Banach space $(X_c, \|\cdot\|_c)$ which has a dense subspace W isomorphic with X. The space X_c is unique except for isomorphisms.*

Concepts of isometries of C^*-algebra-values metric spaces and isomorphisms of C^*-algebra-values normed spaces will be provided in the next section with more general than those of the cone version.

3. Completion of C^*-algebra-valued metric and normed spaces

In this section we verify that a C^*-algebra-valued metric space can be embedded in a complete C^*-algebra-valued metric space as a dense subspace. The theorem in a version of a C^*-algebra-valued normed space is also provided. We apply the fact that the C^*-algebra-valued metric (resp. normed) spaces are cone metric (resp. normed) spaces to extend the completion results from [10]. To work with a cone metric space, we need to assume that the interior of a cone is not empty. However, this property does not generally occur for a C^*-algebra as we show in the series of examples below.

Example 3.1. Let a C^*-algebra $\mathbb{A}$ be a complex plane $\mathbb{C}$. Then $\mathbb{A}_+ = [0, \infty)$, so $\mathrm{Int}(\mathbb{A}_+)$ is empty in $\mathbb{C}$. Observe that $\mathrm{Int}(\mathbb{A}_+)$ is not empty in $\mathbb{R}$, the set of hermitian elements of $\mathbb{C}$. ■

Example 3.2. In this example we consider $\mathbb{A}$ as a C^*-algebra of all bounded complex sequences ℓ^∞ with operations defined as follows:

$$\begin{aligned}
(\xi_n) + (\eta_n) &= (\xi_n + \eta_n), \\
(\xi_n)(\eta_n) &= (\xi_n \eta_n), \\
\lambda(\xi_n) &= (\lambda \xi_n), \\
(\xi_n)^* &= (\bar{\xi_n}), \\
\|(\xi_n)\|_{\mathbb{A}} &= \sup_{n \in \mathbb{N}} |\xi_n|,
\end{aligned}$$

for every $(\xi_n), (\eta_n) \in \ell^\infty$ and every $\lambda \in \mathbb{C}$. We have

$$\ell^\infty_h = \big\{ a \in \ell^\infty : a^* = a \big\} = \big\{ (\xi_n) \in \ell^\infty : \xi_n \in \mathbb{R} \text{ for all } n \in \mathbb{N} \big\}$$

and

$$\ell^\infty_+ = \big\{ a \in \ell^\infty_h : \sigma(a) \subseteq \mathbb{R}_+ \big\} = \big\{ (\xi_n) \in \ell^\infty : \xi_n \in \mathbb{R}_+ \text{ for all } n \in \mathbb{N} \big\}.$$

To show that $\mathrm{Int}(\ell^\infty_+) = \emptyset$, we let $a = (\xi_n) \in \ell^\infty_+$ and $\varepsilon > 0$. Then choose $b = (\xi_1 - i\frac{\varepsilon}{2}, \xi_2, \xi_3, \ldots)$. Clearly, b is in $\ell^\infty \setminus \ell^\infty_+$ such that $\|a - b\|_{\mathbb{A}} = \frac{\varepsilon}{2} < \varepsilon$. This implies that $b \in B(a, \varepsilon)$, the open ball in ℓ^∞ of radius ε centered at a. Since ε is arbitrary, the element a is not an interior point of ℓ^∞_+. Therefore $\mathrm{Int}(\ell^\infty_+) = \emptyset$. ■

Example 3.3 (A C^*-algebra-valued metric space with the empty interior of $\mathbb{A}_+$**).**

In this example we replace X and $\mathbb{A}$ by $\mathbb{C}$ and $\mathbb{C}^2$, respectively. By the same operations used in the previous example, the space $\mathbb{C}^2$ can be considered as a C^*-subalgebra of ℓ^∞ with $\mathrm{Int}(\mathbb{C}^2_+) = \emptyset$. Let $d : \mathbb{C} \times \mathbb{C} \to \mathbb{C}^2$ be a function defined by

$$d(a, b) = (|a - b|, \alpha|a - b|),$$

for every $a, b \in \mathbb{C}$ and α is a fixed positive real number. Therefore, $(\mathbb{C}, \mathbb{C}^2, d)$ is a C^*-algebra-valued metric space. ■

Although the situation in the previous example can occur, the assumption of the nonempty interior of $\mathbb{A}_+$ is not necessary. There exists a suitable real Banach subspace of $\widetilde{\mathbb{A}}$ containing $\widetilde{\mathbb{A}}_+$ with a nonempty interior under the topology on the Banach subspace restricted from $\widetilde{\mathbb{A}}$, and so, we will work on the subspace instead.

Proposition 3.4. *$\mathbb{A}_h$ is a real Banach subspace of a C^*-algebra $\mathbb{A}$.*

Proof. We know that $\mathbb{A}_h \subseteq \mathbb{A}$, $0_{\mathbb{A}} \in \mathbb{A}_h$ and $(\alpha a + b)^* = \alpha a + b$ for all $\alpha \in \mathbb{R}$ and all $a, b \in \mathbb{A}_h$. Then $\mathbb{A}_h$ is a real normed space. Let $\{a_n\}$ be a sequence in $\mathbb{A}_h$ converging to $a \in \mathbb{A}$. Since $\|a_n - a\|_{\mathbb{A}} = \|(a_n - a)^*\|_{\mathbb{A}} = \|a_n^* - a^*\|_{\mathbb{A}} = \|a_n - a^*\|_{\mathbb{A}}$, $\{a_n\}$ converges to a^*. By the uniqueness of limits of a convergent sequence, we have $a = a^*$, i.e. $a \in \mathbb{A}_h$. Therefore, $\mathbb{A}_h$ is closed in $\mathbb{A}$, and so $\mathbb{A}_h$ is a real Banach subspace of $\mathbb{A}$. ■

Proposition 3.5. *If $\mathbb{A}$ is a unital C^*-algebra, then $\mathrm{Int}_{A_h}(\mathbb{A}_+) \neq \emptyset$.*

Proof. Let I be a unit of $\mathbb{A}$ and $B(I, 1) = \{a \in \mathbb{A}_h : \|a - I\|_{\mathbb{A}} < 1\}$. Then Theorem 2.4 implies that $B(I, 1) \subseteq \mathbb{A}_+$. Hence, $I \in \mathrm{Int}_{A_h}(\mathbb{A}_+)$, so $\mathrm{Int}_{A_h}(\mathbb{A}_+) \neq \emptyset$. ■

Corollary 3.6. *If $\mathbb{A}$ is a unital C^*-algebra and $\mathbb{A} = \mathbb{A}_h$, then $\mathrm{Int}(\mathbb{A}_+) \neq \emptyset$.*

Corollary 3.7. $\mathrm{Int}_{\widetilde{\mathbb{A}}_h}(\widetilde{\mathbb{A}}_+) \neq \emptyset$.

In the previous section, we show that a unital C^*-algebra $\mathbb{A}$ contains $\mathbb{A}_+$ as a normal cone. Then so does $\mathbb{A}_h$. Therefore a C^*-algebra-valued metric space $(X, \mathbb{A}, d)$ is a cone metric space $(X, \widetilde{\mathbb{A}}_h, d)$ with a normal cone $\widetilde{\mathbb{A}}_+$ such that $\mathrm{Int}_{\widetilde{\mathbb{A}}_h}(\widetilde{\mathbb{A}}_+) \neq \emptyset$. Finally, we obtain Lemma 2.14 in a version of a C^*-algebra-valued metric space $(X, \mathbb{A}, d)$, equivalent definitions of convergent and Cauchy sequences, stated in the following theorem.

Theorem 3.8. *Let $\{x_n\}$ be a sequence in a C^*-algebra-valued metric space $(X, \mathbb{A}, d)$. Then the following statements are satisfied.*

(1) $\{x_n\}$ converges to $x \in \mathbb{X}$ (in the sense of Definition 2.9) if and only if for every $c \in \widetilde{\mathbb{A}}_h$ with $c \gg 0$ there is a positive integer N such that for every integer $n \geq N$ we have $d(x_n, x) \ll c$.

(2) $\{x_n\}$ is Cauchy (in the sense of Definition 2.9) if and only if for every $c \in \widetilde{\mathbb{A}}_h$ with $c \gg 0$ there is a positive integer N such that for every integer $n, m \geq N$ we have $d(x_n, x_m) \ll c$.

Proof. We prove only the case of convergence, the other can be proved similarly. Suppose that $\{x_n\}$ converges to an element x of $(X, \mathbb{A}, d)$. Then $\{x_n\}$ converges to an element x of $(X, \widetilde{\mathbb{A}}, d)$, and so, converges in $(X, \widetilde{\mathbb{A}}_h, d)$. Then the forward implication is obtained after applying Lemma 2.14. For the converse implication, we suppose that the condition holds. Then Lemma 2.14 implies that $\lim_{n\to\infty} \|d(x_n, x)\|_{\widetilde{\mathbb{A}}_h} = 0$. Since $d(x_n, x)$ belongs to $\mathbb{A}$, we have $\lim_{n\to\infty} \|d(x_n, x)\|_{\mathbb{A}} = 0$. Therefore, $\{x_n\}$ converges to an element x of $(X, \mathbb{A}, d)$. ■

Let X and Y are metric spaces. A function $T : X \to Y$ is an isometry if an only if

$$(d_X(x, y)) = d_Y(T(x), T(y)),$$

for every $x, y \in X$. In this case, $d_X(x, y) \mapsto d_Y(T(x), T(y))$ is an identity mapping which admits several properties including the preservation of the norm for $\mathbb{R}$. In case of C^*-algebra-valued metric spaces, the $d_X(x, y)$ and $d_Y(T(x), T(y))$ may be in different C^* algebras which are not related to each other. Thus a function $f : d_X(X, X) \to d_Y(Y, Y)$ is needed and the norms' preservation seems to be straightforward to add to the non-identity function f. Additionally, algebraic operations of C^*-algebra must be preserved, so f will be assumed as a linear map from a real linear span of $\mathrm{span}(d_X(X, X))$ to $\mathrm{span}(d_Y(Y, Y))$. Then f is always injective, and f^{-1} is also an injective norm preserving linear operator.

Observe that $\text{span}(d_X(X,X))$ and $\text{span}(d_Y(Y,Y))$ are linear subspace of the real Banach spaces of hermitian elements in C^*-algebras.

Now assume that $(X, \mathbb{A}, d_X)$ and $(Y, \mathbb{B}, d_Y)$ are C^*-algebra-valued metric spaces. Let $T : X \to Y$ be a function. If there exists norm-preserving linear operator f from $\text{span}(d_X(X,X))$ to $\text{span}(d_Y(Y,Y))$ such that

$$f(d_X(x,y)) = d_Y(T(x), T(y)),$$

for every $x, y \in X$, the function T is called an *isometry from X to Y with respect to f*. The space X and Y are said to be *isometric* if there exists a bijective isometry (with respect to f) from X to Y. We note that the phrase "with respect to f" may be omitted if not confused.

Proposition 3.9. *An isometry between C^*-algebra-valued metric spaces is always injective.*

Proof. Suppose that $(X, \mathbb{A}, d_X)$ and $(Y, \mathbb{B}, d_Y)$ are C^*-algebra-valued metric spaces and T is an isometry from X to Y with respect to f. Let $x, y \in X$ such that $T(x) = T(y)$. Then $f(d_X(x,y)) = d_Y(T(x), T(y)) = 0_{\mathbb{A}}$. Since f is norm-preserving, $\|d_X(x,y)\|_{\mathbb{A}} = \|f(d_X(x,y))\|_{\mathbb{B}} = 0$. Now we have $d_X(x,y) = 0_{\mathbb{A}}$, and so $x = y$. Therefore, T is injective. ∎

Let $\sim$ be a relation on a family of C^*-algebra-valued metric spaces which indicates that two C^*-algebra-valued metric spaces are isometric. For more precisely, $X \sim Y$ if and only if X is isometric with Y. The next proposition show that $\sim$ is an equivalence relation.

Lemma 3.10. *Let T be an isometry from $(X, \mathbb{A}, d_X)$ to $(Y, \mathbb{B}, d_Y)$ with respect to f. If T is surjective, then f is also surjective, i.e., $f(\text{span}(d_X(X,X))) = \text{span}(d_Y(Y,Y))$.*

Proof. Let $u, v \in T(X)$ and T be surjective, that is, $T(X) = Y$. Thus there are $x, y \in X$ such that $T(x) = u$ and $T(y) = v$. Then

$$f(d_X(x,y)) = d_Y(T(x), T(y)) = d_Y(u, v).$$

After applying linearity of f, we have f is surjective. ∎

Proposition 3.11. *The relation $\sim$ determined by isometries is an equivalence relation on the family of all C^*-algebra-valued metric spaces.*

Proof. Let $(X, \mathbb{A}, d_X), (Y, \mathbb{B}, d_Y)$ and $(Z, \mathbb{C}, d_Z)$ are C^*-algebra-valued metric spaces. We see that the identity function is a bijective isometry on X. Then $X \sim X$, so $\sim$ is reflexive. Next suppose that $X \sim Y$. Then there is a bijective isometry T form X to Y with respect to a norm-preserving linear operator f. By the previous lemma, f becomes a bijective norm-preserving linear operator, so $f^{-1} : \text{span}(d_Y(Y,Y)) \to \text{span}(d_X(X,X))$ is also a bijective norm-preserving linear operator such that

$$f^{-1}(d_Y(x,y)) = d_X(T^{-1}(x), T^{-1}(y)),$$

for every $x, y \in Y$. Thus T^{-1} is bijective isometry form Y to X with respect to f^{-1}, so $Y \sim X$. This verifies that $\sim$ is symmetric. To investigate the transitive property we additionally assume that $Y \sim Z$. Thus there exists a bijective isometry S from Y to Z with respect to g. Then $S \circ T : X \to Z$ is bijective, and $g \circ f : \text{span}(d_X(X,X)) \to \text{span}(d_Z(Z,Z))$ is a norm-preserving linear operator. Hence $S \circ T$ is a bijective isometry form X to Z with respect to a norm-preserving function $g \circ f$. Thus $X \sim Z$, and so $\sim$

is transitive. Therefore $\sim$ is an equivalence relation on the class of all C^*-algebra-valued metric spaces. ∎

Traditionally, denseness of a subset in a topological space is determined using neighborhoods or open balls in the space. It is equivalent to the definition described by sequences. In case of a C^*-algebra-valued metric space, we provide a definition using open balls. Also an equivalent definition using sequences will be assigned.

Definition 3.12. Let $(X, \mathbb{A}, d)$ be a C*-algebra-valued metric space. For any $\varepsilon > 0$, we define

$$B(x, \varepsilon) = \{y \in X : \|d(x, y)\|_{\mathbb{A}} < \varepsilon\},$$

Let M be a subset of X, the set of all limit points or *closure* of M is determined by

$$\mathrm{Cl}(M) = \{x \in X : B(x, \varepsilon) \cap M \neq \emptyset \text{ for every } \varepsilon > 0\}.$$

If $\mathrm{Cl}(M) = X$, we say that M is *dense* in X.

Because of Theorem 3.8, an equivalent definition of closure of the set M is obtained, that is,

$$\mathrm{Cl}(M) = \{x \in X : B_1(x, c) \cap M \neq \emptyset \text{ for every } c \gg 0\},$$

where $B_1(x, c) = \{y \in X : d(x, y) < c\}$ with $c \in \mathbb{A}$ such that $c \gg 0$.

Theorem 3.13. *The subset M of a C^*-algebra-valued metric space $(X, \mathbb{A}, d)$ is dense in X if and only if for every $x \in X$ there is a sequence $\{x_n\}$ in M converging to x.*

Proof. Assume that M is dense X and $x \in X$. Then there exits $x_n \in B(x, \frac{1}{n}) \cap M \neq \emptyset$ for every $n \in \mathbb{N}$. Thus we can form a sequence $\{x_n\}$ of elements of M. Let $\varepsilon > 0$. There is a positive integer N such that $\frac{1}{N} < \varepsilon$. Then for every $n \geq N$, $x_n \in B(x, \varepsilon)$. This means that $\{x\}$ converges to x.

For the converse implication we assume the condition holds. We show that $\mathrm{Cl}(M) = X$. Let $x \in X$. Then there is a sequence $\{x_n\}$ in M converging to x. For a given positive real number ε, there is an integer N such that $\|d(x_n, x)\|_{\mathbb{A}} < \varepsilon$ for every $n \geq N$. Thus $x_N \in B(x, \varepsilon)$. This shows that $B(x, \varepsilon) \cap M \neq \emptyset$ for every $\varepsilon > 0$, so $x \in \mathrm{Cl}(M)$. Hence $\mathrm{Cl}(M) = X$, so M is dense in X. ∎

We have shown that any C^*-algebra-valued metric space $(X, \mathbb{A}, d)$ can be considered as the cone metric space $(X, \widetilde{\mathbb{A}}_h, d)$ with the normal cone $\widetilde{\mathbb{A}}_+$ such that $\mathrm{Int}_{\widetilde{\mathbb{A}}_h}(\widetilde{\mathbb{A}}_+) \neq \emptyset$. Thus, we can work on the cone metric space instead, and obtain the metric completion theorem for $(X, \mathbb{A}_h, d)$ after applying Theorem 2.17. Since the values of d belong to $\mathbb{A}$, the C^*-algebra-valued metric space $(X, \mathbb{A}, d)$ is actuary contained in the acquired space as a dense subspace. We conclude this result in the following theorem.

Theorem 3.14 (Completion of C^*-algebra-valued metric spaces)**.**
For any C^*-algebra-valued metric space $(X, \mathbb{A}, d)$, there exists a complete C^*-algebra-valued metric space $(X_c, \mathbb{A}, d_c)$ which contains a dense subspace W isometric with X. The space X_c is unique except for isometries.

Proof. Consider $(X, \mathbb{A}, d)$ is a cone metric space $(X, \widetilde{\mathbb{A}}_h, d)$ with the normal cone $\widetilde{\mathbb{A}}_+$ such that $\mathrm{Int}_{\widetilde{\mathbb{A}}_h}(\widetilde{\mathbb{A}}_+) \neq \emptyset$. Then Theorem 2.17 implies that there is a complete cone metric space $(X_c, \widetilde{\mathbb{A}}_h, d_c)$ which contains a dense subspace W isometric (in the sense of cone metric spaces) with X. We see that $(X_c, \widetilde{\mathbb{A}}, d_c)$ is also a C^*-algebra-valued metric space.

We will verify that d_c is an $\mathbb{A}$-valued metric for X_c, in fact, after taking the composition with the inverse of the mapping $a \mapsto (a, 0)$ from $\mathbb{A}$ to $\widetilde{\mathbb{A}}$.

Let $x, y \in X_c$. Since W is dense in X_c, there exist sequences $\{x_n\}$ and $\{y_n\}$ in W converging to x and y, respectively. By Lemma 2.15, we have

$$d_c(x, y) = \lim_{n\to\infty} d_c(x_n, y_n). \tag{3.1}$$

Let T be a bijective isometry of a cone metric space from W to $(X, \widetilde{\mathbb{A}}_h, d)$. We see that the values of d is initially in $\mathbb{A}$, so

$$d_c(x_n, y_n) = d(T(x_n), T(y_n)) \in \mathbb{A},$$

for every $n \in \mathbb{N}$. Since $\mathbb{A}$ is closed in $\widetilde{\mathbb{A}}$, we have

$$d_c(x, y) = \lim_{n\to\infty} d_c(x_n, y_n) \in \mathbb{A}.$$

This implies that d_c is an $\mathbb{A}$-valued metric for X_c, and so W and X are isometric with respect to the identity function $I : \mathrm{span}(d_c(W, W)) \to \mathrm{span}(d(X, X))$.

Next we prove the uniqueness of X_c. Let $(X_b, \mathbb{B}, d_b)$ be another C^*-algebra-valued metric space containing a dense subspace W_b which is isometric to X. Then there is a bijective isometry T_b from X to W_b with respect to f. Thus, $T_b \circ T$ is also a bijective isometry from W to W_b with respect to $f \circ I = f$. If we can extend the operator $T_b \circ T$ to be a bijective isometry from X_c to X_b with respect to the norm-preserving linear extension $\tilde{f}$ of f, then X_c and X_b will be isometric with respect to $\tilde{f}$. To complete this proof we will do the necessary arrangements accordingly:

(1) Prove that $\tilde{f}$ exists.
(2) Extended $T_b \circ T$ to be an isometry from X_c to X_b with respect to $\tilde{f}$.
(3) Verify that the extension of $T_b \circ T$ is bijective.

Let us start with the item 1. We can see that f is a norm-preserving linear operator from $\mathrm{span}(d_c(W, W))$ to $\mathrm{span}(d_b(X_b, X_b))$. Suppose that $a, b \in d_c(X_c, X_c)$ and α, β be scalars. Since W is dense in X_c, after applying Lemma 2.15 we obtain that there are sequences $\{a_n\}$ and $\{b_n\}$ in $d_c(W, W)$ converging to a and b respectively. Next apply continuity of the addition and the scalar multiplication with respect to the norm of $d_c(X_c, X_c)$. We have

$$\lim_{n\to\infty} (\alpha a_n + \beta b_n) = \alpha a + \beta b.$$

This shows that $\mathrm{span}(d_c(W, W))$ is dense in $\mathrm{span}(d_c(X_c, X_c))$. For the completeness of $\mathrm{span}(d_b(X_b, X_b))$, we can show that it is closed in $\mathbb{B}_h$. The proof can be done by the similar arguments applied in the previous one. Now applying the bounded linear extension theorem to obtain the bounded linear extension $\tilde{f} : \mathrm{span}(d_c(X_c, X_c)) \to \mathrm{span}(d_b(X_b, X_b))$ of f. Next we show that $\tilde{f}$ is norm-preserving. Assume that $a \in \mathrm{span}(d_c(X_c, X_c))$. Since $\mathrm{span}(d_c(W, W))$ is dense in $\mathrm{span}(d_c(X_c, X_c))$, there is a sequence $\{a_n\}$ in $\mathrm{span}(d_c(W, W))$ converging to a. By Lemma 2.15 together with the continuity of f and the norms of $\mathbb{A}$ and $\mathbb{B}$, we have

$$\|\tilde{f}(a)\|_{\mathbb{B}} = \lim_{n\to\infty} \|f(a_n)\|_{\mathbb{B}} = \lim_{n\to\infty} \|a_n\|_{\mathbb{A}} = \|a\|_{\mathbb{A}}.$$

Now $\tilde{f}$ is a norm-preserving linear operator.

Next we apply the same setting of (3.1) and additional assume that $x'_n = T_b(T(x_n))$. Then $x'_n \in W_b$ and

$$\|d_c(x_m, x_n)\|_{\mathbb{A}} = \|f(d_c(x_m, x_n))\|_{\mathbb{B}} = \|d_b(x'_m, x'_n)\|_{\mathbb{B}}. \tag{3.2}$$

Since $x_n \to x$, $\{x'_n\}$ is a Cauchy sequence in W_b. Hence there are $x' \in X_b$ such that $x'_n \to x'$. Thus we determine

$$(T_b \circ T)(x) = x'. \tag{3.3}$$

Assume that $\{z_n\}$ be another sequence in W converging to x. Let $z'_n = T_b(T(z_n))$. By the same method above we can show that the sequence $\{z'_n\}$ converges in X_b. Suppose that $z'_n \to z'$. For every $n \in \mathbb{N}$, we have

$$\begin{aligned}
\|d_b(x', z')\|_{\mathbb{B}} &\le \|d_b(x', x'_n)\|_{\mathbb{B}} + \|d_b(x'_n, z'_n)\|_{\mathbb{B}} + \|d_b(z'_n, z')\|_{\mathbb{B}} \\
&= \|d_b(x', x'_n)\|_{\mathbb{B}} + \|d_b(T_b(T(x_n)), T_b(T(z_n)))\|_{\mathbb{B}} + \|d_b(z'_n, z')\|_{\mathbb{B}} \\
&= \|d_b(x', x'_n)\|_{\mathbb{B}} + \|f(d_c(x_n, z_n))\|_{\mathbb{B}} + \|d_b(z'_n, z')\|_{\mathbb{B}} \\
&= \|d_b(x', x'_n)\|_{\mathbb{B}} + \|d_c(x_n, z_n)\|_{\mathbb{A}} + \|d_b(z'_n, z')\|_{\mathbb{B}} \\
&\le \|d_b(x', x'_n)\|_{\mathbb{B}} + \|d_c(x_n, x)\|_{\mathbb{A}} + \|d_c(x, z_n)\|_{\mathbb{A}} + \|d_b(z'_n, z')\|_{\mathbb{B}}.
\end{aligned}$$

This implies that $x' = z'$. Hence the extension of $T_b \circ T$ from X_c to X_b is a well-defined. Now we additionally assume that $y \in X_c$ and $\{y_n\}$ is a sequence W converging to y. By Lemma 2.15 together with (3.3), we obtain

$$\begin{aligned}
\tilde{f}(d_c(x, y)) &= \lim_{n\to\infty} f(d_c(x_n, y_n)) \\
&= \lim_{n\to\infty} d_b((T_b \circ T)(x_n), (T_b \circ T)(y_n)) \\
&= d_b((T_b \circ T)(x), (T_b \circ T)(y)).
\end{aligned}$$

$T_b \circ T$ is now an isometry from X_c to X_b with respect to $\tilde{f}$.

Finally, we verify that the extension of $T_b \circ T$ is bijective. Let $x' \in X_b$. Since W_b is dense in X_b, there is a sequence $\{x'_n\}$ in W_b converging to x'. Since $T_b \circ T : W \to W_b$ is surjective, there are $x_n \in W$ such that $T_b(T(x_n)) = x'_n$ for all $n \in \mathbb{N}$. By applying the equation (3.2), $\{x_n\}$ is Cauchy in X_c, so it converges to an element x in X_c. Thus

$$\begin{aligned}
\|d_b(T_b(T(x)), x')\|_{\mathbb{B}} &\le \|d_b(T_b(T(x)), T_b(T(x_n)))\|_{\mathbb{B}} + \|d_b(x'_n, x')\|_{\mathbb{B}} \\
&= \|d_c(x, x_n)\|_{\mathbb{A}} + \|d_b(x'_n, x')\|_{\mathbb{B}}.
\end{aligned}$$

This implies that $T_b(T(x)) = x'$. Hence $T_b \circ T$ is surjective, and so, bijective after applying Proposition 3.9. Consequently, $T_b \circ T$ is a bijective isometry from X_c to X_b with respect to $\tilde{f}$. Therefore X_c and X_b are isometric. ∎

Next, we focus on a C^*-algebra-valued normed space. Let X be a vector space over the real or complex fields and $\mathbb{A}$ be a C^*-algebra. A triple $(X, \mathbb{A}, \|\cdot\|_X)$ is called a C^*-algebra-valued normed space if $\|\cdot\|_X$ is a function from X to A_+ satisfying the following properties:

(1) $\|x\|_X = 0_{\mathbb{A}}$ if and only if $x = 0_X$,
(2) $\|\alpha x\|_X = |\alpha|\|x\|_X$,
(3) $\|x + y\|_X \le \|x\|_X + \|y\|_X$,

for every $x, y \in X$ and every scalar α. Notice that $0_{\mathbb{A}}$ and 0_X are zeros in $\mathbb{A}$ and X respectively.

By the definition of a C^*-algebra-valued norm, we can investigate that the function $d : X \times X \to \mathbb{A}$ determined by $d(x, y) = \|x - y\|_X$ is a C^*-algebra-valued metric. We call it the *C^*-algebra-valued metric* induced by the norm $\|\cdot\|_X$ We conclude this fact in the proposition below

Proposition 3.15. *A C^*-algebra-valued normed space $(X, \mathbb{A}, \|\cdot\|_X)$ is a C^*-algebra-valued metric space with a metric $d : X \times X \to \mathbb{A}$ given by $d(x, y) = \|x - y\|_X$.*

A complete C^*-algebra-valued normed space under the metric induced by the C^*-algebra-valued norm is called a *C^*-algebra-valued Banach Space*. In the next example, we show that every commutative C^*-algebra is a C^*-algebra-valued normed space.

Lemma 3.16. *Let A be commutative C^*-algebra. Then $\mathbb{A}_h$ is a closed $*$-subalgebra of $\mathbb{A}$ over the real field. Moreover, if $a, b \in \mathbb{A}_+$, then $ab \in \mathbb{A}_+$ and $(ab)^{1/2} = a^{1/2}b^{1/2}$.*

Proof. Since $\mathbb{A}$ is commutative, $(ab)^* = a^*b^* = ab$ for every $a, b \in \mathbb{A}_h$. Combine with Proposition 3.4, $\mathbb{A}_h$ is a real $*$-subalgebra of $\mathbb{A}$.

Next, suppose that $a, b \in \mathbb{A}_+$ Theorem 2.4 implies that $a = c^*c$ for some $c \in \mathbb{A}$. Thus, we have $0_{\mathbb{A}} = c^*0_{\mathbb{A}}c \leq c^*bc = c^*cb = ab$, so ab is positive. By the same way, $a^{1/2}b^{1/2}$ is also positive. Since $(a^{1/2}b^{1/2})^2 = ab$, Theorem 2.2 implies that $a^{1/2}b^{1/2} = (ab)^{1/2}$. ∎

Example 3.17. Let $\mathbb{A}$ be a commutative C^*-algebra and $X = \mathbb{A}$. By using Proposition 2.1, every element $x \in \mathbb{A}$ can be uniquely decomposed as $x = a + bi$ for some $a, b \in \mathbb{A}_h$. Then we define $\|\cdot\|_0 : X \to \mathbb{A}_+$ by

$$\|x\|_0 = (a^2 + b^2)^{1/2}.$$

We will show that $(X, \|\cdot\|_0, \mathbb{A})$ is a C^*-algebra-valued normed space.

Since a and b are hermitian, Theorem 2.4 implies that a^2 and b^2 are positive. Thus, $(a^2 + b^2)^{1/2}$ is also positive after applying Proposition 2.3 and Theorem 2.2, respectively. We now obtain that $\|\cdot\|_0$ is an $\mathbb{A}_+$ valued function. Since $x = 0_X$ if and only if $a = b = 0_X$, we obtain that $\|x\|_0 = 0_{\mathbb{A}}$ if and only if $x = 0_X$. Next, suppose that $\gamma = \alpha + \beta i$ where $\alpha, \beta \in \mathbb{R}$. Hence, $\gamma x = (\alpha + \beta i)(a + bi) = (\alpha a - \beta b) + (\beta a + \alpha b)i$, so

$$\begin{aligned}\|\gamma x\|_0^2 &= (\alpha a - \beta b)^2 + (\beta a + \alpha b)^2 \\ &= \alpha^2 a^2 + \beta^2 b^2 + \beta^2 a^2 + \alpha^2 b^2 \\ &= (\alpha^2 + \beta^2)(a^2 + b^2).\end{aligned}$$

Theorem 2.2 and Proposition 2.5 imply that $\|\gamma x\|_0 = ((\alpha^2 + \beta^2)(a^2 + b^2))^{1/2} = |\alpha|\|x\|_0$.

Finally, we prove the triangle inequality. We additionally assume that $y \in X$ is uniquely represented by $c + di$ where $c, d \in \mathbb{A}_h$. Consider

$$\begin{aligned}\|x + y\|_0^2 &= \|(a + c) + (b + d)i\|_0^2 \\ &= (a + c)^2 + (b + d)^2 \\ &= (a^2 + 2ac + c^2) + (b^2 + 2bd + b^2) \\ &= (a^2 + b^2 + c^2 + d^2) + 2(ac + bd),\end{aligned}$$

and

$$\begin{aligned}(\|x\|_0 + \|y\|_0)^2 &= \|x\|_0^2 + 2\|x\|_0\|y\|_0 + \|y\|_0^2 \\ &= (a^2+b^2) + 2(a^2+b^2)^{1/2}(c^2+d^2)^{1/2} + (c^2+d^2) \\ &= (a^2+b^2+c^2+d^2) + 2(a^2+b^2)^{1/2}(c^2+d^2)^{1/2}.\end{aligned}$$

Hence after verifying that $ac+bd \leq (a^2+b^2)^{1/2}(c^2+d^2)^{1/2}$ and then apply Theorem 2.6, we will obtain the inequality $\|x+y\|_0^2 \leq (\|x\|_0 + \|y\|_0)^2$.

We may see that $0_{\mathbb{A}} \leq (ad-bc)^2 = (ad)^2 - 2abcd + (bc)^2$, so $2abcd \leq (ad)^2 + (bc)^2$. Therefore,

$$\begin{aligned}(ac+bd)^2 &= (ac)^2 + 2abcd + (bd)^2 \\ &\leq (ac)^2 + (ad)^2 + (bc)^2 + (bd)^2 \\ &= (a^2+b^2)(c^2+d^2).\end{aligned}$$

Theorem 2.6 implies $((ac+bd)^2)^{1/2} \leq ((a^2+b^2)(c^2+d^2))^{1/2}$. Then apply Theorem 2.2 and Lemma 3.16 to the left and right sides of the inequality, respectively. Thus we obtain $ac+bd \leq (a^2+b^2)^{1/2}(c^2+d^2)^{1/2}$. Now, the triangle inequality of $\|\cdot\|_0$ is investigated. Consequently, $\|\cdot\|_0$ is an $\mathbb{A}$-valued norm for $\mathbb{A}$. ■

Remark 3.18. $\|a\|_0 = a$ for every positive element a of a C^*-algebra $\mathbb{A}$.

Consider an additive operator between traditional normed spaces, it is an isometry if and only if it is norm-preserving. Thus, an isomorphism between normed spaces is always isometry. The similar arguments will be applied to define an isomorphism of C^*-algebra-valued normed spaces and the phrase "with respect to f" is also applied to the terminologies determined below. However, we may leave out it for convenience if there is not any confusion.

Assume that $(X, \mathbb{A}, \|\cdot\|_X)$ and $(Y, \mathbb{B}, \|\cdot\|_Y)$ be C^*-algebra-valued normed spaces and $T : X \to Y$ be an operator. If there exists a norm-preserving linear operator $f : \mathrm{span}(\|X\|_X) \to \mathrm{span}(\|Y\|_Y)$ such that

$$f(\|x\|_X) = \|T(x)\|_Y,$$

for every $x \in X$, the operator T is called a *C^*-valued-norm-preserving operator from X to Y with respect to f*. Additionally, a bijective C^*-valued-norm-preserving linear operator with respect to f is called an *isomorphism with respect to f*. We say that the C^*-valued spaces X and Y are *isomorphic* if there exists an isomorphism (with respect to f) from X to Y.

We know that any incomplete normed space can be embedded in a Banach space. In [10], the author defined a cone normed space and verified the existence of its completion. The existence of the C^*-algebra-valued Banach completion will be verified in the next theorem.

Theorem 3.19 (Completion of C^*-algebra-valued normed spaces)**.**
For any C^*-algebra-valued normed space $(X, \mathbb{A}, \|\cdot\|)$, there exists a C^*-algebra-valued Banach space $(X_c, \mathbb{A}, \|\cdot\|_c)$ which contains a dense subspace W isomorphic with X. The space X_c is unique except for isomorphism.

Proof. The process of the proof follows from the completion theorem for the metric version. Similar to the case of metric, we can consider $(X, \mathbb{A}, \|\cdot\|)$ as a cone normed space

$(X, \widetilde{\mathbb{A}}_h, \|\cdot\|)$. Then apply Theorem 2.19 to obtain a cone Banach space $(X_c, \widetilde{\mathbb{A}}_h, \|\cdot\|_c)$ containing a dense subspace W which is isomorphic with X. Let $x \in X_c$. Then there is a sequence $\{x_n\}$ in W converging to x. Now consider X_c as an $\widetilde{\mathbb{A}}$-valued cone metric space with the metric $d(x, y) = \|x - y\|_c$. By Lemma 2.15, we have

$$\|x\|_c = d(x, 0_X) = \lim_{n\to\infty} d(x_n, 0_x) = \lim_{n\to\infty} \|x_n\|_c. \tag{3.4}$$

Let T be an isomorphism (a norm-preserving linear operator) of a cone normed space from W to $(X, \widetilde{\mathbb{A}}_h, \|\cdot\|)$. Since the values of $\|\cdot\|$ is initiated in a C^*-algebra $\mathbb{A}$, we have

$$\|x\|_c = \lim_{n\to\infty} \|x_n\|_c = \lim_{n\to\infty} \|T(x_n)\| \in \mathbb{A}.$$

Now $\|\cdot\|_c$ is an $\mathbb{A}$-valued norm for X_c. In addition, W and X are isomorphic to each other with respect to the identity function $I : \operatorname{span}(\|W\|_c) \to \operatorname{span}(\|X\|)$.

For the uniqueness of X_c, we suppose that $(X_b, \mathbb{B}, \|\cdot\|_b)$ is an other C^*-algebra-valued normed space containing a dense subspace W_b and $T_b : X \to W_b$ are isomorphism with respect to f. We can show that $T_b \circ T$ is an isomorphism from W to W_b with respect to $f \circ I = f$. To investigate that X_c and X_b are isometric, we need to complete the following tasks to extend the operator $T_b \circ T$ to be an isomorphism from X_c to X_b with respect to the norm-preserving linear extension $\tilde{f}$ of f:

(1) Prove that $\tilde{f}$ exists.
(2) Extended $T_b \circ T$ to be an isometry from X_c to X_b with respect to $\tilde{f}$.
(3) Verify that the extension of $T_b \circ T$ is bijective.
(4) Verify that the extension of $T_b \circ T$ is a linear operator.

For the first three items, the similar methodology of the proof in Theorem 3.14 will be applied to $\|\cdot\|_c$ and $\|\cdot\|_b$. This implies that we can extend $T_b \circ T$ to be a bijective isometry from X_c to X_b with respect to $\tilde{f}$ as follows. For every $x \in X_c$

$$f(\|x\|_c) = \|x'\|_b, \quad (T_b \circ T)(x) = x',$$

where x' is the limit of the sequence $\{x'_n\}$ such that $x'_n = T_b(T(x_n))$ and $\{x_n\}$ is a sequence in W converging to x. Let α and β be scalars, and $x, y \in X_c$. Then there are sequences $\{x_n\}$ and $\{y_n\}$ in W converging to x and y, respectively. Since $T_b \circ T$ is linear on W, we have

$$(T_b \circ T)(\alpha x_n + \beta y_n) = \alpha(T_b \circ T)(x_n) + \beta(T_b \circ T)(y_n),$$

for every positive integer n. Since addition and scalar multiplication of a C^*-algebra-valued normed space are continuous with respect to the C^*-algebra-valued norm. We have

$$(T_b \circ T)(\alpha x_n + \beta y_n) \to (T_b \circ T)(\alpha x + \beta y)$$

and

$$\alpha(T_b \circ T)(x_n) + \beta(T_b \circ T)(y_n) \to \alpha(T_b \circ T)(x) + \beta(T_b \circ T)(y).$$

By the uniqueness of limits,

$$(T_b \circ T)(\alpha x + \beta y) = \alpha(T_b \circ T)(x) + \beta(T_b \circ T)(y).$$

This means that $T_b \circ T$ is a linear operator from X_c to X_b. Now we obtain that $T_b \circ T$ is an isomorphism from X_c to X_b, so X_c and X_b are isomorphic. The proof of the theorem is now complete. ∎

4. Connection with Hilbert C^*-modules

This section provides certain relationships between concepts of a C^*-algebra-valued metric space and an inner-product C^*-module which is a generalization of an inner product space. The concept of inner-product C^*-module was first introduced in [13], the study of I. Kaplansky in 1953, to develop the theory for commutative unital algebras. In the 1970s, the definition was extended to the case of noncommutative C^*-algebra, see more details in [14, 15]. Let $\mathbb{A}$ be a C^*-algebra and X be a complex vector space which is a right $\mathbb{A}$-module with compatible scalar multiplication:

$$\alpha(xa) = (\alpha x)a = x(\alpha a), \tag{4.1}$$

for every $\alpha \in \mathbb{C}, x \in X$ and $a \in \mathbb{A}$. The triple $(X, \mathbb{A}, \langle\cdot,\cdot\rangle)$ is called an *inner product* $\mathbb{A}$*-module* if the mapping $\langle\cdot,\cdot\rangle : X \times X \to \mathbb{A}$ satisfies the following conditions:

(1) $\langle x, \alpha y + \beta z\rangle = \alpha\langle x, y\rangle + \beta\langle x, z\rangle$,
(2) $\langle x, ya\rangle = \langle x, y\rangle a$,
(3) $\langle y, x\rangle = \langle x, y\rangle^*$,
(4) $\langle x, x\rangle \geq 0_{\mathbb{A}}$,
(5) if $\langle x, x\rangle = 0_{\mathbb{A}}$, then $x = 0_X$,

for every $\alpha \in \mathbb{C}$, $x, y \in X$ and $a \in \mathbb{A}$. It is known that any inner product C^*-module $(X, \mathbb{A}, \langle\cdot,\cdot\rangle)$ is a normed space with a scalar-valued norm $\|\cdot\|_m$ given by

$$\|x\|_m = \|\langle x, x\rangle\|_{\mathbb{A}}^{1/2},$$

for every $x \in X$ where $\|\cdot\|_{\mathbb{A}}$ is a norm on $\mathbb{A}$. It is called a *Hilbert* C^**-module* if the induced norm is complete.

The concept of completion is also extended to inner product C^*-modules. It is mentioned in [16] that for any inner product C^*-module X over a C^*-algebra $\mathbb{A}$, one can form its completion X_c, a Hilbert $\mathbb{A}$-module, using a similar way to the case of the scalar-valued inner product space. That is, for given sequences $\{x_n\}$ and $\{y_n\}$ in X converging to x and y in X_c, we define

$$\langle x, y\rangle := \lim_{n\to\infty} \langle x_n, y_n\rangle.$$

It is an $\mathbb{A}$-valued inner product on X_c constructed from that of X using the completeness of $\mathbb{A}$ to confirm that the limit exists.

Next, we provide a connection between the concept of C^*-algebra-valued metric completion and the completion of an inner product C^*-module. We show that this two concepts are identical if a C^*-algebra-valued inner product can induce a C^*-algebra-valued norm. Similar to the case of a traditional inner product space, for any inner product C^*-module $(X, \mathbb{A}, \langle\cdot,\cdot\rangle)$ we determine a function $\|\cdot\|_X : X \to \mathbb{A}$ by

$$\|x\|_X = \langle x, x\rangle^{1/2}. \tag{4.2}$$

Can the $\mathbb{A}$-valued function $\|\cdot\|_X$ be an $\mathbb{A}$-valued norm on X? To answer the question we consider $\mathbb{A}$ as a right module over itself, so it becomes an inner product $\mathbb{A}$-module together with an $\mathbb{A}$-valued inner product defined by

$$\langle x, y\rangle = x^* y,$$

for every $x, y \in \mathbb{A}$. In this case we have

$$\|x\|_X = (x^* x)^{1/2},$$

and the result given by R. Harte in [17] implies that $\|\cdot\|_X$ does not satisfy the triangle inequality in certain cases. Therefore the $\mathbb{A}$-valued function may not become an $\mathbb{A}$-valued norm in general. However, R. Jiang provide sufficient conditions in [18] to make the triangle inequality hold, so $\|\cdot\|_X$ becomes an $\mathbb{A}$-valued norm of X. The $\mathbb{A}$-valued metric completion concept for an inner product C^*-module will be studied in this case.

Proposition 4.1. *Let $(X, \mathbb{A}, \langle\cdot,\cdot\rangle)$ be an inner product C^*-module. If the function $\|\cdot\|_X$ defined above satisfies the triangle inequality, then it is an $\mathbb{A}$-valued norm on X.*

Proof. The proof can be obtained directly from the definition of an $\mathbb{A}$-valued norm and an $\mathbb{A}$-valued inner product. ■

If the inner product C^*-module $(X, \mathbb{A}, \langle\cdot,\cdot\rangle)$ is a C^*-algebra-valued normed space, and so a C^*-algebra-valued metric space, we can consider whether the space is complete by using a C^*-algebra-valued metric.

Theorem 4.2. *Assume that an inner product C^*-module $(X, \mathbb{A}, \langle\cdot,\cdot\rangle)$ is a C^*-algebra-valued norm space with an $\mathbb{A}$-valued norm $\|\cdot\|_X$ induced by $\langle\cdot,\cdot\rangle$. Then it is a Hilbert C^*-module if and only if it is a C^*-algebra-valued Banach space.*

Proof. Let x be any element of X and $\|\cdot\|_X$ be an $\mathbb{A}$-valued norm on X induced by $\langle\cdot,\cdot\rangle$. Since $\|x\|_X^2 = \langle x, x\rangle$, we have

$$\big\|\langle x, x\rangle\big\|_{\mathbb{A}} = \big\|\|x\|_X^2\big\|_{\mathbb{A}} = \big\|\|x\|_X\big\|_{\mathbb{A}}^2.$$

Thus,

$$\|x\|_m = \big\|\langle x, x\rangle\big\|_{\mathbb{A}}^{1/2} = \big\|\|x\|_X\big\|_{\mathbb{A}}.$$

Then by Definition 2.9 we obtain that the two concepts of convergence of any sequence $\{x_n\}$ in X by $\|\cdot\|_X$ and $\|\cdot\|_m$ are equivalent. Therefore, X is a Hilbert C^*-module if and only if it is a C^*-algebra-valued Banach space. ■

Now we consider the conditions which makes $(X, \mathbb{A}, \langle\cdot,\cdot\rangle)$ to be an $\mathbb{A}$-valued normed space. For any C^*-algebra $\mathbb{A}$, we let $\mathbb{A}''$ be the enveloping Von Neumann algebra of $\mathbb{A}$. Proposition 2.3 in [18] concludes that $\mathbb{A}$ is commutative if and only if $\mathbb{A}''$ is commutative. It is a useful fact which is applied to prove the triangle inequality for $\|\cdot\|_X$ defined by (4.2). By the sake of Gelfand-Naimark Theorem, we can consider $\mathbb{A}$ as C^*-subalgebra of $B(H)$ for some Hilbert space H. Then $\overline{\langle X, X\rangle}''$ also lies in $B(H)$ where $\overline{\langle X, X\rangle}$ is a closed two-side ideal of $\mathbb{A}$ generated by $\langle X, X\rangle$. In Lemma 3.5 of [18], the author provides an important fact for a Hilbert $\mathbb{A}$-module. Since the proof of the lemma does not require the completeness of the Hilbert $\mathbb{A}$-module, we can remove the condition and apply the new version of the lemma to the reverse implication of Theorem 3.6 in [18]. Therefore we have the sufficient conditions to make the C^*-valued function $\|\cdot\|_X$ induced by $\langle\cdot,\cdot\rangle$ satisfy the triangle inequality.

Proposition 4.3. *Let $(X, \mathbb{A}, \langle\cdot,\cdot\rangle)$ be an inner product C^*-module. If the closed two-side ideal $\overline{\langle X, X\rangle}$ of $\mathbb{A}$ is commutative, then $\|\cdot\|_X$ satisfies the triangle inequality.*

Corollary 4.4. *Let $(X, \mathbb{A}, \langle\cdot,\cdot\rangle)$ be an inner product C^*-module with a commutative C^*-algebra $\mathbb{A}$. Then $\|\cdot\|_X$ satisfies the triangle inequality.*

Theorem 4.5. *Let $(X, \mathbb{A}, \langle\cdot,\cdot\rangle)$ be an inner product C^*-module. If the closed two-side ideal $\overline{\langle X, X\rangle}$ of $\mathbb{A}$ is commutative, then the completion of $(X, \mathbb{A}, \langle\cdot,\cdot\rangle)$ is an $\mathbb{A}$-valued Banach completion of $(X, \mathbb{A}, \|\cdot\|_X)$.*

Proof. By Proposition 4.3, $(X, \mathbb{A}, \langle\cdot,\cdot\rangle)$ become an $\mathbb{A}$-valued norm space with the $\mathbb{A}$-valued norm $\|\cdot\|_X$ induced by $\langle\cdot,\cdot\rangle$. Let $(X_c, \mathbb{A}, \langle\cdot,\cdot\rangle_c)$ be a completion of $(X, \mathbb{A}, \langle\cdot,\cdot\rangle)$. Now apply denseness of X in X_c together with joint continuity of multiplication on $\mathbb{A}$, so $\overline{\langle X_c, X_c\rangle_c}$ is commutative. Then $(X_c, \mathbb{A}, \langle\cdot,\cdot\rangle_c)$ becomes an $\mathbb{A}$-valued norm space with the $\mathbb{A}$-valued norm $\|\cdot\|_{X_c}$ induced by $\langle\cdot,\cdot\rangle_c$. By Theorem 4.2 $(X_c, \mathbb{A}, \|\cdot\|_{X_c})$ is actually an $\mathbb{A}$-valued Banach space. Suppose that $x \in X_c$ and $\{x_n\}$ is a sequence in X converging to x by the norm on X extended from $\|\cdot\|_m$. Then

$$\|x\|_{X_c} = \langle x, x\rangle_c^{1/2} = \lim_{n\to\infty} \langle x_n, x_n\rangle^{1/2} = \lim_{n\to\infty} \|x_n\|_X.$$

Certainly, the equality implies that the norm $\|\cdot\|_X$ is the restriction of $\|\cdot\|_{X_c}$ on X. Also the inequality implies the denseness of X in X_c under the norm $\|\cdot\|_{X_c}$. Therefore $(X_c, \mathbb{A}, \|\cdot\|_{X_c})$ is a $\mathbb{A}$-valued Banach completion of $(X, \mathbb{A}, \|\cdot\|_X)$. ∎

Acknowledgements

We would like to thank the referees for their comments and suggestions on the manuscript. This research was supported by Nakhon Sawan Rajabhat University, Thailand.

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