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Coincidence and Common Fixed Point Results in G-Metric Spaces using Generalized Cyclic Contraction

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Abstract Here, we have established the generalized cyclic contractive condition in G-metric spaces which can't be reduced to the contractive condition in standard metric spaces. The coincidence and common fixed point results are obtained for the pair of (A, B)-weakly increasing mappings in G-metric spaces.

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Keywords: G-metric spaces; cyclic maps; common fixed point

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1. Introduction

In 2006, Mustafa and Sims [1] introduced the notion of G-metric spaces. After that many researchers established fixed point and common fixed point results in G-metric spaces. Jleli and Samet [2], Samet et al. [3] have shown that G-metric space has a quasi-metric type structure and then many results on such spaces are derived from quasi-metric spaces.

The notion of cyclic mappings was introduced by Kirk et al. [4] and proved fixed point results for cyclic mappings. Such results are generalized by Shatanawi and Postolache [5] by introducing the notion of (A, B)-weakly increasing maps.

Shatanawi and Abodayeh [6] introduced new contractive condition and proved fixed point and common fixed point results in G-metric spaces for which the techniques of Jleli and Samet [2], Samet et al. [3] can't be used to reduce the contractive condition to metric spaces.

In this paper, we have dropped the continuity condition and used $\psi \in \Psi$ instead of $\psi \in \Phi$ and generalized the contractive condition of Shatanawi and Abodayeh [6] for cyclic mappings and proved common fixed point result in G-metric spaces for the pair of (A, B)-weakly increasing mappings and some illustrative examples are given. Note that

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the generalized cyclic contractive condition can't be reduced to contractive conditions in standard metric spaces.

2. Preliminaries

Notations:

(1) Ψ is the family of all mappings $\psi : [0,\infty) \to [0,\infty)$ verifying: if $\{t_m\}_{m\in\mathbb{N}} \subset [0,\infty)$ and $\psi(t_m) \to 0$ then $t_m \to 0$.
(ii) Φ is the family of all altering distance functions.

Definition 2.1. An altering distance function is a continuous, non-decreasing mapping $\phi : [0,\infty) \to [0,\infty)$ such that $\phi^{-1}(0) = 0$.

Remark 2.2. $\Phi \subset \Psi$.

Lemma 2.3 ([7])**.** *Let $\phi \in \Phi, \psi \in \Psi$ and $t_n \subset [0,\infty)$ be a sequence such that $\phi(t_{n+1}) \leq \phi(t_n) - \psi(t_n)$, for all $n \in \mathbb{N}$, then $t_n \to 0$.*

Definition 2.4 ([1])**.** Let X be a nonempty set. Let $G : X \times X \times X \to \mathbb{R}^+$ be a function satisfying the following properties:

(G_1) $G(x,y,z) = 0$, if $x = y = z$,
(G_2) $G(x,x,y) > 0$, for all $x,y \in X$ with $x \neq y$,
(G_3) $G(x,x,y) \leq G(x,y,z)$; for all $x,y,z \in X$ with $z \neq y$,
(G_4) $G(x,y,z) = G(x,z,y) = G(y,z,x) = ...$ (symmetry in all three variables),
(G_5) $G(x,y,z) \leq G(x,a,a) + G(a,y,z)$; for all $x,y,z,a \in X$(rectangle inequality).

The function G is called G-metric on X and the pair (X,G) is called a G-metric space.

Definition 2.5. A G-metric space (X,G) is said to be symmetric if $G(x,y,y) = G(y,x,x)$; for all $x,y \in X$.

Lemma 2.6. *If (X,G) is a G-metric space, then*

$$G(x,y,y) \leq 2G(y,x,x), \;\; for\ all\ x,y \in X.$$

Definition 2.7. Let (X,G) be a G-metric space, let $x \in X$ be a point and let $\{x_n\} \subseteq X$ be a sequence. We say that:

(1) $\{x_n\}$ G-converges to x, and we write $\{x_n\} \to x$, if $\lim\limits_{n,m\to\infty} G(x_n,x_m,x) = 0$, that is, for all $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ satisfying $G(x_n,x_m,x) \leq \varepsilon$ for all $n,m \geq n_0$. (In such a case, x is the G-limit of x_n).
(2) $\{x_n\}$ is G-cauchy if $\lim\limits_{n,m,k\to\infty} G(x_n,x_m,x_k) = 0$, that is, for all $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ satisfying $G(x_n,x_m,x_k) \leq \varepsilon$ for all $n,m,k \geq n_0$.
(3) (X,G) is complete if every G-Cauchy sequence in X is G-convergent in X.

Proposition 2.8. *Let (X,G) be a G-metric space, let $\{x_n\} \subseteq X$ be a sequence and let $x \in X$. Then the following are equivalent.*

(a) $\{x_n\}$ *G-converges to* x,
(b) $\lim\limits_{n\to\infty} G(x_n,x_n,x) = 0$,
(c) $\lim\limits_{n\to\infty} G(x_n,x,x) = 0$.

Proposition 2.9. *Let (X,G) be a G-metric space, let $\{x_n\} \subseteq X$ be a sequence and let $x \in X$. Then the following are equivalent.*

(a) $\{x_n\}$ *is* G*-Cauchy,*
(b) $\lim_{n,m\to\infty} G(x_n, x_m, x_m) = 0.$

Definition 2.10. A sequence $\{x_n\}$ in a G-metric space (X, G) is asymptotically regular if $\lim_{n\to\infty} G(x_n, x_{n+1}, x_{n+1}) = 0$.

Lemma 2.11 ([8],Lemma 4.1.5)**.** *Let* $\{x_n\}$ *be an asymptotically regular sequence in a* G*-metric space* (X, G) *and suppose that* $\{x_n\}$ *is not Cauchy. Then there exists a positive real number* $\varepsilon > 0$ *and two subsequences* $\{x_{n_k}\}$ *and* $\{x_{m_k}\}$ *of* $\{x_n\}$ *such that, for all* $k \in \mathbb{N}$,

$$k \le n_k < m_k < n_{k+1},$$

$$G(x_{n_k}, x_{n_k+1}, x_{m_k-1}) \le \varepsilon < G(x_{n_k}, x_{n_k+1}, x_{m_k})$$

and also, for all given $p_1, p_2, p_3 \in \mathbb{Z}$,

$$\lim_{n\to\infty} G(x_{n_k+p_1}, x_{m_k+p_2}, x_{m_k+p_3}) = \varepsilon.$$

Definition 2.12. Let (X, G) be a G-metric space. We say that a mapping $T : X \to X$ is G-continuous at $x \in X$ if $\{Tx_m\} \to Tx$ for all sequence $\{x_m\} \subseteq X$ such that $\{x_m\} \to x$.

In 2013, Shatanawi and Postolache [5] introduced (A, B)-weakly increasing functions for pair of mappings.

Definition 2.13. Let $(X, \preceq)$ be a partially ordered set and A, B be two closed subsets of X with $X = A \cup B$. Let $f, g : X \to X$ be two mappings. Then the pair (f, g) is said to be (A, B)-weakly increasing if $fx \preceq gfx$ for all $x \in A$ and $gx \preceq fgx$ for all $x \in B$.

Shatanawi and Abodayeh [6] introduced a new contractive condition by utilizing the notion of (A, B)- weakly increasing mappings and using auxiliary functions from Φ, proved the following common fixed point result in G-metric spaces.

Theorem 2.14. *Let* $\preceq$ *be an ordered relation in a set* X*. Let* (X, G) *be a complete* G*-metric space and* $X = A \cup B$*, where* A *and* B *are nonempty closed subsets of* X*. Let* f, g *be self mappings on* X *that satisfy the following conditions:*

(1) The pair (f, g) *is* (A, B)*-weakly increasing.*
(2) $f(A) \subseteq B$ *and* $g(B) \subseteq A$*.*
(3) There exist two functions $\phi, \psi \in \Phi$ *such that*

$$\phi(G(fx, gfx, gy)) \le \phi(G(x, fx, y)) - \psi(G(x, fx, y))$$

holds for all comparative elements $x, y \in X$ *with* $x \in A$ *and* $y \in B$ *and*

$$\phi(G(gx, fgx, fy)) \le \phi(G((x, gx, y))) - \psi(G(x, gx, y))$$

holds for all comparative elements $x, y \in X$ *with* $x \in B$ *and* $y \in A$*.*

(4) f *or* g *is continuous.*

Then, f *and* g *have a common fixed point in* $A \cap B$*.*

3. Main results

Here, we have considered functions $\psi \in \Psi$ and generalized the contractivity condition of Theorem 2.14 and proved common fixed point theorems in G-metric spaces.

Theorem 3.1. *Let $\preceq$ be an ordered relation in a set X. Let (X, G) be a complete G-metric space and $X = A \cup B$, where A and B are nonempty closed subsets of X. Let f, g be self mappings on X that satisfy the following conditions:*

(1) The pair (f, g) is (A, B)-weakly increasing.
(2) $f(A) \subseteq B$ and $g(B) \subseteq A$.
(3) There exist two functions $\phi \in \Phi, \psi \in \Psi$ such that

$$\phi(G(fx, gfx, gy)) \leq \phi(M(x, y)) - \psi(M(x, y)) \tag{3.1}$$

holds for all comparative elements $x, y \in X$ with $x \in A$ and $y \in B$ and

$$\phi(G(gx, fgx, fy)) \leq \phi(M'(x, y)) - \psi(M'(x, y)) \tag{3.2}$$

holds for all comparative elements $x, y \in X$ with $x \in B$ and $y \in A$, where

$$\begin{aligned} M(x, y) =max\Big\{ & G(x, fx, y), G(x, fx, fx), G(y, gy, gy), \\ & \frac{1}{2}\Big(G(fx, fx, gy), G(x, gfx, gy), G(fx, gfx, y)\Big)\Big\} \end{aligned}$$

and

$$\begin{aligned} M'(x, y) =max\Big\{ & G(x, gx, y), G(x, gx, gx), G(y, fy, fy), \\ & \frac{1}{2}\Big(G(gx, gx, fy), G(x, fgx, fy), G(gx, fgx, y)\Big)\Big\}. \end{aligned}$$

(4) f or g is continuous.

Then, f and g have a common fixed point in $A \cap B$.

Proof. Since A is nonempty, start with $x_0 \in A$. In view of condition (2), we can construct a sequence $\{x_n\}$ in X such that $fx_{2n} = x_{2n+1},$ *for* $x_{2n} \in A$ *and* $gx_{2n+1} = x_{2n+2},$ *for* $x_{2n+1} \in B, n \in \mathbb{N}$.
By condition (1), we have $x_n \preceq x_{n+1}$, for all $n \in \mathbb{N}$. If $x_{2n_0} = x_{2n_0+1}$ for some $n_0 \in \mathbb{N}$, then x_{2n_0} is a fixed point of f in $A \cap B$. Since $x_{2n_0} \preceq x_{2n_0+1}$, by condition (3), we have

$$\begin{aligned} \phi(G(x_{2n_0+1}, x_{2n_0+2}, x_{2n_0+2})) &= \phi(G(fx_{2n_0}, gfx_{2n_0}, gx_{2n_0+1})) \\ &\leq \phi(M(x_{2n_0}, x_{2n_0+1})) - \psi(M(x_{2n_0}, x_{2n_0+1})), \end{aligned} \tag{3.3}$$

where

$$\begin{aligned} &M(x_{2n_0}, x_{2n_0+1}) \\ &= max\Big\{G(x_{2n_0}, fx_{2n_0}, x_{2n_0+1}), G(x_{2n_0}, fx_{2n_0}, fx_{2n_0}), \\ &\qquad G(x_{2n_0+1}, gx_{2n_0+1}, gx_{2n_0+1}), \frac{1}{2}\Big(G(fx_{2n_0}, fx_{2n_0}, gx_{2n_0+1}), \\ &\qquad G(x_{2n_0}, gfx_{2n_0}, gx_{2n_0+1}), G(fx_{2n_0}, gfx_{2n_0}, x_{2n_0+1})\Big)\Big\} \\ &= max\Big\{G(x_{2n_0}, x_{2n_0+1}, x_{2n_0+1}), G(x_{2n_0+1}, x_{2n_0+2}, x_{2n_0+2}), \\ &\qquad \frac{1}{2}\Big(G(x_{2n_0+1}, x_{2n_0+1}, x_{2n_0+2}), G(x_{2n_0}, x_{2n_0+2}, x_{2n_0+2})\Big)\Big\}. \end{aligned}$$

Using Lemma 2.6, we obtain

$$G(x_{2n_0+1}, x_{2n_0+1}, x_{2n_0+2}) \leq 2G(x_{2n_0+1}, x_{2n_0+2}, x_{2n_0+2}),$$

and by rectangle inequality (G_5), we get

$$G(x_{2n_0}, x_{2n_0+2}, x_{2n_0+2}) \leq G(x_{2n_0}, x_{2n_0+1}, x_{2n_0+1}) + G(x_{2n_0+1}, x_{2n_0+2}, x_{2n_0+2}).$$

Then,

$$\begin{aligned} M(x_{2n_0}, x_{2n_0+1}) &= max\{G(x_{2n_0}, x_{2n_0+1}, x_{2n_0+1}), G(x_{2n_0+1}, x_{2n_0+2}, x_{2n_0+2})\} \\ &= G(x_{2n_0+1}, x_{2n_0+2}, x_{2n_0+2}). \end{aligned}$$

From (3.3), we have

$$\begin{aligned} \phi(G(x_{2n_0+1}, x_{2n_0+2}, x_{2n_0+2})) \leq &\phi(G(x_{2n_0+1}, x_{2n_0+2}, x_{2n_0+2})) \\ &- \psi(G(x_{2n_0+1}, x_{2n_0+2}, x_{2n_0+2})). \end{aligned}$$

Implies $\psi(G(x_{2n_0+1}, x_{2n_0+2}, x_{2n_0+2})) = 0$. Since $\psi \in \Psi$, we have

$$G(x_{2n_0+1}, x_{2n_0+2}, x_{2n_0+2}) = 0$$

and $x_{2n_0+1} = x_{2n_0+2}$. So, we get $x_{2n_0} = x_{2n_0+1} = x_{2n_0+2}$. Therefore, x_{2n_0} is a fixed point of g in $A \cap B$. Hence, x_{2n_0} is a common fixed point of f and g in $A \cap B$.
Now, we assume that $x_{n+1} \neq x_n$, for all $n \in \mathbb{N}$. Since, $x_{2n} \preceq x_{2n+1}$, for all $n \in \mathbb{N}$, by condition (3) we have

$$\begin{aligned} \phi(G(x_{2n+1}, x_{2n+2}, x_{2n+2})) &= \phi(G(fx_{2n}, gfx_{2n}, gx_{2n+1})) \\ &\leq \phi(M(x_{2n}, x_{2n+1})) - \psi(M(x_{2n}, x_{2n+1})), \end{aligned} \tag{3.4}$$

where

$$M(x_{2n}, x_{2n+1}) = max\left\{G(x_{2n}, x_{2n+1}, x_{2n+1}), G(x_{2n+1}, x_{2n+2}, x_{2n+2})\right\}.$$

Case i: If $M(x_{2n}, x_{2n+1}) = G(x_{2n+1}, x_{2n+2}, x_{2n+2})$, then by (3.4), we get

$$\begin{aligned} \phi(G(x_{2n+1}, x_{2n+2}, x_{2n+2})) \leq &\phi(G(x_{2n+1}, x_{2n+2}, x_{2n+2})) \\ &- \psi(G(x_{2n+1}, x_{2n+2}, x_{2n+2})). \end{aligned}$$

Therefore, $\psi(G(x_{2n+1}, x_{2n+2}, x_{2n+2})) = 0$, for all $n \in \mathbb{N}$. By taking limit as $n \to \infty$, we get

$$\lim_{n\to\infty} \psi(G(x_{2n+1}, x_{2n+2}, x_{2n+2})) = 0.$$

Since $\psi \in \Psi$, we have

$$\lim_{n\to\infty} G(x_{2n+1}, x_{2n+2}, x_{2n+2}) = 0. \tag{3.5}$$

Case ii: If $M(x_{2n}, x_{2n+1}) = G(x_{2n}, x_{2n+1}, x_{2n+1})$. From (3.4), we have

$$\phi(G(x_{2n+1}, x_{2n+2}, x_{2n+2})) \leq \phi(G(x_{2n}, x_{2n+1}, x_{2n+1})) - \psi(G(x_{2n}, x_{2n+1}, x_{2n+1})). \tag{3.6}$$

By Lemma 2.3, we get

$$\lim_{n\to\infty} G(x_{2n}, x_{2n+1}, x_{2n+1}) = 0. \tag{3.7}$$

From (3.5) and (3.7), we obtain that for all $n \in \mathbb{N}$

$$\lim_{n\to\infty} G(x_n, x_{n+1}, x_{n+1}) = 0. \tag{3.8}$$

From definition of G-metric spaces, we have

$$\lim_{n\to\infty} G(x_n, x_n, x_{n+1}) = 0. \tag{3.9}$$

That is, $\{x_n\}$ is asymptotically regular sequence. Now, we prove that $\{x_n\}$ is G-Cauchy. It is sufficient to show that $\{x_{2n}\}$ is a G-Cauchy sequence. Suppose on contrary that is not. Then by (3.8), (3.9) and Lemma 2.11 there exists $\varepsilon > 0$ and two subsequences $\{x_{2n_k}\}$ and $\{x_{2m_k}\}$ of $\{x_{2n}\}$ such that, for all $k \in \mathbb{N}$, $k \leq 2n_k < 2m_k < 2n_{k+1}$ and for all given $p_1, p_2, p_3 \in \mathbb{Z}$,

$$\lim_{n\to\infty} G(x_{2n_k+p_1}, x_{2m_k+p_2}, x_{2m_k+p_3}) = \varepsilon. \tag{3.10}$$

Since, $x_{2m_k} \preceq x_{2n_k+1}$, by using condition (3), we get

$$\begin{aligned}\phi(G(x_{2m_k+1}, x_{2m_k+2}, x_{2n_k+2})) &= \phi(G(fx_{2m_k}, gfx_{2m_k}, gx_{2n_k+1}))\\ &\leq \phi(M(x_{2m_k}, x_{2n_k+1})) - \psi(M(x_{2m_k}, x_{2n_k+1})),\end{aligned} \tag{3.11}$$

where

$$\begin{aligned}&M(x_{2m_k}, x_{2n_k+1})\\ &= max\Big\{G(x_{2m_k}, fx_{2m_k}, x_{2n_k+1}), G(x_{2m_k}, fx_{2m_k}, fx_{2m_k}),\\ &\qquad G(x_{2n_k+1}, gx_{2n_k+1}, gx_{2n_k+1}), \frac{1}{2}\Big(G(fx_{2m_k}, fx_{2m_k}, gx_{2n_k+1}),\\ &\qquad G(x_{2m_k}, gfx_{2m_k}, gx_{2n_k+1}), G(fx_{2m_k}, gfx_{2m_k}, x_{2n_k+1})\Big)\Big\}\\ &= max\Big\{G(x_{2m_k}, x_{2m_k+1}, x_{2n_k+1}), G(x_{2m_k}, x_{2m_k+1}, x_{2m_k+1}),\\ &\qquad G(x_{2n_k+1}, x_{2n_k+2}, x_{2n_k+2}), \frac{1}{2}\Big(G(x_{2m_k+1}, x_{2m_k+1}, x_{2n_k+2}),\\ &\qquad G(x_{2m_k}, x_{2m_k+2}, x_{2n_k+2}), G(x_{2m_k+1}, x_{2m_k+2}, x_{2n_k+1})\Big)\Big\}.\end{aligned}$$

By using (3.8), (3.9) and (3.10), we get $\lim_{k\to\infty} M(x_{2m_k}, x_{2n_k+1}) = max\{\varepsilon, 0, \frac{\varepsilon}{2}\} = \varepsilon$. Take $\{t_k = G(x_{2m_k+1}, x_{2m_k+2}, x_{2n_k+2})\}, \{s_k = M(x_{2m_k}, x_{2n_k+1})\}$. Then $\{t_k\}$ and $\{s_k\}$ are sequences converging to the same limit ε and they satisfy $\phi(t_k) \leq \phi(s_k) - \psi(s_k)$, for

all k.
Therefore, $\psi(s_k) \leq \phi(s_k) - \phi(t_k)$.
By taking limit as $k \to \infty$, since $\phi \in \Phi$, we have

$$\lim_{k\to\infty} \psi(s_k) \leq \phi(\varepsilon) - \phi(\varepsilon) = 0.$$

Since $\psi \in \Psi$, $\lim_{k\to\infty} s_k = 0$. This implies that $\varepsilon = 0$, a contradiction. Thus, $\{x_{2n}\}$ is G-Cauchy. So, sequence $\{x_n\}$ is G-Cauchy. Since (X, G) is complete, there exists $u \in X$ such that $\{x_n\}$ is G-convergent to u. Therefore, the subsequences $\{x_{2n+1}\}$ *and* $\{x_{2n}\}$ are G-convergent to u.
Since $\{x_{2n}\} \subseteq A$ and A is closed, implies $u \in A$. Also, $\{x_{2n+1}\} \subseteq B$ and B is closed, implies $u \in B$. We may assume that f is continuous. So, we have $\lim_{n\to\infty} fx_{2n} = fu$ *and* $\lim_{n\to\infty} fx_{2n} = \lim_{n\to\infty} x_{2n+1} = u$. By uniqueness of the limit we have $fu = u$. Since $u \preceq u$, by condition (3) we have

$$\begin{aligned}\phi(G(u, gu, gu)) &= \phi(G(fu, gfu, gu)) \\ &\leq \phi(M(u, u)) - \psi(M(u, u)), \end{aligned} \tag{3.12}$$

where

$$\begin{aligned} M(u, u) = max&\Big\{G(u, fu, u), G(u, fu, fu), G(u, gu, gu), \\ &\frac{1}{2}\Big(G(fu, fu, gu), G(u, gfu, gu), G(fu, gfu, u)\Big)\Big\} \\ &= max\{G(u, u, u), G(u, gu, gu), \frac{1}{2}(G(u, u, gu), G(u, gu, gu), G(u, gu, u))\} \\ &= max\{G(u, gu, gu), \frac{1}{2}G(u, gu, gu)\} \\ &= G(u, gu, gu). \end{aligned}$$

Using (3.12), we obtain

$$\begin{aligned}\phi(G(u, gu, gu)) &= \phi(G(fu, fgu, gu)) \\ &\leq \phi(G(u, gu, gu)) - \psi(G(u, gu, gu)). \end{aligned}$$

Therefore, $\psi(G(u, gu, gu)) = 0$. Implies, $G(u, gu, gu) = 0$. Hence, $gu = u$. Thus, u is a common fixed point of f and g in $A \cap B$. ∎

Corollary 3.2. *Let $\preceq$ be an ordered relation in a set X. Let (X, G) be a complete G-metric space and $X = A \cup B$, where A and B are nonempty closed subsets of X. Let f be a continuous self map on X that satisfy the following conditions:*

(1) $fx \preceq f^2x$, for all $x \in X$.
(2) $f(A) \subseteq B$ and $f(B) \subseteq A$.
(3) There exist two functions $\phi \in \Phi, \psi \in \Psi$ such that

$$\phi(G(fx, f^2x, fy)) \leq \phi(M(x, y)) - \psi(M(x, y)) \tag{3.13}$$

holds for all comparative elements $x, y \in X$, where

$$\begin{aligned} M(x, y) = max\{&G(x, fx, y), G(x, fx, fx), G(y, fy, fy), \\ &\frac{1}{2}(G(fx, fx, fy), G(x, f^2x, fy), G(fx, f^2x, y))\}. \end{aligned}$$

Then, f has a fixed point in $A \cap B$.

Proof. The proof follows from Theorem 3.1 by taking $g = f$. ∎

To support the usability of our result, following example is stated.

Example 3.3. Let $X = [0,1]$ and let $f : X \to X$ be given as $f(x) = \frac{x^2}{1+x}$. Take $A = [0, \frac{1}{2}]$ *and* $B = [0,1]$. Define the function $G : X \times X \times X \to [0,\infty)$ as

$$G(x,y,z) = \begin{cases} 0, & if\ x = y = z, \\ max\{x,y,z\}, & otherwise. \end{cases}$$

Clearly, G is a complete G-metric on X. We introduce a relation on X by $x \preceq y$ if and only if $y \leq x$. Also, define the functions $\phi, \psi : [0,\infty) \to [0,\infty)$ *by* $\phi(t) = 2t$ *and* $\psi(t) = \frac{t}{1+2t}$.

Note that $fA = [0, \frac{1}{6}] \subseteq B$ *and* $fB = [0, \frac{1}{2}] \subseteq A$.

To prove (1), given $x \in X$,

$$f^2x = \frac{x^2}{(1+x)} \frac{x^2}{(1+x+x^2)}.$$

Since $x \in [0,1]$, $\frac{x^2}{(1+x+x^2)} < 1$. Thus, $f^2x \leq fx$ and hence $fx \preceq f^2x$ for all $x \in X$. To prove (3), given $x, y \in X$ with $x \geq y$. Then,

$$G(fx, f^2x, fy) = max\left\{\frac{x^2}{(1+x)}, \frac{x^2}{(1+x)} \frac{x^2}{(1+x+x^2)}, \frac{y^2}{(1+y)}\right\} = \frac{x^2}{(1+x)}$$

and

$$M(x,y) = max\left\{x, y, \frac{x^2}{2(1+x)}, \frac{x}{2}\right\} = x.$$

Since

$$\frac{2x^2}{(1+x)} \leq 2x - \frac{x}{(1+2x)},$$

we have

$$\phi(G(fx, f^2x, fy)) \leq \phi(M(x,y)) - \psi(M(x,y)).$$

Hence, all the conditions of Corollary 3.2 are satisfied. Notice that 0 is the unique fixed point of f.

In Theorem 2.14, we drop the condition of continuity and $\psi(0) = 0$ and replace $\psi \in \Phi$ with $\psi \in \Psi$, then we get the following result.

Theorem 3.4. *Let $\preceq$ be an ordered relation in a set X. Let (X,G) be a complete G-metric space and $X = A \cup B$, where A and B are nonempty closed subsets of X. Let f, g be self mappings on X that satisfy the following conditions:*

(1) The pair (f,g) is (A,B)-weakly increasing.
(2) $f(A) \subseteq B$ and $g(B) \subseteq A$.
(3) There exist two functions $\phi \in \Phi, \psi \in \Psi$ such that

$$\phi(G(fx, gfx, gy)) \leq \phi(G(x, fx, y)) - \psi(G(x, fx, y))$$

holds for all comparative elements $x, y \in X$ with $x \in A$ and $y \in B$ and

$$\phi(G(gx, fgx, fy)) \leq \phi(G((x, gx, y))) - \psi(G(x, gx, y))$$

holds for all comparative elements $x, y \in X$ with $x \in B$ and $y \in A$.

(4) f or g is continuous.

Then, f and g have a common fixed point in $A \cap B$.

Proof. By taking $M(x,y) = G(x, fx, y)$ and $M'(x,y) = G(x, gx, y)$ in Theorem 3.1 and using similar argument the result can be proved.

■

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