Thai Journal of Mathematics
Volume 20 Number 3 (2022)
Pages 1099–1107

http://thaijmath.in.cmu.ac.th

ISSN 1686-0209

On Some Fixed Point Theorem in Generalized Complex Valued Metric Spaces for BKC-Contraction

Shantanu Bhaumik[1,*], **Devnarayan Yadav**[2] **and Surendra Kumar Tiwari**[1]

[1] *Department of Mathematics, Dr. C. V. Raman University, Kota, Bilaspur (Chattisgarh)-India*
e-mail : santbhk@gmail.com (S. Bhaumik); sk10tiwari@gmail.com (S.K. Tiwari)

[2] *Assistant Teacher, Mathmatics, Govt. School Shilphari, Bilaspur*
e-mail : dnyadavbsp@gmail.com (D. Yadav)

Abstract In this paper, we introduce a new concept of BKC-contraction in generalized complex valued metric spaces for some partial order and establish a fixed point theorem. In addition an example is present which illustrates our main result.

MSC: 74H10; 54H25

Keywords: fixed point; BKC-contraction; metric space; generalized complex valued metric space

Submission date: 27.10.2021 / Acceptance date: 11.01.2022

1. Introduction and Basic Concepts

The first important and significant result was proved by Banach [1] in 1922 for contraction mapping in complete metric space. The Banach principle asserts, if (X, d) is complete metric space and $T : X \to X$ satisfies

$$d(Tx, Ty) \leq \alpha d(x, y), \tag{1.1}$$

for all $x, y \in X$, with $\alpha \in [0, 1)$, then T has a unique fixed point.
This principle has been improved and extended by several mathematicians in different directions. Some of them are as follows:
Let T be a mapping on a metric space(X, d) and $x, y \in X$, then T is said to be a Kannan type contraction [2], if there exists a number $\lambda \in [0, \frac{1}{2})$ such that

$$d(Tx, Ty) \leq \lambda[d(x, Tx) + d(y, Ty)]. \tag{1.2}$$

Chatterjea type contraction [3], if there exists a number $\lambda \in [0, \frac{1}{2})$ such that

$$d(Tx, Ty) \leq \lambda[d(x, Ty) + d(y, Tx)]. \tag{1.3}$$

*Corresponding author.

Published by The Mathematical Association of Thailand.

Reich type contraction [4], if there exists a number $\alpha, \mu, \gamma \in [0,1)$ with $\lambda + \mu + \gamma < 1$, such that

$$d(Tx, Ty) \leq \lambda d(x, y) + \mu d(x, Tx) + \gamma d(y, Ty). \tag{1.4}$$

Inspired by the impact of this natural idea, the above contractions have been extended and generalized by several researchers in various spaces such as quasi metric space, Cone metric space, G-metric space, Partial metric space, and Complex valued metric space and so on. Following this trend in 2011, Azam et al. [5], introduced complex valued metric spaces and obtained common fixed point theorem on complex valued metric space satisfying a contractive condition.

Recently, Jleli and Samet [6] introduced concept of generalized metric space, which covers different well known structures. In 2017, Elkouch and Marhrani [7] extended the Theorems (1.2) and (1.3) with $\lambda = \frac{1}{3}$ in (X, d) and they proved that a mapping T has a fixed point x and also they introduced Hardy-Rogers contraction in this space.
Very recently, Issara Inchan and Urairat Deepan [8] defined and introduced a very interesting concept of a generalized complex valued metric space and considered the general Hardy-Rogers contraction and proved some fixed point results on X.

In particular, motivated by Issara and Deepan [8], we define and introduce the BKC-contraction for some partial order relation, if there exists non negative real number $\lambda \in [0, \frac{1}{2})$ such that for any $x, y \in X$ satisfying

$$D(fx, fy) \leq \lambda max\{2D(x, y), d(x, Tx) + D(y, Ty), D(x, Ty) + d(y, Ty)\}. \tag{1.5}$$

Then we claim that a mapping $f : X \to X$ satisfying (1.5) has a fixed point on X.
Next, we begin the notations, definitions for this work.

Let $\mathbb{C}$ be the set of complex numbers and $z_1, z_2, \in \mathbb{C}$. Define a partial order relation $\precsim$ on $\mathbb{C}$ as follows:
$z_1 \precsim z_2$ if and only if $Re(z_1) \leq Re(z_2)$ and $Im(z_1) \leq Im(z_2)$. Thus we can say that $z_1 \precsim z_2$ if one of the following condition hold

(i) $Re(z_1) = Re(z_2)$ and $Im(z_1) = Im(z_2)$.
(ii) $Re(z_1) < Re(z_2)$ and $Im(z_1) = Im(z_2)$.
(iii) $Re(z_1) = Re(z_2)$ and $Im(z_1) < Im(z_2)$.
(iv) $Re(z_1) < Re(z_2)$ and $Im(z_1) < Im(z_2)$.

In particular, we will write $z_1 \precnsim z_2$ if $z_1 \neq z_2$ and one of (ii),(iii) and (iv) is satisfied and we will write $z_1 \prec z_2$ only (iv) is satisfied.

Remark 1.1. We can easily to check the following:

(1) If $a, b \in R; 0 \leq a \leq b$ and $z_1 \precsim z_2$ then $az_1 \precsim bz_2$, for all $z_1, z_2 \in \mathbb{C}$.
(2) $0 \precsim z_1 \precnsim z_2 \Rightarrow |z_1| < |z_2|$.
(3) $z_1 \precsim z_2$ and $z_2 \prec z_3 \Rightarrow z_1 \prec z_3$.

Azam et al. [5] defined the complex valued metric space in the following way:

Definition 1.2. ([5]) Let X be a nonempty set. Suppose that the mapping $d : X \times X \to C$ satisfies the following the following conditions :

(1) $0 \precsim d(x, y)$, for all $x, y \in X$ and $d(x, y) = 0$.
(2) $d(x, y) = d(y, x)$, for all $x, y \in X$.
(3) $d(x, y) \precsim d(x, z) + d(z, y)$; for all $x, y, z \in X$.

Then d is called a complex valued metric on X and (X, d) is called a complex valued metric space.
Elkouch and Marhrani [7] defined a new class metric space as follows: Let X be non empty set, $D : X \times X \to [0, +\infty)$ be a given mapping. For every $x \in X$, define the set

$$C(D, X, x) = \Big\{\{x_n\} \subseteq X : \lim_{n\to\infty} D(x_n, x) = 0\Big\}.$$

Definition 1.3. ([7]) A mapping $D : X \times X \to X$ is called a generalized metric if it satisfies the following conditions:

(1) For every $(x, y) \in X \times X$, we have
$$D(x, y) = 0 \Rightarrow x = y.$$

(2) For every $(x, y) \in X \times X$, we have
$$D(x, y) = D(y, x).$$

(3) There exists a real constant $c > 0$ such that for all $(x, y) \in X \times X$ and $\{x_n\} \in C(D, X, x)$ we have
$$D(x, y) \leq c \lim_{n\to\infty} sup D(x_n, y).$$

The pair (X, D) is called a generalized metric spaces.

In this work, we consider a nonempty set X, and $D : X \times X \to \mathbb{C}$ be a given mapping. For every $x \in X$, we define the set

$$C(D, X, x) = \{\{x_n\} \subseteq X : \lim_{n\to\infty} |D(x_n, x)| = 0\}.$$

Definition 1.4. ([8]) Let X be non empty set, a mapping $D : X \times X \to \mathbb{C}$ is called a generalized complex valued metric if it satisfied the following conditions:

(C_1) For every $x, y \in X$, we have
$$0 \precsim D(x, y).$$

(C_2) For every $x, y \in X$, we have
$$D(x, y) = 0 \Rightarrow x = y.$$

(C_3) For all $x, y, \in X$, we have
$$D(x, y) = D(y, x).$$

(C_4) There exists a complex constant $0 \prec r$ such that for all $x, y \in X$ and $\{x_n\} \in C(D, X, x)$, we have
$$D(x, y) \leq r \lim_{n\to\infty} \sup |D(x_n, y)|.$$

Then, the pair (X, D) is called a generalized complex valued metric space.

Definition 1.5. ([8]) Let (X, D) be a generalized complex valued metric space, let $\{x_n\}$ be a sequence in X, and let $x \in X$. We say that $\{x_n\}$ is D-converges to x in X, if $\{x_n\} \in C(D, X, x)$. We denote $\lim_{n\to\infty} x_n = x$.

Definition 1.6. ([8]) Let (X, D) be a generalized complex valued metric space. Then a sequence $\{x_n\}$ in X is said to be D-Cauchy sequence in X, if $\lim_{n\to\infty} |D(x_n, x_{n+m})| = 0$.

Definition 1.7. ([8]) Let (X, D) be a generalized complex valued metric space. If every D-Cauchy sequence in X is D-converges in X, then (X, D) is called a D-complete complex valued metric space.

Now, we define BKC-contraction which is generalization of Banach contraction, Kannan and Chatterjea contractions on generalized complex valued metric space as follows:

Definition 1.8. Led (X, D) be a generalized complex valued metric space. A self mapping $T : X \to X$ is called BKC-contraction, if there exists non negative real constant $\lambda \in [0, \frac{1}{2})$ such that for any $x, y \in X$, satisfying

$$D(Tx, Ty) \precsim \lambda\, max\{2D(x, y),\ D(x, Tx) + D(y, Ty),\ D(x, Ty) + D(y, Tx)\}. \tag{1.6}$$

2. Main Results

In this section first we prove some propositions for use in the main theorem and then prove some fixed point theorem in generalized complex valued metric space.

Proposition 2.1. *Let (X, D) be a generalized complex valued metric space, and let $T : X \to X$ be a BKC-contraction. Then any fixed point $u \in X$ of T satisfies*

$$|D(u, u)| < \infty \Rightarrow D(u, u) = 0.$$

Proof. : Let $u \in X$ be a fixed point of T such that $|D(u, u)| < \infty$ and $Tu = u$. To show that $D(u, u) = 0$.
Now let

$$\begin{aligned} D(u, u) &= D(Tu, Tu) \\ &\precsim \lambda\, max\,\{2D(u, u), D(u, Tu) + D(u, Tu), D(u, Tu) + D(u, Tu)\} \\ &= 2\lambda D(u, u). \end{aligned}$$

By Remark 1.1(2), we have

$$|D(u, u)| \leq 2\lambda |D(u, u)|.$$

Since $2\lambda \in [0, 1)$ so, $|D(u, u)| = 0$. Hence $D(u, u) = 0$. ■

For every $x \in X$, we define

$$\delta(D, T, x) = sup\{|D(T^i x, T^j x)| : i, j \in \mathbb{N}\}.$$

Theorem 2.2. *: Let (X, D) be a D-complete generalized complex valued metric space and let $T : X \to X$ be a self mapping on X satisfying (1.6). Let $|r|\lambda < \frac{1}{2}$, and there exists an element $x_0 \in X$ such that $\delta(D, T, x_0) < \infty$. Then the sequence $\{T^n x_0\}$ converges to some $u \in X$ and u is a fixed point of T. Moreover, for each fixed point v of T in X such that $|D(v, v)| < \infty$, we have $u = v$.*

Proof. : Let $n \in \mathbb{N}$ with $n \geq 2$, and $i, j \in \mathbb{N}$, we obtain that

$$D\left(T^{n+i} x_0, T^{n+j} x_0\right) = D\left(T(T^{n+i-1} x_0), T(T^{n+j-1} x_0)\right).$$

By (1.6), we have

$$D(T^{n+i}x_0, T^{n+j}x_0) \precsim \lambda max\Big\{2D(T^{n+i-1}x_0, T^{n+j-1}x_0), \\ D(T^{n+i-1}x_0, T^{n+i}x_0) + D(T^{n+j-1}x_0, T^{n+j}x_0), \\ D(T^{n+i-1}x_0, T^{n+j}x_0) + D(T^{n+j-1}x_0, T^{n+i}x_0)\Big\}.$$

By Remark 1.1(2), we have

$$\begin{aligned} |D(T^{n+i}x_0, T^{n+j}x_0)| \leq \lambda max\Big\{&2\left|D(T^{n+i-1}x_0, T^{n+j-1}x_0)\right|, \\ &|D(T^{n+i-1}x_0, T^{n+i}x_0) + D(T^{n+j-1}x_0, T^{n+j}x_0)|, \\ &|D(T^{n+i-1}x_0, T^{n+j}x_0) + D(T^{n+j-1}x_0, T^{n+i}x_0)|\Big\} \\ = 2\lambda\delta(D, T, T^{n-1}x_0).& \end{aligned}$$

We have

$$|D(T^{n+i}x_0, T^{n+j}x_0)| \leq 2\lambda\delta(D, T, T^{n-1}x_0), \tag{2.1}$$

by (2.1), we see that $2\lambda\delta(D, T, T^{n-1}x_0)$ is upper bound of the set $\{|D(T^{n+i}x_0, T^{n+j}x_0)| : i, j \in \mathbb{N}\}$. Since $\delta(D, T, T^n x_0) = sup\{|D(T^{n+i}x_0, T^{n+j}x_0)| : i, j \in \mathbb{N}\}$ and $\delta(D, T, T^n x_0)$ is least upper bound of $\big\{|D(T^{n+i}x_0, T^{n+j}x_0)|\big\}$ so it follow that

$$\delta(D, T, T^n x_0) \leq 2\lambda\delta(D, T, T^{n-1}x_0). \tag{2.2}$$

Hence, by induction we get

$$\begin{aligned} \delta(D, T, T^n x_0) &\leq 2\lambda\delta(D, T, T^{n-1}x_0) \\ &\leq (2\lambda)^2\delta(D, T, T^{n-2}x_0) \\ &\;\vdots \\ &\leq (2\lambda)^n\delta(D, T, x_0). \end{aligned}$$

Let $m, n \in \mathbb{N}$ and $m > n$, then

$$\begin{aligned} |D(T^n x_0, T^{n+m}x_0)| &= \left|D\left(T(T^{n-1}x_0), T^{m+1}(T^{n-1}x_0)\right)\right| \\ &\leq \delta\left(D, T, T^{n-1}x_0\right) \\ &\;\vdots \\ &\leq (2\lambda)^{n-1}\delta(D, T, x_0), \end{aligned}$$

for all integer m. Since $\delta(D, T, x_0) < \infty$ and $2\lambda < 1$, we have

$$\lim_{n\to\infty} (2\lambda)^{n-1}\delta(D, T, x_0) = 0.$$

Thus

$$\lim_{n\to\infty} |D(T^n x_0, T^{n+m}x_0)| = 0. \tag{2.3}$$

Which implies that $\{T^n x_0\}$ is a Cauchy sequence. Since X be a D-complete generalized complex valued metric space so, there exists $u \in X$ such that

$$\lim_{n\to\infty} |D(T^n x_0, u)| = 0. \tag{2.4}$$

By Definition 1.4(4), we have

$$D(Tu, u) \precsim r \lim_{n\to\infty} \sup |D(Tu, T^n x_0)|. \tag{2.5}$$

By Remarks 1.1(2), we have

$$|D(Tu, u)| \leq |r| \lim_{n\to\infty} \sup |D(Tu, T^{n+1} x_0)|. \tag{2.6}$$

By (1.6), we have

$$D(T^{n+1}x_0, Tu) \precsim \lambda max\Big\{2D(T^n x_0, u), \quad D(T^n x_0, T^{n+1}x_0) + D(u, Tu), \\ D(T^n x_0, Tu) + D(u, T^{n+1}x_0)\Big\}.$$

By Remarks 1.1(2)

$$\left|D(T^{n+1}x_0, Tu)\right| \leq \lambda max\Big\{ |2D(T^n x_0, u)|, \left|D(T^n x_0, T^{n+1}x_0) + D(u, Tu)\right|, \\ \left|D(T^n x_0, Tu) + D(u, T^{n+1}x_0)\right| \Big\}. \tag{2.7}$$

By (2.3), (2.4) and (2.7) we have

$$\lim_{n\to\infty} \sup \left|D(T^{n+1}x_0, Tu)\right| \leq \lambda max\Big\{ |D(u, Tu)|, \lim_{n\to\infty} |D(T^n x_0, Tu)| \Big\}. \tag{2.8}$$

Now the following two cases arise
Case–I: Let $max\Big\{|D(u, Tu)|, \lim_{n\to\infty} |D(T^n x_0, Tu)|\Big\} = |D(u, Tu)|$.
Then from (2.6), (2.8) we have

$$|D(Tu, u)| \leq |r| \lim_{n\to\infty} \sup |D(Tu, T^{n+1}x_0)| \leq |r|\lambda |D(Tu, u)|.$$

Since $0 < |r|\lambda < \frac{1}{2}$, and $\lambda \in [0, \frac{1}{2})$ so, $|D(Tu, u)| = 0$. Thus, $D(Tu, u) = 0$ it follows that

$$Tu = u.$$

Therefore, u is a fixed point of T.
Case–II: Let $max\Big\{|D(u, Tu)|, \lim_{n\to\infty} |D(T^n x_0, Tu)|\Big\} = \lim_{n\to\infty} |D(T^n x_0, Tu)|$.
Then from (2.6), (2.8) we have

$$\begin{aligned} |D(Tu, u)| &\leq |r| \lim_{n\to\infty} \sup |D(T^{n+1}x_0, Tu)| \\ &\leq |r|\lambda \lim_{n\to\infty} |D(T^n x_0, Tu)| \\ &\vdots \\ &\leq \lim_{n\to\infty} |r|(\lambda)^{n+1} |D(x_0, Tu)|. \end{aligned}$$

Since $|r|\lambda < \frac{1}{2}$, and $\lambda \in [0, \frac{1}{2})$ so, $\lim_{n\to\infty} |r|(\lambda)^{n+1} = 0$ therefore $|D(Tu, u)| = 0$. Thus, $D(Tu, u) = 0$ it follows that

$$Tu = u.$$

If v is any fixed point of T such that $|D(v,v)| < \infty$ and $|D(u,u)| < \infty$ then by (1.6), we have

$$\begin{aligned} D(u,v) &= D(Tu,Tv) \\ &\precsim \lambda max\Big\{2D(u,v), D(u,Tu)+D(v,Tv), D(u,Tv)+D(v,Tu)\Big\} \\ &= 2\lambda D(u,v). \end{aligned}$$

Then from 1.1(2) we have

$$|D(u,v)| \leq 2\lambda |D(u,v)|.$$

Since $\lambda \in [0, \frac{1}{2})$ so, $|D(u,v)| = 0$. Therefore $D(u,v) = 0$ and we have $u = v$. ■

Example 2.3. : Let $X = [0,1)$ and let $D : X \times X \to \mathbb{C}$ be the mapping defined by

$$\begin{cases} D(x,y) = \frac{1}{2}(x+y)i & \text{if } x \neq 0 \ \text{ and } y \neq 0, \\ D(x,0) = D(0,x) = \frac{xi}{4} & \text{for all}\, x \in X. \end{cases} \tag{2.9}$$

Hence the conditions (C_1),(C_2) and (C_3) are trivial. Now for condition (C_4), let $\left\{x_n = \frac{(n-1)x}{n} : n \in \mathbb{N}\right\} \subseteq X$, and put $r = i$, then we have

$$\lim_{n\to\infty} \sup |D(x_n, y)| = \lim_{n\to\infty} \sup \left\{ \sqrt{\frac{1}{4}\left\{\frac{(n-1)x}{n} + y\right\}^2} \right\} = \frac{1}{2}(x+y).$$

Hence,

$$D(x,y) = \frac{1}{2}(x+y)i \precsim \frac{1}{2}(x+y)i = r \lim_{n\to\infty} \sup |D(x_n,y)|.$$

Also, let $\left\{x_n = \frac{x}{n} : n \in \mathbb{N}\right\} \subseteq X$, then $\lim\limits_{n\to\infty} |D(x_n, 0)| = 0$ so,

$$D(0,y) = \frac{y}{4}i \ \text{ and } \ \lim_{n\to\infty} \sup |D(x_n,y)| = \lim_{n\to\infty} \sup \left\{ \sqrt{\frac{1}{4}\left\{\frac{x}{n} + y\right\}^2} \right\} = \frac{1}{2}(x+y).$$

Hence,

$$D(0,y) = \frac{y}{4}i \precsim \frac{1}{2}(x+y)i = r \lim_{n\to\infty} \sup |D(x_n,y)|.$$

It follows that (X, D) is a generalized complex valuedmetric space.
Furthermore, let $T : X \to X$ be defined by

$$T(x) = \frac{x}{x+3} \ \text{ for all } x \in X.$$

Now the following two cases arise
Case-I Let $x, y \in X - \{0\}$ then

$$D(Tx,Ty) = \frac{1}{2}\left(\frac{x}{x+3} + \frac{y}{y+3}\right)i,$$

and

$$max\{2D(x,y),\, D(x,Tx)+D(y,Ty),\, D(x,Ty)+D(y,Tx)\} = (x+y)i.$$

Hence

$$D(Tx,Ty) \precsim \frac{1}{4} max\{2D(x,y),\, D(x,Tx)+D(y,Ty),\, D(x,Ty)+D(y,Tx)\}.$$

Case-II Let $x \in X$ and $y = 0$ then

$$D(Tx, T0) = \frac{xi}{4(x+3)},$$

and

$$max\{2D(x,0),\, D(x,Tx)+D(0,T0),\, D(x,T0)+D(0,Tx)\} = \frac{1}{2}\left\{x + \frac{x}{x+3}\right\}i.$$

Hence

$$D(Tx,Ty) \precsim \frac{1}{4}\, max\{2D(x,y),\, D(x,Tx)+D(y,Ty),\, D(x,Ty)+D(y,Tx)\}.$$

Therefore T is a BKC-contraction with $\lambda = \frac{1}{4}$.

Now, we choose $x_0 = 1$, so that

$\delta(D,T,1) = sup\{|D(T^i1, T^j1)| : i,j \in \mathbb{N}\} = \frac{1}{4} < \infty$.

The hypotheses of Theorem 2.2 are satisfied. Therefore T has a unique fixed point since D is bounded; note that $T(0) = 0$.

3. Conclusion

In this article, we introduce a new contraction which is known as BKC-contraction, in generalized complex valued metric spaces and obtain fixed point theorem in these space. For the usability of our result, we give a suitable example, which extends the further scope.

Acknowledgements

We would like to thank anonymous reviewers for their comments on the manuscript which helped us very much in improving and presenting the original version of this paper.

References

[1] S. Banach, Sur les opérations dans les ensembles abstraits et leurs applications auxéquations intégrales. Fund Math. 3 (1922) 133–181.

[2] R. Kannan, Some results on fixed points, Bull. Calcutta Math. Soc. 6 (1968) 71–78.

[3] S. K. Chatterjea, Fixed points theorems, C. R. Acad. Bulgare Sci. 25 (1972) 727–730.

[4] S. Reich, Some remarks concerning contraction mappings, Canad. Math. Bull. 14 (1971) 121–124.

[5] A. Azam, F. Brain, M. Khan. Common fixed point theorems in complex valued metric space, Numer. Funct. Anal. Optim. 32 (3) (2011) 243–253.

[6] M. Jleli, B. Samet, A generalized metric space and related fixed point theorems. Fixed Point Theory Appl. 2015 (2015) 61 1–17, https://doi.org/10.1186/s13663-015-0312-7

[7] Y. Elkouch, E. M. Marhrani, On some fixed point theorems in generalized metric space, Fixed Point Theory Appl. 23 (2017) 121–124. DOI: 10.1186/s13663-017-0617-9.

[8] I. Inchan, U. Deepan, Some fixed point of Hardy-Rogers contraction in generalized complex valued metric spaces, Communications in Mathematics and Applications, 10 (2) (2019) 257–265. DOI: 10.26713/cma.v10i2.1077

[9] V. Berinde, Iterative Approximation of Fixed Points, Lecture Notes in Mathematics, 2nd edition, Springer, Berlin (2007), DOI: 10.1007/978-3-540-72234-2.