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Convergence Theorems by Using a Projection Method without the Monotonicity in Hilbert Spaces

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Abstract In this paper, we present a projection iterative algorithm for finding the common solution of variational inequality problem without monotonicity, fixed point problem of a nonexpansive mapping, and zero point problem of the sum of two monotone mappings in Hilbert spaces. When setting the solution set of the dual variational inequality is nonempty, the strong convergence theorem is established under some suitable control conditions. Finally, we reduce some mappings in our main result to study several problems.

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1. Introduction

Throughout of this paper, we suppose that H is a real Hilbert space with a norm $\|\cdot\|$ and an inner product $\langle\cdot,\cdot\rangle$ and $C \subset H$ is a nonempty, closed, and convex. The fixed point problem is to find a point $x \in C$ such that $x = Tx$ where a mapping $T : C \to C$. We denote that $F(S)$ is the set of fixed point of S where a mapping $S : C \to C$. Mathematicians are interesting the fixed point problem because there are many applications of this problem such as variational inequality problems, saddle point problems, minimax problems (see in [1–4]). In 1953, Mann [5] introduced an iteration scheme for finding the fixed point of the mapping T:

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n T x_n \tag{1.1}$$

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where $\{\alpha_n\} \subset (0,1)$ and $\sum_{n=0}^{\infty} \alpha_n(1-\alpha_n) = \infty$. When T is a nonexpansive mapping, it can prove that the sequence $\{x_n\}$ generated by (1.1) converges weakly to a fixed point of T. Moreover, x which is a fixed point of the mapping $P_C(I - rA)$ when $r > 0$, I stands for the identity mapping, A stands for the continuous mapping from C to H, and P_C stands for the metric projection iff a point $x \in C$ such that

$$\langle Ax, y - x\rangle \geq 0 \quad \forall y \in C \tag{1.2}$$

It is known that the classical variational inequality problem (for short, $VI(C, A)$). Suppose that $SOL(C, A)$ is the solution of $VI(C, A)$ and the solution set of the dual variational inequality:

$$SOL_D(C, A) := \{x \in C | \langle Ay, y - x\rangle \geq 0, \ \forall y \in C\}. \tag{1.3}$$

Obviously, $SOL_D(C, A) \subset SOL(C, A)$ because A is continuous and C is convex. Many authors attaches great importance to establish the projection type algorithms for finding $SOL(C, A)$ such as Goldstein-Levitin-Polyak projection methods [6, 7]; combined relaxation methods [8–10]; proximal point methods [11]; extragradient projection methods [12–22]; double projection methods [23]. These methods have the common assumption $SOL(C, A) \subset SOL_D(C, A)$. In the sense of Karamardian [24], this assumption is a direct consequence of pseudomonotonicity of A. In 2015, Ye and He [25] presented a double projection method for solving $SOL(C, A)$ without monotonicity of A. They assume only assumption $SOL_D(C, A) \neq \emptyset$ and show that $SOL(C, A) \subset SOL_D(C, A)$ imply $SOL_D(C, A) \neq \emptyset$ (but not converse). Furthermore, the sequence $\{x_n\}$ generated by their method conveges to $SOL(C, A)$.

Recently, fixed point problem, variational inequalities, and zero point problems have been investigated by authors besed on iterative methods (see in [26–31]). One of iterative methods for solving the common solution was presented by Feng and Jing in [26]. They proposed the projection methods base on a hybrid projection iterative algorithm and proved strong convergence theorems without any compact assumptions. The methods can find the common solution of fixed point problems of a nonexpansive mapping, monotone variational inequality problems, and zero point problems of the sum of a maximal monotone operator and an inverse-strongly monotone mapping in Hilbert spaces.

Inspired and motivated by [25] and [26], we propose a projection iterative algorithm base on a projection method [26] and a double projection method [25] for finding the common solution of fixed point problems of a nonexpansive mapping, variational inequality problems without a monotonicity of A, and zero point problems of the sum of a maximal monotone operator and an inverse-strongly monotone mapping in Hilbert spaces. When setting the solution set of the dual variational inequality is nonempty, the strong convergence theorem of the proposed iterative algorithm is established under some suitable control conditions. In addition, we reduce our main result to study several problems.

2. Preliminaries

In this section, we collect essential equipments for using in section 3. The projection from $x \in H$ onto C is defined by $P_C(x) := \arg\min\{\|y - x\| \,|\, y \in C\}$. The natural residual function $\gamma_\mu(\cdot)$ is defined by $\gamma_\mu(x) := x - P_C(x - \mu Ax)$ where $\mu > 0$ is a parameter. If $\mu = 1$, we denote $\gamma(x)$.

Lemma 2.1 ([26]). *Let C be a nonempty, closed, and convex subset of H. Then*

$$\|x - P_C x\|^2 + \|y - P_C x\|^2 \leq \|x - y\|^2, \quad \forall x \in H, y \in C.$$

We recall that a set-valued mapping $M : H \rightrightarrows H$ is said to be monotone iff, for every $x, y \in H$, $f \in Mx$ and $g \in My$ imply $\langle x - y, f - g\rangle > 0$. A monotone mapping M is maximal iff the graph $Graph(M)$ of R is not properly contained in the graph of any other monotone mapping. It is known that a monotone mapping M is maximal if and only if, for any $(x, f) \in H \times H$, $\langle x - y, f - g\rangle \geq 0$, for every $(y, g) \in Graph(M)$ implies $f \in Rx$.

For a maximal monotone operator M on H and $r > 0$, we assume the single valued resovent $J_r : H \to D(M)$, where $D(M)$ denotes the domain of M. It is known that J_r is firmly nonexpansive, and $M^{-1}(0) = F(J_r)$, where $F(J_r) := \{x \in D(M) : x = J_r x\}$ and $M^{-1}(0) := \{x \in H : 0 \in Mx\}$.

Lemma 2.2 ([26]). *Let C be a nonempty, closed, and convex subset of H, $B : C \to H$ be a mapping, and $M : H \rightrightarrows H$ be a maximal monotone operator. Then $F(J_r(I - sB)) = (B + M)^{-1}(0)$.*

Lemma 2.3 ([26]). *Let C be a nonempty, closed, and convex subset of H. Let $S : C \to C$ be a nonexpansive mapping. Then the mapping $I - S$ is demiclosed at zero, that is, if $\{x_n\}$ is a sequence in C such that $x_n \rightharpoonup \bar{x}$ and $x_n - Sx_n \to 0$, then $\bar{x} \in F(S)$.*

Lemma 2.4 ([15]). *Let C be a closed convex subset of H, h be a real-valued function, and $K := \{x \in: h(x) \leq 0\}$. If K is nonempty and h is Lipschitz continuous on C with modolus $\theta > 0$, then*

$$dist(x, K) \leq \theta^{-1} \max\{h(x), 0\}, \ \forall x \in C \tag{2.1}$$

Remark 2.5. If we set $K := K \cap C$ and $K \cap C \neq \emptyset$, then (2.1) holds. Note that C and $K \cap C$ are closed, so there exist $\min_{y \in K \cap C} \|x - y\|$ and $\min_{y \in K} \|x - y\|$ which $\min_{y \in K \cap C} \|x - y\| \leq \min_{y \in K} \|x - y\|$, that is, $dist(x, K) \leq dist(x, C \cap K)$

Lemma 2.6. *Let the function h_n be defined by Step 5 and $\{x_n\}$ be generated by Algorithm 1. If $SOL_D(C, A) \neq \emptyset$, then $h_n(x_n) \geq (1 - \sigma)\|r_{\lambda_n}(x_n)\|^2 > 0$ for every n. If $x^* \in SOL_D(C, A)$, then $h_n(x^*) \leq 0$ for every n.*

Lemma 2.7 ([15]). *$x^* \in SOL(C, A)$ if and only if $\|r_\mu(x^*)\| = 0$.*

Lemma 2.8. *If $\tilde{x}$ is any accumulation point of $\{x_n\}$ which generated by Algorithm 1 , then $\tilde{x} \in \cap_{n=1}^{\infty} H_n$.*

Proof. Suppose that l is nonnegative integer and $\tilde{x}$ is an accumulation point of $\{x_n\}$. There is a subsequence $\{x_{n_m}\}$ of $\{x_n\}$ which $\lim_{m\to\infty} x_{n_m} = \tilde{x}$. We have $x_{n_m} = P_{C_{n_i-1} \cap \tilde{H}_{n_m-1}} x_1$ and $\tilde{H}_{n_m-1} = \cap_{j=1}^{j=n_m-1} H_j$. It obtains that $x_{n_m} \in H_l$ for every $m \geq l + 1$. From H_l is closed and $\lim_{m\to\infty} x_{n_m} = \tilde{x}$, we get $\tilde{x} \in H_l$. This completes the proof ∎

3. Main results

In this section, we present our algorithms and reduce our strong convergence theorems for studying several problems.

Algorithm 1. Setting $J_{s_n} = (I + s_n M)^{-1}$, $\{\rho_n\} \in (0, \frac{1}{\alpha})$, $\{s_n\} \in (0, 2\beta)$, $\{\alpha_n\} \in (0, 1)$, $\sigma \in (0, 1)$ and $\lambda \in (0, 1)$. Choose $x_1 \in C$ and $C_1 = C$ as an initial point. Set $n = 1$.
Step 1. Compute $z_n := P_C(J_{s_n}(x_n - s_n Bx_n) - \rho_n AJ_{s_n}(x_n - s_n Bx_n))$.
Step 2. Compute $\gamma(x_n) = x_n - z_n$. If $\gamma(x_n) = 0$, stop. Otherwise, go to Step 3.

Step 3. Compute $u_n = x_n - \eta_n \gamma(x_n)$ where $\eta_n = \lambda_{m_n}$ with m_n is the smallest nonnegative integer satisfying

$$\langle A(x_n) - A(x_n - \lambda_m \gamma(x_n)), \gamma(x_n)\rangle \leq \sigma \|\gamma(x_n)\|^2 . \tag{3.1}$$

Step 4. Compute $y_n = \alpha_n x_n + (1-\alpha_n) SP_C(J_{s_n}(x_n - s_n Bx_n) - \rho_n Az_n)$.
Step 5. Compute $x_{n+1} = P_{C_{n+1}\cap \tilde{H}_{n+1}}(x_1)$, where

$$C_{n+1} = \{v \in C_n : \|y_n - v\| \leq \|x_n - v\|\}$$

and $\tilde{H}_{n+1} := \cap_{j=1}^{j=n+1} H_j$ with $H_j := \{v : h_j(v) \leq 0\}$ is a half space defined by the function $h_j(v) := \langle A(u_j), v - u_j\rangle$.

Theorem 3.1. *Let $A : C \to H$ be an α-Lipschitz continuous mapping, $S : C \to C$ be a nonexpansive mapping with a nonempty fixed point set, $B : C \to H$ be a β-inverse strongly monotone mapping, and $M : H \rightrightarrows H$ be a maximal monotone operator with $D(M) \subset C$. Suppose that $SOL_D(C, A)$ and $\Theta := SOL(C, A) \cap F(S) \cap (B + M)^{-1}(0)$ are nonempty sets. Then $\{x_n\}$ which a sequence generated by Algorithm 1 converges strongly to $P_\Theta x_1$ where $a, b, c, d,$ and e are real constants and satisfy*
(A) $0 < a \leq \rho_n \leq b < \frac{1}{\alpha}$,
(B) $0 < c \leq s_n \leq d < 2\beta$, *and*
(C) $0 \leq \alpha_n \leq e < 1$.

Proof. We are going to show that $C_n \cap \tilde{H}_n$ is closed and convex for every $n \geq 1$. By hypothesis, we have $C_1 = C$ which is closed and convex. Thus $C_1 \cap \tilde{H}_1$ is closed and convex. Assume that $C_k \cap \tilde{H}_k$ is closed and convex for some $k \geq 1$. We show that $C_{k+1} \cap \tilde{H}_{k+1}$ is closed and convex for some k. Let $v_1, v_2 \in C_{k+1} \cap \tilde{H}_{k+1}$. Suppose that $v = tv_1 + (1-t)v_2$ where $t \in (0, 1)$. By $h_{k+1}(v_1) \leq 0$ and $h_{k+1}(v_2) \leq 0$, we have

$$\begin{aligned} h_{k+1}(v) &= \langle A(u_{k+1}), tv_1 + v_2 - tv_2 - u_{k+1} + tu_{k+1} - tu_{k+1}\rangle \\ &= \langle A(u_{k+1}), v_2 - u_{k+1}\rangle + \bar{t}\langle A(u_{k+1}), v_1 - u_{k+1}\rangle - \bar{t}\langle A(u_{k+1}), v_2 - u_{k+1}\rangle \\ &\leq 0 \end{aligned}$$

and

$$\|y_k - v\| \leq \|x_k - v\| \iff \|y_k\|^2 - \|x_k\|^2 - 2\langle v, y_k - x_k\rangle \geq 0. \tag{3.2}$$

It can obtain that $v \in C_{k+1} \cap \tilde{H}_{k+1}$. Therefore $C_n \cap \tilde{H}_n$ is closed and convex for every $k \geq 1$. Hereon, we will show that $\Theta \subset C_n \cap \tilde{H}_n$ for every $n \geq 1$. Set $w_n = P_C(v_n - \rho_n Az_n)$ when $v_n = J_{s_n}(x_n - s_n Bx_n)$. By the assumption and $S_D \neq \emptyset$, we obtain that $\Theta \subset C_1 \cap \tilde{H}_1$. Assume that $\Theta \subset C_k \cap \tilde{H}_k$ for some $k \geq 1$. From Lemma 2.1, we see that for every $p \in \Theta \subset C_k \cap \tilde{H}_k$,

$$\begin{aligned} \|w_n - p\| &\leq \|v_k - \rho_k Az_k - p\|^2 - \|v_k - \rho_k Az_k - w_k\|^2 \\ &= \|v_k - p\|^2 - \|v_k - w_k\|^2 + 2\rho_k\langle Az_m, p - w_k\rangle \\ &= \|v_k - p\|^2 - \|v_k - w_k\|^2 + 2\rho_k(\langle Az_k - Ap, p - z_m\rangle + \langle Ap, p - z_k\rangle \\ &\quad + \langle Az_k, z_k - w_k\rangle) \\ &\leq \|v_k - p\|^2 - \|v_k - z_k + z_k - w_m\|^2 + 2\rho_k\langle Az_k, z_k - w_k\rangle \\ &= \|v_k - p\|^2 - \|v_k - z_k\|^2 - \|z_m - w_m\|^2 \\ &\quad + 2\langle v_k - z_k - \rho_k Az_k, w_k - z_k\rangle. \end{aligned} \tag{3.3}$$

Since A is Lipschitz continuous and $z_k = P_C(v_k - \rho_k Av_k)$, it can imply

$$\begin{aligned}\langle v_k - z_k - \rho_k Az_k, w_k - z_m\rangle &= \langle v_k - z_k - \rho_k Av_k, w_k - z_m\rangle \\ &\quad +\langle \rho_k Av_k - \rho_k Az_k, w_k - z_m\rangle \\ &\leq \rho_k \alpha\|v_k - z_m\|\|w_k - z_k\|. \end{aligned} \tag{3.4}$$

Thank to (3.3) and (3.4), we see that

$$\begin{aligned}\|w_k - p\|^2 &\leq \|v_k - p\|^2 - \|v_k - z_k\|^2 - \|z_k - w_k\|^2 + 2\rho_k\alpha\|v_k - z_k\|\|w_k - z_k\| \\ &\leq \|v_k - p\|^2 - (1 - \rho_k^2\alpha^2)\|v_k - z_k\|^2.\end{aligned}$$

According to (A), we have

$$\begin{aligned}\|y_k - p\|^2 &\leq \alpha_k\|x_k - p\|^2 + (1-\alpha_k)\|Sw_k - p\|^2 \\ &\leq \alpha_k\|x_k - p\|^2 + (1-\alpha_k)\|w_k - p\|^2 \\ &\leq \alpha_k\|x_k - p\|^2 + (1-\alpha_k)(\|v_k - p\|^2 - (1-\rho_k^2\alpha^2)\|v_k - z_k\|^2) \\ &\leq \|x_k - p\|^2 - (1-\alpha_k)(1-\rho_k^2\alpha^2)\|v_k - z_k\|^2 \\ &\leq \|x_k - p\|^2. \end{aligned} \tag{3.5}$$

This implies that $p \in C_{k+1} \cap \tilde{H}_{k+1}$. So $\Theta \subset C_n \cap \tilde{H}_n$ for all $n \geq 1$. We denote that $x_n = P_{C_n \cap \tilde{H}_n} x_1$. Thus $\|x_1 - x_n\| \leq \|x_1 - p\|$ for all $p \in \Theta$. Since B is inverse-strongly monotone and Lemma 2.2, it can obtain that $(B+M)^{-1}(0)$ is closed and convex. Since A is Lipschitz continuous, we obtain that $VI(C, A)$ is closed and convex. Obviously, Θ is also closed and convex. Thus

$$\|x_1 - x_n\| \leq \|x_1 - P_\Theta x_1\|. \tag{3.6}$$

We conclude that $\{x_n\}$ is bounded. Since $x_n = P_{C_n \cap \tilde{H}_n} x_1$ and $x_{n+1} = P_{C_{n+1} \cap \tilde{H}_{n+1}} x_1$, we get

$$\begin{aligned} 0 &\leq \langle x_1 - x_n, x_n - x_{n+1}\rangle \\ &= \langle x_1 - x_n, x_n - x_1 + x_1 - x_{n+1}\rangle \\ &\leq -\|x_1 - x_n\|^2 + \|x_1 - x_n\|\|x_1 - x_{n+1}\|. \end{aligned}$$

Therefore $\|x_n - x_1\| \leq \|x_{n+1} - x_1\|$. This implies that $\lim_{n\to\infty}\|x_n - x_1\|$ exists. We note that

$$\begin{aligned}\|x_n - x_{n+1}\| &= \|x_n - x_1\|^2 + 2\langle x_n - x_1, x_1 - x_{n+1}\rangle + \|x_1 - x_{n+1}\|^2 \\ &= \|x_n - x_1\|^2 - 2\|x_n - x_1\|^2 + 2\langle x_n - x_1, x_n - x_{n+1}\rangle \\ &\quad +\|x_1 - x_{n+1}\|^2 \\ &\leq \|x_1 - x_{n+1}\|^2 - \|x_n - x_1\|^2.\end{aligned}$$

Then $\lim_{n\to\infty}\|x_n - x_{n+1}\| = 0$. By $x_{n+1} = P_{C_{n+1} \cap \tilde{H}_{n+1}} x_1 \in C \cap \tilde{H}_{n+1}$, we have

$$\|y_n - x_{n+1}\| \leq \|x_n - x_{n+1}\|. \tag{3.7}$$

So $\|y_n - x_n\| \leq \|y_n - x_{n+1}\| + \|x_n - x_{n+1}\| \leq 2\|x_n - x_{n+1}\|$.
We know that $\lim_{n\to\infty}\|x_n - x_{n+1}\| = 0$. It can obtain

$$\lim_{n\to\infty}\|x_n - y_n\| = 0. \tag{3.8}$$

Since B is β- inverse strongly monotone and (B), we have

$$\begin{aligned}\|(I-s_nB)x-(I-s_nB)y\|^2 &= \|x-y\|^2-2s_n\langle x-y,Bx-By\rangle+s_n^2\|Bx-By\|^2\\ &\leq \|x-y\|^2-s_n(2\beta-s_n)\|Bx-By\|^2\\ &\leq \|x-y\|^2,\end{aligned}$$

for every $x,y\in C$. According to (3.5), we get

$$\begin{aligned}\|y_n-p\|^2 &\leq \alpha_n\|x_n-p\|^2+(1-\alpha_n)\|v_n-p\|^2\\ &= \alpha_n\|x_n-p\|^2+(1-\alpha_n)\|J_{s_n}(p-s_nBp)\|^2\\ &\leq \|x_n-p\|^2-(1-\alpha_n)s_n(2\beta-s_n)\|Bx_n-Bp\|^2.\end{aligned}$$

This implies that

$$\begin{aligned}(1-\alpha_n)s_n(2\beta-s_n)\|Bx_n-Bp\|^2 &\leq \|x_n-p\|^2-\|y_n-p\|^2\\ &\leq \|x_n-y_n\|(\|x_n-p\|+\|y_n-p\|).\end{aligned}$$

By means of (B) and (C), we receive

$$\lim_{n\to\infty}\|Bx_n-Bp\|=0. \tag{3.9}$$

Since J_{s_n} is firmly nonexpansive, it can imply that

$$\begin{aligned}\|v_n-p\|^2 &= \|J_{s_n}(x_n-s_nBx_n)-J_{s_n}(p-s_nBp)\|^2\\ &\leq \langle v_n-p,(x_n-s_nBx_n)-(p-s_nBp)\rangle\\ &= \frac{1}{2}\|v_n-p\|^2+\|(x_n-s_nBx_n)-(p-s_nBp)\|^2\\ &\quad -\|(v_n-p)-((x_n-s_nBx_n)-(p-s_nBp))\|^2\\ &\leq \frac{1}{2}(\|v_n-p\|^2+\|x_n-p\|^2-\|v_n-x_n+s_n(Bx_n-Bp)\|^2)\\ &= \frac{1}{2}(\|v_n-p\|^2+\|x_n-p\|^2-\|v_n-x_n\|^2-s_n^2\|Bx_n-Bp\|^2\\ &\quad -2s_n\langle v_n-x_n,Bx_n-Bp\rangle)\\ &\leq \frac{1}{2}(\|v_n-p\|^2+\|x_n-p\|^2-\|v_n-x_n\|^2+2s_n\|v_n-x_n\|\|Bx_n-Bp\|).\end{aligned}$$

Therefore,

$$\|v_n-p\|^2\leq\|x_n-p\|^2-\|v_n-x_0\|^2+2s_n\|v_n-x_n\|\|Bx_n-Bp\|. \tag{3.10}$$

Combining (3.5) with (3.10), the previous inequality becomes

$$\begin{aligned}\|v_n-p\|^2 &= \alpha_n\|x_n-p\|^2+(1-\alpha_n)\|v_n-p\|^2\\ &\leq \|x_n-p\|^2-(1-\alpha_n)\|v_n-x_n\|^2+2s_n\|v_n-x_n\|\|Bx_n-Bp\|.\end{aligned}$$

It obtains that

$$\begin{aligned}(1-\alpha_n)\|v_n-x_n\|^2 &= \|x_n-p\|^2-\|y_n-p\|^2+2s_n\|v_n-x_n\|\|Bx_n-Bp\|\\ &\leq \|x_n-y_n\|(\|x_n-p\|+\|y_n-p\|)+2s_n\|v_n-x_n\|\|Bx_n-Bp\|.\end{aligned}$$

By (3.8), (3.9), and (C), we have

$$\lim_{n\to\infty}\|v_n-x_n\|=0. \tag{3.11}$$

As a consequence of (3.5), we find that

$$\begin{aligned}(1-\alpha_n)(1-\rho_n^2\alpha^2)\|v_n - z_n\|^2 &\leq \|x_n - p\|^2 - \|y_n - p\|^2 \\ &\leq \|x_n - y_n\|(\|x_n - p\| + \|y_n - p\|).\end{aligned}$$

From (A), (C), and (3.8), it obtains

$$\lim_{n\to\infty} \|v_n - z_n\| = 0. \tag{3.12}$$

We denote that

$$\begin{aligned}\|w_n - z_n\|^2 &= \|P_C(v_n - rAz_n) - P_C(v_n - \rho_n Av_n)\|^2 \\ &\leq \|(v_n - \rho_n Az_n) - (v_n - \rho_n Av_n)\|^2 \\ &\leq \rho_n^2\alpha^2\|z_n - v_n\|^2.\end{aligned}$$

By (3.12), it implies

$$\lim_{n\to\infty} \|u_n - z_n\| = 0. \tag{3.13}$$

Note that

$$\begin{aligned}\|x_n - Sx_n\| &\leq \|x_n - Sw_n\| + \|Sw_n - Sx_n\| \\ &\leq \frac{\|x_n - y_n\|}{1-\alpha_n} + \|w_n - x_n\| \\ &\leq \frac{\|x_n - y_n\|}{1-\alpha_n} + \|w_n - z_n\| + \|z_n - v_n\| + \|v_n - x_n\|.\end{aligned}$$

Thank to (3.8), (3.11), (3.12), (3.13), and (C), we get

$$\lim_{n\to\infty} \|x_n - Sx_n\| = 0. \tag{3.14}$$

Since $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ which $x_{n_i} \rightharpoonup q \in C$. By Lemma 2.3, we can conclude that $q \in F(S)$.

Afterwards, we are going to show that $q \in SOL(C, A)$. Since A and γ are continuous, it can imply that $\{z_n\}, \{\gamma(x_n)\}$ and $\{y_n\}$ are bounded. Furthermore, the continuity of A implies that $\{Ay_n\}$ is bounded, that is, for some $W > 0$

$$\|Ay_n\| \leq W, \quad \forall\, n. \tag{3.15}$$

Form the definition of $\tilde{H}_{n+1}$, we note that $\tilde{H}_{n+1} \subseteq H_{n+1}$ for every n. Therefore

$$\operatorname{dist}(x_{n+1}, C_{n+1} \cap \tilde{H}_{n+1}) \geq \operatorname{dist}(x_{n+1}, C_{n+1} \cap H_{n+1}). \tag{3.16}$$

Taking the limit $n \to \infty$ in (3.16), we have

$$\lim_{n\to\infty} \operatorname{dist}(x_{n+1}, C_{n+1} \cap H_{n+1}) = 0. \tag{3.17}$$

Obviously, every h_{n+1} is Lipschitz continuous on C with modulus W. By using Lemma 2.4 and Lemma 2.2, we get that

$$\operatorname{dist}(x_{n+1}, C_{n+1} \cap H_{n+1}) \geq W^{-1} h_{n+1}(x_{n+1}) \geq W^{-1}(1-\sigma)\eta_n\|\gamma(x_{n+1})\|^2. \tag{3.18}$$

Thank to (3.17) and (3.18), it follows that $\lim_{n\to\infty} \eta_n\|\gamma(x_{n+1})\|^2 = 0$.

If $\limsup_{n\to\infty} \eta_n > 0$, then $\liminf_{n\to\infty} \|\gamma(x_n)\| = 0$. Since $\{x_n\}$ is bounded and γ is continuous, there is an accumulation point q of $\{x_n\}$ which $\gamma(q) = 0$. According to Lemma 2.7 and Lemma 2.8, it implies $q \cap_{n=1}^{\infty} (H_n \cap SOL(C, A))$. Thus $q \in SOL(C, A)$.

If $\limsup_{n\to\infty} \eta_n = 0$, then $\lim_{n\to\infty} \eta_n = 0$. We suppose that $\bar{q}$ is any accumulation point of $\{x_n\}$. There is a subsequence x_{n_j} which converges to $\bar{q}$. By the condition of η_n, 3.1 is not satisfied for $m_n - 1$. So

$$\langle Ax_{n_j} - A(x_{n_j} - \lambda^{-1}\eta_{n_j}\gamma(x_{n_j})), \gamma(x_{n_j})\rangle \geq \sigma\|\gamma(x_{n_j})\|^2. \tag{3.19}$$

Taking the limit in (3.19), we have

$$0 \geq \sigma\|\gamma(\bar{q})\|^2 \geq 0. \tag{3.20}$$

It obtains $\gamma(\bar{q}) = 0$. Thus $\bar{q} \in \cap_{n=1}^{\infty}(H_n \cap SOL(C,A))$. We can conclude that $\bar{q} \in SOL(C,A)$.

Later, we will prove that $q \in (B+M)^{-1}(0)$. We denote that $x_n - s_nBx_n \in v_n + s_nMv_n$. Thus

$$\frac{x_n - v_n}{s_n} - Bx_n \in Mv_n. \tag{3.21}$$

Let $\tau \in Mv$. We know that M is monotone. From (3.21), it follows that

$$\langle \frac{x_n - v_b}{s_n} - Bx_n - \tau, v_n - v\rangle \geq 0. \tag{3.22}$$

By dint of (B), we have $\langle -Bq - \tau, q - v\rangle \geq 0$. So $-Bq \in Mq$. This means that $q \in (B+M)^{-1}(0)$. We conclude that $q \in \Theta$. Suppose that there is another subsequence $\{x_{n_j}\}$ of $\{x_n\}$ which $x_{n_j} \rightharpoonup \tilde{q} \in \Theta$. By Opial's condition, $\tilde{q} = q$.

Eventually, we will show that $q = P_\Theta x_1$ and $x_n \to q$. From (3.6) and the lower semicontinuity of norm, we find that

$$\|x_1 - P_\Theta x_1\| \leq \|x_1 - q\| \leq \liminf_{n\to\infty} \|x_1 - x_n\| \leq \limsup_{n\to\infty} \|x_1 - x_n\| \leq \|x_1 - P_\Theta x_1\|. \tag{3.23}$$

This implies that $\lim_{n\to\infty} \|x_1 - x_n\| = \|x_1 - P_\Theta x_1\| = \|x_1 - q\|$. Hence, the sequence $\{x_n\}$ converges strongly to $P_\Theta x_1$. ∎

Theorem 3.1 is reduced by setting $B = 0$. We receive the following algorithm.

Algorithm 2. Setting $J_{s_n} = (I + s_nM)^{-1}$, $\{\rho_n\} \in (0, \frac{1}{\alpha})$, $\{s_n\} \in (0,\infty)$, $\{\alpha_n\} \in (0,1)$, $\sigma \in (0,1)$ and $\lambda \in (0,1)$. Choose $x_1 \in C$ and $C_1 = C$ as an initial point. Set $n = 1$.
Step 1. Compute $z_n := P_C(J_{s_n}x_n - \rho_nAJ_{s_n}x_n)$.
Step 2. Compute $\gamma(x_n) = x_n - z_n$. If $\gamma(x_n) = 0$, stop. Otherwise, go to Step 3.
Step 3. Compute $u_n = x_n - \eta_n\gamma(x_n)$ where $\eta_n = \lambda_{m_n}$ with m_n is the smallest nonnegative integer satisfying

$$\langle A(x_n) - A(x_n - \lambda_m\gamma(x_n)), \gamma(x_n)\rangle \leq \sigma \|\gamma(x_n)\|^2.$$

Step 4. Compute $y_n = \alpha_nx_n + (1-\alpha_n)SP_C(J_{s_n}x_n - \rho_nAz_n)$.
Step 5. Compute $x_{n+1} = P_{C_{n+1}\cap\tilde{H}_{n+1}}(x_1)$, where

$$C_{n+1} = \{v \in C_n : \|y_n - v\| \leq \|x_n - v\|\}$$

and $\tilde{H}_{n+1} := \cap_{j=1}^{j=n+1}H_j$ with $H_j := \{v : h_j(v) \leq 0\}$ is a half space defined by the function $h_j(v) := \langle A(u_j), v - u_j\rangle$.

Corollary 3.2. *Let $A : C \to H$ be an α-Lipschitz continuous mapping, $S : C \to C$ be a nonexpansive mapping with a nonempty fixed point set, and $M : H \rightrightarrows H$ be a maximal monotone operator with $D(M) \subset C$. Suppose that $SOL_D(C,A)$ and $\kappa := SOL(C,A) \cap F(S) \cap (M)^{-1}(0)$ are nonempty sets. Then the sequence $\{x_n\}$, generated by Algorithm 2,* *converges strongly to* $P_\Theta x_1$ *where* $a, b, c, d,$ *and* e *are real constants and satisfy*

(A) $0 < a \leq \rho_n \leq b < \frac{1}{\alpha}$,
(B) $0 < c \leq s_n \leq d < \infty$, *and*
(C) $0 \leq \alpha_n \leq e < 1$.

When setting $M = 0$ and $J_{s_n} = I$, Corollary 3.2 can reduce to the following algorithm.

Algorithm 3. Setting $J_{s_n} = (I + s_n M)^{-1}, \{\rho_n\} \in (0, \frac{1}{\alpha}), \{\alpha_n\} \in (0,1)$, $\sigma \in (0,1)$ and $\lambda \in (0,1)$. Choose $x_1 \in C$ and $C_1 = C$ as an initial point. Set $n = 1$.
Step 1. Compute $z_n := P_C(x_n - \rho_n A x_n)$.
Step 2. Compute $\gamma(x_n) = x_n - z_n$. If $\gamma(x_n) = 0$, stop. Otherwise, go to Step 3.
Step 3. Compute $u_n = x_n - \eta_n \gamma(x_n)$ where $\eta_n = \lambda_{m_n}$ with m_n is the smallest nonnegative integer satisfying

$$\langle A(x_n) - A(x_n - \lambda_m \gamma(x_n)), \gamma(x_n) \rangle \leq \sigma \|\gamma(x_n)\|^2 .$$

Step 4. Compute $y_n = \alpha_n x_n + (1 - \alpha_n) S P_C(x_n - \rho_n A z_n)$.
Step 5. Compute $x_{n+1} = P_{C_{n+1} \cap \tilde{H}_{n+1}}(x_1)$, where

$$C_{n+1} = \{v \in C_n : \|y_n - v\| \leq \|x_n - v\|\}$$

and $\tilde{H}_{n+1} := \cap_{j=1}^{j=n+1} H_j$ with $H_j := \{v : h_j(v) \leq 0\}$ is a half space defined by the function $h_j(v) := \langle A(u_j), v - u_j \rangle$.

Corollary 3.3. *Let* $A : C \to H$ *be an* α*-Lipschitz continuous mapping,* $S : C \to C$ *be a nonexpansive mapping with a nonempty fixed point set, and* $M : H \rightrightarrows H$ *be a maximal monotone operator with* $D(M) \subset C$*. Suppose that* SOL_D *and* $\psi := SOL(C, A) \cap F(S)$ *are nonempty sets. Then* $\{x_n\}$ *which a sequence generated by Algorithm 1 converges strongly to* $P_\Theta x_1$ *where* a, b *and* c *are real constants and satisfy*

(A) $0 < a \leq \rho_n \leq b < \frac{1}{\alpha}$ *and*
(B) $0 \leq \alpha_n \leq c < 1$.

4. Conclusions

In this work, we propose projection iterative algorithms combining both a projection iterative algorithm and a double projection iterative algorithm to solve the common solution of variational inequality problem without monotonicity of A, fixed point problem of a nonexpansive mapping, and zero point problem of the sum of two monotone mappings in Hilbert spaces. Furthermore, the strong convergence theorem is generated by the proposed algorithm under some suitable control conditions. Finally, we use our main result to solve the several problems by reducing some mappings.

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