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Direct Powers of Hyperalgebras Carried by Good Homomorphisms

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Abstract In the present paper, we study the direct power of hyperalgebras obtained as the subhyperalgebra of their direct product carried by good homomorphisms. We give sufficient conditions for the validity of the second exponential law and a weak form of the first exponential law. Moreover, we restrict the concepts of the direct powers from hyperalgebras to algebras and give sufficient conditions for the validity of the first and the second exponential laws for algebras.

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1. Introduction

In his pioneering papers [1, 2], G. Birkhoff introduced the concept of the cardinal (i.e., direct) arithmetic of partially ordered sets and showed that it behaves analogously to the arithmetic of natural numbers. In particular, he proved that for any given partially ordered sets $\mathbf{G}$, $\mathbf{H}$ and $\mathbf{K}$, the following three laws are fulfilled ($\times$ and $\prod$ denote the direct product, $\sum$ the direct sum and $\cong$ the isomorphism of partially ordered sets):

$(\mathbf{G}^{\mathbf{H}})^{\mathbf{K}} \cong \mathbf{G}^{\mathbf{H}\times\mathbf{K}}$ - *the first exponential law*,

$\prod_{i\in I}\mathbf{G}_i^{\mathbf{H}} \cong (\prod_{i\in I}\mathbf{G}_i)^{\mathbf{H}}$ - *the second exponential law*,

$\prod_{i\in I}\mathbf{G}^{\mathbf{H}_i} \cong \mathbf{G}^{\sum_{i\in I}\mathbf{H}_i}$ (if $\mathbf{H}_i$, $i \in I$, are pairwise disjoint) - *the third exponential law*.

Birkhoff's arithmetic of ordered sets was generalized in [3]. Several authors then extended the cardinal arithmetics to relational systems - see e.g.[4, 5]. Conversely, the cardinal arithmetics were restricted from relational systems to universal algebras in [6, 7], to partial algebras in [8] and to hyperalgebras in [9, 10]. In the present paper, we continue the study of direct arithmetic of hyperalgebras based on [10].

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The concept of good homomorphisms of hyperalgebras was introduced in [11] by restricting the concept of strong homomorphisms of relational systems in [4] to hyperalgebras. In this paper, we study the subsystem of the direct power of hyperalgebras carried by the set of all good homomorphisms. With respect to good homomorphisms, we find conditions satisfying the weak first and the second exponential laws for direct power of hyperalgebras. We focus especially on investigating the direct powers of algebras. The results obained are generalizations of those of [7].

The concept of hyperalgebras was proved to be useful for many application in various branches of mathematics and computer science (automata theory). This leads to a rapid development of the theory of hyperalgebras since the beginning of 90's of the last century-see e.g. [12–14]. The aim of this paper is also to contribute to the development of the theory of hyperalgebras by using good homomorphisms.

2. Preliminaries

For any given sets G and H, we denote by G^H the set of all mappings of H into G. It is easy to see that there is a bijection $\alpha\colon G^{H\times K} \to (G^H)^K$(where $\times$ denotes the Cartesian product) given by $\alpha(h)(y)(x) = h(x, y)$ whenever $h \in G^{H\times K}, x \in H$ and $y \in K$. The bijection α will be called *canonical*.

Throughout the paper, a mapping $f\colon G \to H$ $(G, H$ sets) will often be denoted as indexed family $(f_i; i \in G)$ where $f_i \in H$ for every $i \in G$, so that f_i means $f(i)$ for every $i \in G$.

While arities of hyperoperations are usually considered to be natural numbers, we consider them to be arbitrary sets. Therefore, hyperalgebras are defined as follows. Let Ω be a nonempty set. A family $\tau = (K_\lambda; \lambda \in \Omega)$ of sets will be called a *type*. By a *universal hyperalgebra* (briefly, a *hyperalgebra*) of type τ, we mean a pair $\mathbf{G} = \langle G, (p_\lambda; \lambda \in \Omega)\rangle$ where G is a nonempty set, the so-called *carrier* of $\mathbf{G}$, and $p_\lambda\colon G^{K_\lambda} \to \exp G \setminus \{\emptyset\}$ is a map, the so called K_λ-*ary hyperoperation* on G, for every $\lambda \in \Omega$. Of course, if $K_\lambda = \emptyset$, then p_λ is nothing but a nonempty subset of G. To avoid some nonwanted singularities, all hyperalgebras $\mathbf{G} = \langle G, (p_\lambda; \lambda \in \Omega)\rangle$ are considered to have a type $(K_\lambda; \lambda \in \Omega)$ with $K_\lambda \neq \emptyset$ for every $\lambda \in \Omega$. A hyperalgebra $\mathbf{G} = \langle G, (p_\lambda; \lambda \in \Omega)\rangle$ with the property card $p_\lambda(x_i; i \in K_\lambda) = 1$ for every $\lambda \in \Omega$ and every $(x_i; i \in K_\lambda) \in G^{K_\lambda}$ is called a (*universal*) *algebra*. In the case $\tau = (K)$ where card $K = 2$, hyperalgebras of type τ are usually called *hypergroupiods*.

Given a hyperalgebra $\mathbf{G} = \langle G, (p_\lambda; \lambda \in \Omega)\rangle$ of type $\tau = (K_\lambda; \lambda \in \Omega)$, $\lambda \in \Omega$ and a family $X_i, i \in K_\lambda$, of subsets of G, we put $p_\lambda(X_i; i \in K_\lambda) = \bigcup\{p_\lambda(x_i; i \in K_\lambda) \mid x_i \in X_i$ for every $i \in K_\lambda\}$.

Let $\mathbf{H} = \langle H, (q_\lambda; \lambda \in \Omega)\rangle, \mathbf{G} = \langle G, (p_\lambda; \lambda \in \Omega)\rangle$ be a pair of hyperalgebras of type τ. Then $\mathbf{G}$ is called a *subhyperalgebra* of $\mathbf{H}$ provided that $G \subseteq H$ and $p_\lambda(x_i; i \in K_\lambda) = q_\lambda(x_i; i \in K_\lambda)$ whenever $\lambda \in \Omega$ and $x_i \in H$ for each $i \in K_\lambda$.

A map $f\colon H \to G$ is called a *homomorphism* of $\mathbf{H}$ into $\mathbf{G}$ if, for each $\lambda \in \Omega$, $f(q_\lambda(x_i; i \in K_\lambda)) \subseteq p_\lambda(f(x_i); i \in K_\lambda)$. A homomorphisms f of $\mathbf{H}$ into $\mathbf{G}$ is said to be a good homomorphism of $\mathbf{H}$ into $\mathbf{G}$ if, for each $\lambda \in \Omega$, $f(q_\lambda(x_i; i \in K_\lambda)) = p_\lambda(f(x_i); i \in K_\lambda)$. We denote by $Hom(\mathbf{H}, \mathbf{G})$ the set of all homomorphisms of $\mathbf{H}$ into $\mathbf{G}$ and $GHom(\mathbf{H}, \mathbf{G})$ the set of all good homomorphisms of $\mathbf{H}$ into $\mathbf{G}$. If f is a bijection of H onto G and a good homomorphisms of $\mathbf{H}$ into $\mathbf{G}$, then f is called an *isomorphism* of $\mathbf{H}$ onto $\mathbf{G}$. In other words, an isomorphism of $\mathbf{H}$ onto $\mathbf{G}$ is a bijection $f\colon H \to G$ such that f is a homomorphism of $\mathbf{H}$ into $\mathbf{G}$ and f^{-1} is a homomorphism of $\mathbf{G}$ into $\mathbf{H}$. If there is an isomorphism of $\mathbf{G}$ onto $\mathbf{H}$, then we write $\mathbf{H} \cong \mathbf{G}$ and say that $\mathbf{H}$ and $\mathbf{G}$ are *isomorphic*. We say that $\mathbf{H}$ may be *embedded* into $\mathbf{G}$ and write $\mathbf{H} \preccurlyeq \mathbf{G}$ if there exists a subhyperalgebra $\mathbf{G}'$ of $\mathbf{G}$ such that $\mathbf{H} \cong \mathbf{G}'$.

The *direct product* of a family $\mathbf{G}_i = \langle G_i, (p^i_\lambda; \lambda \in \Omega)\rangle, i \in I$, of hyperalgebras of type $\tau = (K_\lambda; \lambda \in \Omega)$ is the hyperalgebra $\prod_{i\in I} \mathbf{G}_i = \langle \prod_{i\in I} G_i, (q_\lambda; \lambda \in \Omega)\rangle$ of type τ where $\prod_{i\in I} G_i$ denotes the Cartesian product and, for any $\lambda \in \Omega$ and any $(f_k; k \in K_\lambda) \in (\prod_{i\in I} G_i)^{K_\lambda}$, $q_\lambda(f_k; k \in K_\lambda) = \prod_{i\in I} p^i_\lambda(f_k(i); k \in K_\lambda)$. If the set I is finite, say $I = \{1, \ldots, m\}$, then we write $\mathbf{G}_1 \times \ldots \times \mathbf{G}_m$ instead of $\prod_{i\in I} G_i$. If $\mathbf{G}_i = \mathbf{G}$ for every $i \in I$, then we write $\mathbf{G}^I$ instead of $\prod_{i\in I} \mathbf{G}_i$.

The *direct sum* of a family $\mathbf{G}_i = \langle G, (p^i_\lambda; \lambda \in \Omega)\rangle, i \in I$, of hyperalgebras of type τ is the hyperalgebra $\sum_{i\in I} \mathbf{G}_i = \langle G, (q_\lambda; \lambda \in \Omega)\rangle$ where, for every $\lambda \in \Omega$ and every $(x_i; i \in K_\lambda) \in G^{K_\lambda}, q_\lambda(x_i; i \in K_\lambda) = \bigcup_{i\in I} p^i_\lambda(x_i; i \in K_\lambda)$. If the set I is finite, say $I = \{1, \ldots, m\}$, then we write $\mathbf{G}_1 \uplus \ldots \uplus \mathbf{G}_m$ instead of $\sum_{i\in I} \mathbf{G}_i$.

Let $\mathbf{G} = \langle G, (p_\lambda; \lambda \in \Omega)\rangle$ be a hyperalgebra of type $\tau = (K_\lambda; \lambda \in \Omega)$ and let $\lambda_0 \in \Omega$. An element $x \in G$ is called an *idempotent* with respect to the hyperoperation p_{λ_0} if $x \in p_{\lambda_0}(x_i; i \in K_{\lambda_0})$ whenever $x_i = x$ for every $i \in K_{\lambda_0}$. If every element of G is an idempotent with respect to each hyperoperation $p_\lambda, \lambda \in \Omega$, then the hyperalgebra $\mathbf{G}$ is said to be *idempotent*.

3. Direct powers of hyperalgebras

Generally, given a pair of hyperalgebras $\mathbf{G}, \mathbf{H}$, the subhyperalgebra $\mathbf{K}$ of the direct product $\mathbf{G}^{\mathbf{H}}$ such that $\mathbf{K} = GHom(\mathbf{H}, \mathbf{G})$ need not exist.

Example 3.1. Let $\mathbf{H} = (\{1, 2\}, q)$, $\mathbf{G} = (\{a, b\}, p)$ be binary hyperalgebras and p, q are given by the following tables:

q	1	2
1	{1,2}	{1}
2	{1}	{1,2}

p	a	b
a	$\{a,b\}$	$\{a\}$
b	$\{a\}$	$\{a,b\}$

Let $f\colon \{1, 2\} \to \{a, b\}$ be a map given by $f(1) = a$ and $f(2) = b$ and $g\colon \{1, 2\} \to \{a, b\}$ be a map given by $g(1) = a$ and $g(2) = a$. Then $f \in GHom(\mathbf{H}, \mathbf{G})$ and $g(1) \in p(f(1), f(1))$ and $g(2) \in p(f(2), f(2))$. But, $g \notin GHom(\mathbf{H}, \mathbf{G})$.

For hyperalgebras $\mathbf{G}, \mathbf{H}$, we give conditions for finding the subhyperalgebra $GHom(\mathbf{H}, \mathbf{G})$ of the direct product $\mathbf{G}^H$.

In the sequel, we work with $K \times L$-matrices over G where K, L, G are nonempty sets. These matrices are nothing but the maps $x\colon K \times L \to G$, i.e., the indexed sets $(x_{ij}; i \in K, j \in L)$ denoted briefly by (x_{ij}).

Here we recall some definitions.

Definition 3.2 ([9])**.** A hyperalgebra $\langle G, (p_\lambda; \lambda \in \Omega)\rangle$ of type $\tau = (K_\lambda, \lambda \in \Omega)$ is called *medial* if, for every $\mu, \nu \in \Omega$ and every $K_\mu \times K_\nu$-matrix (a_{ij}) over G, from $x_i \in p_\nu(a_{ij}; j \in K_\nu)$ for each $i \in K_\mu$ and $y_j \in p_\mu(a_{ij}; i \in K_\mu)$ for each $j \in K_\nu$ it follows that $p_\nu(x_i; i \in K_\mu) = p_\mu(y_j; j \in K_\nu)$.

Example 3.3. ([9]) Let $(X, \leq)$ be a partially ordered set with a least element 0 and let A be the set of all atoms of $(X, \leq)$. Put $0' = \{0\}$ and, for any $x \in X$ with $x \neq 0$, put $x' = \{y \in X \mid y < x \text{ and } y \in A \cup \{0\}\}$. Further, for any pair $x, y \in X$ put $x * y = x' \cap y'$. Then $(X, ', *)$ is a medial hyperalgebra of type $(1, 2)$. Indeed, it can easily be seen that, for any $x \in X$, we have $a' = \{0\} = b'$ whenever $a \in x'$ and $b \in x'$. Further, for any $a, b, c, d \in X$, we have $x * y = f * g$ whenever $x \in a * b$, $y \in c * d$, $f \in a * c$ and $g \in b * d$. Finally, for any $a, b \in X$, we have $x' = y * z$ whenever $x \in a * b$, $y \in a'$ and $z \in b'$.

Definition 3.4. Let $\mathbf{H} = \langle H, (q_\lambda; \lambda \in \Omega)\rangle$ and $\mathbf{G} = \langle G, (p_\lambda; \lambda \in \Omega)\rangle$ be hyperalgebras of the same type τ. We say that $\mathbf{G}$ is *powerable* by $\mathbf{H}$ if there exists a hyperalgebra $\mathbf{G}^{\diamond\mathbf{H}} = \langle GHom(\mathbf{H}, \mathbf{G}), (r_\lambda; \lambda \in \Omega)\rangle$ such that $r_\lambda(f_i; i \in K_\lambda) = \{f \in GHom(\mathbf{H}, \mathbf{G}) \mid f(x) \in p_\lambda(f_i(x); i \in K_\lambda)\}$ whenever $\lambda \in \Omega$ and $(f_i; i \in K_\lambda) \in (GHom(\mathbf{H}, \mathbf{G}))^{K_\lambda}$. The hyperalgebra $\mathbf{G}^{\diamond\mathbf{H}}$ is called the *direct power* of $\mathbf{G}$ and $\mathbf{H}$.

It is easy to see that the direct power $\mathbf{G}^{\diamond\mathbf{H}}$ of hyperalgebras of the same type is idempotent whenever $\mathbf{G}$ is idempotent (and medial).

Example 3.5. Let $\mathbf{H} = (\{a, b\}, q)$, $\mathbf{G} = (\{1, 2, 3\}, p)$ be binary hyperalgebras and p, q be given by the following tables:

q	a	b
a	$\{a,b\}$	$\{a,b\}$
b	$\{a,b\}$	$\{a,b\}$

p	1	2	3
1	{1,2}	{1,2}	{1,2}
2	{1,2}	{1,2}	{1,2}
3	{1,2}	{1,2}	{1}

Let $f_1 \colon \{a, b\} \to \{1, 2, 3\}$ be a map given by $f_1(a) = 1$ and $f_1(b) = 2$ and $f_2 \colon \{a, b\} \to \{1, 2, 3\}$ be a map given by $f_2(a) = 2$ and $f_2(b) = 1$. Then $GHom(\mathbf{H}, \mathbf{G}) = \{f_1, f_2\}$. We have $f_i(x) \in p(f_j(x), f_k(x))$ for any $x \in \{a, b\}$ and for any $i, j, k \in \{1, 2\}$. Then there exists a hyperalgebra $\mathbf{G}^{\diamond\mathbf{H}} = \langle GHom(\mathbf{H}, \mathbf{G}), r\rangle$ such that $f_i \in r(f_j, f_k)$ for any $i, j, k \in \{1, 2\}$. Thus $\mathbf{G}$ is powerable by $\mathbf{H}$.

Example 3.6. Let $\mathbf{H} = (\{a, b\}, q)$, $\mathbf{G} = (\{1, 2\}, p)$ be binary hyperalgebras and p, q be given by the following tables:

q	a	b
a	$\{a\}$	$\{a\}$
b	$\{a\}$	$\{a,b\}$

p	1	2
1	{1}	{1}
2	{1}	{1,2}

Let $f \colon \{a, b\} \to \{1, 2\}$ be a map given by $f(a) = 1$ and $f(b) = 2$ and $g \colon \{a, b\} \to \{1, 2\}$ be a map given by $g(a) = 1$ and $g(b) = 1$. Then $GHom(\mathbf{H}, \mathbf{G}) = \{f, g\}$. We have

$f(x) \in p(f(x), f(x))$ for all $x \in \{a, b\}$,
$g(x) \in p(f(x), g(x))$ for all $x \in \{a, b\}$,
$g(x) \in p(g(x), f(x))$ for all $x \in \{a, b\}$, and
$g(x) \in p(g(x), g(x))$ for all $x \in \{a, b\}$.

Then there exists a hyperalgebra $\mathbf{G}^{\diamond\mathbf{H}} = \langle GHom(\mathbf{H}, \mathbf{G}), r\rangle$ such that $f \in r(f, f), g \in r(f, g), g \in r(g, f), g \in r(g, g)$. Thus $\mathbf{G}$ is powerable by $\mathbf{H}$.

Theorem 3.7. *Let $\mathbf{H} = \langle H, (q_\lambda; \lambda \in \Omega)\rangle$ and $\mathbf{G} = \langle G, (p_\lambda; \lambda \in \Omega)\rangle$ be hyperalgebras of the same type τ and $\mathbf{G}$ be medial. Then $\mathbf{G}$ is powerable by $\mathbf{H}$ if and only if for every $(f_i; i \in K_\lambda) \in (GHom(\mathbf{H}, \mathbf{G}))^{K_\lambda}$, there exists a map $f \colon H \to G$ such that the following two conditions are satisfied:*

(i) *$f(x) \in p_\lambda(f_i(x); i \in K_\lambda)$ for every $x \in H$,*
(ii) *$f(q_\mu(x_j; j \in K_\mu)) = p_\lambda(f_i(q_\mu(x_j; j \in K_\mu); i \in K_\lambda)$ for every $\mu \in \Omega$ and for every $(x_j; i \in K_\mu) \in H^{K_\mu}$.*

Proof. Let $\lambda \in \Omega$ and $(f_i; i \in K_\lambda) \in (GHom(\mathbf{H}, \mathbf{G}))^{K_\lambda}$.

$\Rightarrow$ Suppose that $\mathbf{G}$ is powerable by $\mathbf{H}$. Let $f\colon H \to G$ be a map such that $f(x) \in p_\lambda(f_i(x); i \in K_\lambda)$ for every $x \in H$. So we have f is a good homomorphism from $\mathbf{H}$ into $\mathbf{G}$. Let $\mu \in \Omega$ and $(x_j; i \in K_\mu) \in H^{K_\mu}$. We will show that $f(q_\mu(x_j; j \in K_\mu)) = p_\lambda(f_i(q_\mu(x_j; j \in K_\mu); i \in K_\lambda)$. Let $y \in f(q_\mu(x_j; j \in K_\mu))$. Then there exists $z \in q_\mu(x_j; j \in K_\mu)$ such that $y = f(z)$. Since $f \in GHom(\mathbf{H}, \mathbf{G})$, we have $f(q_\mu(x_j; j \in K_\mu)) = p_\mu(f(x_j); j \in K_\mu))$. As $z \in q_\mu(x_j; j \in K_\mu)$ and $f_i \in GHom(\mathbf{H}, \mathbf{G})$ for every $i \in K_\lambda$, we have $f_i(z) \in f_i(q_\mu(x_j; j \in K_\mu)) = p_\mu(f_i(x_j); j \in K_\mu))$ for every $i \in K_\lambda$. By the definition of the map f, we have $f(x_j) \in p_\lambda(f_i(x_j); i \in K_\lambda)$ for every $j \in K_\lambda$. Since $\mathbf{G}$ is medial, we have $p_\lambda(f_i(z); i \in K_\lambda) = p_\mu(f(x_j); j \in K_\mu))$. Therefore,

$$\begin{aligned} y = f(z) &\in f(q_\mu(x_j; j \in K_\mu)) \\ &= p_\mu(f(x_j); j \in K_\mu) \\ &= p_\lambda(f_i(z); i \in K_\lambda) \\ &\subseteq p_\lambda(f_i(q_\mu(x_j; j \in K_\mu); i \in K_\lambda). \end{aligned}$$

Hence, $f(q_\mu(x_j; j \in K_\mu)) \subseteq p_\lambda(f_i(q_\mu(x_j; j \in K_\mu); i \in K_\lambda)$.

Let $y \in p_\lambda(f_i(q_\mu(x_j; j \in K_\mu); i \in K_\lambda) = \bigcup\{p_\lambda(f_i(z_i); i \in K_\lambda); z_i \in q_\mu(x_j; j \in K_\mu)\}$. Then there exists $z_i \in q_\mu(x_j; j \in K_\mu)$ for each $i \in K_\lambda$ such that $y \in p_\lambda(f_i(z_i); i \in K_\lambda)$. As $z_i \in q_\mu(x_j; j \in K_\mu)$ and $f_i \in GHom(\mathbf{H}, \mathbf{G})$ for every $i \in K_\lambda$, we have $f_i(z_i) \in f_i(q_\mu(x_j; j \in K_\mu)) = p_\mu(f_i(x_j); j \in K_\mu))$ for every $i \in K_\lambda$. By the definition of the map f, we have $f(x_j) \in p_\lambda(f_i(x_j); i \in K_\lambda)$ for every $j \in K_\lambda$. As $\mathbf{G}$ is medial, we have $p_\lambda(f_i(z_i); i \in K_\lambda) = p_\mu(f(x_j); j \in K_\mu))$. Then we have

$$\begin{aligned} y &\in p_\lambda(f_i(z_i); i \in K_\lambda) \\ &= p_\mu(f(x_j); j \in K_\mu) \\ &\subseteq f(q_\mu(x_j; j \in K_\mu)). \end{aligned}$$

Consequently, $p_\lambda(f_i(q_\mu(x_j; j \in K_\mu); i \in K_\lambda) \subseteq f(q_\mu(x_j; j \in K_\mu))$.

$\Leftarrow$ Let $\mu \in \Omega$ and let $f\colon H \to G$ be a map such that the following two conditions are satisfied:

- **(i)** $f(x) \in p_\lambda(f_i(x); i \in K_\lambda)$ for every $x \in H$,
- **(ii)** $f(q_\mu(x_j; j \in K_\mu)) = p_\lambda(f_i(q_\mu(x_j; j \in K_\mu); i \in K_\lambda)$ for every $\mu \in \Omega$ and for every $(x_j; i \in K_\mu) \in H^{K_\mu}$.

We will show that f is a good homomorphism from $\mathbf{H}$ into $\mathbf{G}$, i.e., $f(q_\mu(x_j; j \in K_\mu)) = p_\mu(f(x_i); i \in K_\mu)$. Let $y \in f(q_\mu(x_j; j \in K_\mu))$. Then there exists $z \in q_\mu(x_j; j \in K_\mu)$ such that $y = f(z)$. As $z \in q_\mu(x_j; j \in K_\mu)$ and $f_i \in GHom(\mathbf{H}, \mathbf{G})$ for every $i \in K_\lambda$, we have $f_i(z) \in f_i(q_\mu(x_j; j \in K_\mu)) = p_\mu(f_i(x_j); j \in K_\mu))$ for every $i \in K_\lambda$. By the definition of the map f, we have $f(x_j) \in p_\lambda(f_i(x_j); i \in K_\lambda)$ for every $j \in K_\lambda$ and $f(z) \in p_\lambda(f_i(z); i \in K_\lambda)$. Since $\mathbf{G}$ is medial, we have $p_\lambda(f_i(z); i \in K_\lambda) = p_\mu(f(x_j); j \in K_\mu))$. Therefore, $y = f(z) \in p_\lambda(f_i(z); i \in K_\lambda) = p_\mu(f(x_j); j \in K_\mu)$. Hence, $f(q_\mu(x_j; j \in K_\mu)) \subseteq p_\mu(f(x_j); j \in K_\mu)$.

Let $y \in p_\mu(f(x_j); j \in K_\mu)$. By the definition of the map f, we have $f(x_j) \in p_\lambda(f_i(x_j); i \in K_\lambda)$ for every $j \in K_\lambda$. Since $f_i \in GHom(\mathbf{H}, \mathbf{G})$ for every $i \in K_\lambda$, we have $f_i(q_\mu(x_j; j \in K_\mu)) = p_\mu(f_i(x_j); j \in K_\mu))$ for every $i \in K_\lambda$. As $\mathbf{G}$ is medial, we have $p_\lambda(f_i(z_i); i \in K_\lambda) = p_\mu(f(x_j); j \in K_\mu))$ such that $f(z_i) \in p_\mu(f(x_j); j \in K_\mu)$ for some $z_i \in q_\mu(x_j; j \in K_\mu)$ for each $i \in K_\lambda$. Therefore

$$\begin{aligned} y &\in p_\mu(f(x_j); j \in K_\mu) \\ &= p_\lambda(f_i(z_i); i \in K_\lambda) \\ &\subseteq p_\lambda(f_i(q_\mu(x_j; j \in K_\mu); i \in K_\lambda) \\ &= p_\lambda(f(q_\mu(x_j; j \in K_\mu)). \end{aligned}$$

Consequently, $p_\mu(f(x_j); j \in K_\mu) \subseteq f(q_\mu(x_j; j \in K_\mu))$.

∎

Theorem 3.8. *Let* $\mathbf{G}, \mathbf{H}$ *be hyperalgebras of the same type. If* $\mathbf{G}$ *is medial and* $\mathbf{G}$ *is powerable by* $\mathbf{H}$*, then* $\mathbf{G}^{\diamond\mathbf{H}}$ *is medial, too.*

Proof. Let $\mathbf{H} = \langle H, (q_\lambda; \lambda \in \Omega)\rangle$, $\mathbf{G} = \langle G, (p_\lambda; \lambda \in \Omega)\rangle$, $\mathbf{G}^{\diamond\mathbf{H}} = \langle GHom(\mathbf{H}, \mathbf{G}), (r_\lambda; \lambda \in \Omega)\rangle, \mu, \nu \in \Omega$ and let (f_{ij}) be a $K_\mu \times K_\nu$-matrix over $GHom(\mathbf{H}, \mathbf{G})$. Suppose that $h_i \in r_\nu(f_{ij}; j \in K_\nu)$ for all $i \in K_\mu$ and $g_j \in r_\mu(f_{ij}; i \in K_\mu)$ for all $j \in K_\nu$. For every $x \in H$, we have $h_i(x) \in p_\nu(f_{ij}(x); j \in K_\nu)$ for all $i \in K_\mu$ and $g_j(x) \in p_\mu(f_{ij}(x); i \in K_\mu)$ for all $j \in K_\nu$. Since $\mathbf{G}$ is medial, we have $p_\mu(h_i(x); i \in K_\mu) = p_\nu(g_j(x); j \in K_\nu)$ for every $x \in H$. Therefore, $r_\mu(h_i; i \in K_\mu) = r_\nu(h_j; j \in K_\nu)$, so that $\mathbf{G}^{\diamond\mathbf{H}}$ is medial. ∎

Theorem 3.9 and Theorem 3.10 can be proved by using a similar way as in Theorem 3.7.

Theorem 3.9. *Let* $\mathbf{G}, \mathbf{H}, \mathbf{K}$ *be hyperalgebras of the same type,* $\mathbf{H} \times \mathbf{K} = \langle H \times K, (v_\lambda; \lambda \in \Omega)\rangle$ *and* $\mathbf{G} = \langle G, (p_\lambda; \lambda \in \Omega)\rangle$*. If* $\mathbf{G}$ *is medial, then* $\mathbf{G}$ *is powerable by* $\mathbf{H} \times \mathbf{K}$ *if and only if for every* $(f_i; i \in K_\lambda) \in (GHom(\mathbf{H} \times \mathbf{K}, \mathbf{G}))^{K_\lambda}$*, there exists a map* $f \colon H \times K \to G$ *such that the following two conditions are satisfied:*

(i) $f(x, y) \in p_\lambda(f_i(x, y); i \in K_\lambda)$ *for every* $x \in H$ *and* $y \in K$,

(ii) $f(v_\mu((x_j, y_j); j \in K_\mu)) = p_\lambda(f_i(v_\mu((x_j, y_j); j \in K_\mu); i \in K_\lambda)$ *for every* $\mu \in \Omega$ *and for every* $((x_j, y_j); i \in K_\mu) \in (H \times K)^{K_\mu}$.

Theorem 3.10. *Let* $\mathbf{G}, \mathbf{H}, \mathbf{K}$ *be hyperalgebras of the same type,* $\mathbf{K} = \langle K, (s_\lambda; \lambda \in \Omega)\rangle$, $\mathbf{G}^{\diamond\mathbf{H}} = \langle GHom(\mathbf{H}, \mathbf{G}), (r_\lambda; \lambda \in \Omega)\rangle$*. If* $\mathbf{G}$ *is medial and* $\mathbf{G}$ *is powerable by* $\mathbf{H}$*, then* $\mathbf{G}^{\diamond\mathbf{H}}$ *is powerable by* $\mathbf{K}$ *if and only if* $(f_i; i \in K_\lambda) \in (GHom(\mathbf{K}, \mathbf{G}^{\diamond\mathbf{H}}))^{K_\lambda}$*, there exists a map* $f \colon K \to GHom(\mathbf{H}, \mathbf{G})$ *such that the following two conditions are satisfied:*

(i) $f(x) \in r_\lambda(f_i(x); i \in K_\lambda)$ *for every* $x \in K$,

(ii) $f(s_\mu(x_j; j \in K_\mu)) = r_\lambda(f_i(s_\mu(x_j; j \in K_\mu); i \in K_\lambda)$ *for every* $\mu \in \Omega$ *and for every* $(x_j; i \in K_\mu) \in K^{K_\mu}$.

Theorem 3.11. *Let* $\mathbf{G}, \mathbf{H}, \mathbf{K}$ *be hyperalgebras of the same type. If* $\mathbf{G}$ *is medial,* $\mathbf{G}$ *is powerable by* $\mathbf{H}$*,* $\mathbf{G}$ *is powerable by* $\mathbf{H} \times \mathbf{K}$*,* $\mathbf{G}^{\diamond\mathbf{H}}$ *is powerable by* $\mathbf{K}$ *and* $\mathbf{H}$*,* $\mathbf{K}$ *are idempotent, then*

$$\mathbf{G}^{\diamond(\mathbf{H}\times\mathbf{K})} \preccurlyeq (\mathbf{G}^{\diamond\mathbf{H}})^{\diamond\mathbf{K}}.$$

Proof. Let $\mathbf{H} = \langle H, (q_\lambda; \lambda \in \Omega)\rangle, \mathbf{G} = \langle G, (p_\lambda; \lambda \in \Omega)\rangle, \mathbf{K} = \langle K, (s_\lambda; \lambda \in \Omega)\rangle$, $\mathbf{G}^{\diamond\mathbf{H}} = \langle GHom(\mathbf{H}, \mathbf{G}), (r_\lambda; \lambda \in \Omega)\rangle, \mathbf{H}\times\mathbf{K} = \langle H\times K, (v_\lambda; \lambda \in \Omega)\rangle$, $(\mathbf{G}^{\diamond\mathbf{H}})^{\diamond\mathbf{K}} = \langle GHom(\mathbf{K}, \mathbf{G}^{\diamond\mathbf{H}}), (t_\lambda; \lambda \in \Omega)\rangle$, $\mathbf{G}^{\diamond(\mathbf{H}\times\mathbf{K})} = \langle GHom(\mathbf{H} \times \mathbf{K}, \mathbf{G}), (u_\lambda; \lambda \in \Omega)\rangle$.

We define a map $\varphi : GHom(\mathbf{H} \times \mathbf{K}, \mathbf{G}) \to GHom(\mathbf{K}, \mathbf{G}^{\diamond\mathbf{H}})$ by $\varphi(g)(z)(y) = g(y, z)$ whenever $g \in GHom(\mathbf{H} \times \mathbf{K}, \mathbf{G})$, $z \in Z$ and $y \in H$.

Let $g \in GHom(\mathbf{H} \times \mathbf{K}, \mathbf{G})$. We will show that $\varphi(g)(z) \in GHom(\mathbf{H}, \mathbf{G})$ for each $z \in K$, i.e., $\varphi(g)(z)(p_\lambda(y_i; i \in K_\lambda)) = p_\lambda(\varphi(g)(z)(y_i); i \in K_\lambda)$ for each $z \in K$ and each $\lambda \in \Omega$.

Let $z \in K$, $\lambda \in \Omega$, and $f \in \varphi(g)(z)(q_\lambda(y_i; i \in K_\lambda))$. Then there exists $y \in q_\lambda(y_i; i \in K_\lambda)$ such that $f = \varphi(g)(z)(y)$. Since K is idempotent, we have $(y, z) \in v_\lambda((y_i, z); i \in K_\lambda)$. Consequently, $g(y, z) \in g(v_\lambda((y_i, z); i \in K_\lambda))$. As g is a good homomorphism, we have $g(v_\lambda((y_i, z); i \in K_\lambda) = p_\lambda(g(y_i, z); i \in K_\lambda)$. Since $f = \varphi(g)(z)(y) = g(y, z)$, we have $f \in p_\lambda(g(y_i, z); i \in K_\lambda)$. Thus, $\varphi(g)(z)(q_\lambda(y_i; i \in K_\lambda)) \subseteq p_\lambda(\varphi(g)(z)(y_i); i \in K_\lambda)$.

Let $f \in p_\lambda(\varphi(g)(z)(y_i); i \in K_\lambda)$. Then $f \in p_\lambda(g(y_i, z)); i \in K_\lambda)$. Now, as g is a good homomorphism, it follows that $p_\lambda(g(y_i, z); i \in K_\lambda) = g(v_\lambda((y_i, z); i \in K_\lambda))$. Then there exists $(y, z) \in v_\lambda((y_i, z); i \in K_\lambda)$ such that $f = g(y, z)$. Therefore, $f = \varphi(g)(z)(y)$ such that $y \in q_\lambda(y_i; i \in K_\lambda))$. Thus $f \in \varphi(g)(z)(q_\lambda(y_i; i \in K_\lambda))$. Hence $p_\lambda(\varphi(g)(z)(y_i); i \in K_\lambda) \subseteq \varphi(g)(z)(q_\lambda(y_i; i \in K_\lambda))$. Consequently, $\varphi(g)(z) \in GHom(\mathbf{H}, \mathbf{G})$ for each $z \in K$.

Next, we will show that $\varphi(g) \in GHom(\mathbf{K}, \mathbf{G}^{\diamond \mathbf{H}})$. Let $g \in GHom(\mathbf{H} \times \mathbf{K}, \mathbf{G})$, $\lambda \in \Omega$, and $z_i \in K$ for every $i \in K_\lambda$. We will show that $\varphi(g)(s_\lambda(z_i; i \in K_\lambda)) = r_\lambda(\varphi(g)(z_i); i \in K_\lambda)$.

Let $f \in \varphi(g)(s_\lambda(z_i; i \in K_\lambda))$. Then there exists $z \in s_\lambda(z_i; i \in K_\lambda)$ such that $f = \varphi(g)(z)$. Since $\mathbf{H}$ is idempotent, we have $(y, z) \in v_\lambda((y, z_i); i \in K_\lambda)$ for each $y \in H$. Since $g \in GHom(\mathbf{H} \times \mathbf{K}, \mathbf{G})$, we have $g(y, z) \in g(v_\lambda((y, z_i); i \in K_\lambda)) = p_\lambda(g(y, z_i); i \in K_\lambda)$ for each $y \in H$. It follows that $f(y) = \varphi(g)(z)(y) \in p_\lambda(\varphi(g)(z_i)(y); i \in K_\lambda)$ for each $y \in H$, and hence $f = \varphi(g)(z) \in r_\lambda(\varphi(g)(z_i); i \in K_\lambda)$. Therefore, $\varphi(s_\lambda(z_i; i \in K_\lambda) \subseteq r_\lambda(\varphi(g)(z_i); i \in K_\lambda)$.

Let $f \in r_\lambda(\varphi(g)(z_i); i \in K_\lambda)$. Then $f(y) \in p_\lambda(\varphi(g)(z_i)(y); i \in K_\lambda)$ for each $y \in H$. Hence $f(y) \in p_\lambda(g(y, z_i); i \in K_\lambda)$ for each $y \in H$. Now, as g is a good homomorphism, it follows that $p_\lambda(g(y, z_i); i \in K_\lambda) = g(v_\lambda((y, z_i); i \in K_\lambda))$ for each $y \in H$. Then there exists $(y, z) \in v_\lambda((y, z_i); i \in K_\lambda)$ such that $f(y) = g(y, z)$ for each $y \in H$. Therefore, $f(y) = \varphi(g)(z)(y)$ such that $z \in s_\lambda(z_i; i \in K_\lambda)$ for each $y \in H$. Then $f = \varphi(g)(z) \in \varphi(g)(s_\lambda(z_i; i \in K_\lambda))$. Thus $r_\lambda(\varphi(g)(z_i); i \in K_\lambda) \subseteq \varphi(g)(s_\lambda(z_i; i \in K_\lambda))$. Consequently, $\varphi(g) \in GHom(\mathbf{K}, \mathbf{G}^{\diamond \mathbf{H}})$.

Next, we will show that φ is a homomorphisms. Let $\lambda \in \Omega$ and $f \in \varphi(u_\lambda(g_i; i \in K_\lambda))$. Then there exists $g \in u_\lambda(g_i; i \in K_\lambda)$ such that $f = \varphi(g)$. Since $g \in u_\lambda(g_i; i \in K_\lambda))$, we have $g(y, z) \in p_\lambda(g_i(y, z); i \in K_\lambda))$ for every $(y, z) \in H \times K$, which implies $\varphi(g)(z)(y) \in p_\lambda(\varphi(g_i)(z)(y); i \in K_\lambda)$ for every $y \in H$ and every $z \in K$. It follows that $\varphi(g)(z) \in r_\lambda(\varphi(g_i)(z); i \in K_\lambda)$ for every $z \in K$. Therefore, $f = \varphi(g) \in t_\lambda(\varphi(g_i); i \in K_\lambda)$. Then $\varphi(u_\lambda(g_i; i \in K_\lambda)) \subseteq t_\lambda(\varphi(g_i); i \in K_\lambda)$.

Finally, we will show that φ is an injection. Let $\varphi(g_1) = \varphi(g_2)$ for a pair $g_1, g_2 \in GHom(\mathbf{H} \times \mathbf{K}, \mathbf{G})$. Then $\varphi(g_1)(z) = \varphi(g_2)(z)$ for every $z \in K$ and we have $\varphi(g_1)(z)(y) = \varphi(g_2)(z)(y)$ for every $z \in K$ and $y \in H$. Hence $g_1(y, z) = g_2(y, z)$ for every $(y, z) \in H \times K$. Thus, $g_1 = g_2$ and we have shown that φ is an injection.

Therefore φ is an embedding of $\mathbf{G}^{\diamond(\mathbf{H} \times \mathbf{K})}$ into $(\mathbf{G}^{\diamond \mathbf{H}})^{\diamond \mathbf{K}}$ and the proof is complete. ■

Theorem 3.12. *Let* $\mathbf{G}_i = \langle G_i, (p^i_\lambda; \lambda \in \Omega)\rangle, i \in I$, *be a family of hyperalgebras of type* $\tau = (K_\lambda; \lambda \in \Omega)$, $\prod_{i \in I} \mathbf{G}_i = \langle \prod_{i \in I} G_i, (r_\lambda; \lambda \in \Omega)\rangle$, *and let* H *be a hyperalgebra of type* τ. *If* $\mathbf{G}_i$ *is medial for every* $i \in I$, *then* $\prod_{i \in I} \mathbf{G}_i$ *is powerable by* $\mathbf{H}$ *if and only if* $(f_k; k \in K_\lambda) \in (GHom(\mathbf{H}, \prod_{i \in I} \mathbf{G}_i))^{K_\lambda}$, *there exists a map* $f \colon H \to \prod_{i \in I} \mathbf{G}_i$ *such that the following two conditions are satisfied:*

- **(i)** $f(x) \in r_\lambda(f_k(x); k \in K_\lambda)$ *for every* $x \in H$,
- **(ii)** $f(s_\mu(x_j; j \in K_\mu)) = r_\lambda(f_k(s_\mu(x_j; j \in K_\mu); k \in K_\lambda)$ *for every* $\mu \in \Omega$ *and for every* $(x_j; i \in K_\mu) \in H^{K_\mu}$.

Proof. Since $\mathbf{G}_i$ is medial for every $i \in I$, we have $\prod_{i\in I}\mathbf{G_i}$ is medial. We can prove the statement by using the same arguments as in Theorem 3.7. ∎

Theorem 3.13. *Let $\mathbf{G}_i = \langle G_i, (p^i_\lambda; \lambda \in \Omega)\rangle, i \in I$, be a family of hyperalgebras of type $\tau = (K_\lambda; \lambda \in \Omega)$ and let H be a hyperalgebra of type τ. If $\mathbf{G}_i$ is medial , $\mathbf{G}_i$ is powerable by $\mathbf{H}$ for every $i \in I$ and $\prod_{i\in I}\mathbf{G}_i$ is powerable by $\mathbf{H}$ then*

$$\prod_{i\in I}\mathbf{G}_i^{\diamond\mathbf{H}} \cong (\prod_{i\in I}\mathbf{G}_i)^{\diamond\mathbf{H}}.$$

Proof. Let $\mathbf{H} = \langle H, (q_\lambda; \lambda \in \Omega)\rangle$, $\prod_{i\in I}\mathbf{G}_i = \langle\prod_{i\in I}G_i, (r_\lambda; \lambda \in \Omega)\rangle$, $\mathbf{G}_i^{\diamond\mathbf{H}} = \langle GHom(\mathbf{H}, \mathbf{G}_i), (u^i_\lambda; \lambda \in \Omega)\rangle$ for every $i \in I$, $\prod_{i\in I}\mathbf{G}_i{}^{\diamond\mathbf{H}} = \langle\prod_{i\in I}GHom(\mathbf{H}, \mathbf{G}_i), (s_\lambda; \lambda \in \Omega)\rangle$ and $(\prod_{i\in I}\mathbf{G}_i)^{\diamond\mathbf{H}} = \langle GHom(\mathbf{H}, \prod_{i\in I}\mathbf{G}_i), (t_\lambda; \lambda \in \Omega)\rangle$.

We define a map $\alpha\colon \prod_{i\in I}GHom(\mathbf{H}, \mathbf{G}_i) \to (\prod_{i\in I}G_i)^H$ by $\alpha(f^i;\ i \in I)(z) = (f^i(z);\ i \in I)$ for each $(f^i;\ i \in I) \in \prod_{i\in I}GHom(\mathbf{H}, \mathbf{G}_i)$ and each $z \in H$. Let $(f^i;\ i \in I) \in \prod_{i\in I}GHom(\mathbf{H}, \mathbf{G}_i)$ and $\lambda \in \Omega$. We will show that $\alpha(f^i;\ i \in I)(q_\lambda(x_j; j \in K_\lambda)) = r_\lambda(\alpha(f^i;\ i \in I)(x_j); j \in K_\lambda)$. Let $h \in \alpha(f^i;\ i \in I)(q_\lambda(x_j; j \in K_\lambda))$. Then $h = \alpha(f^i;\ i \in I)(x)$ where $x \in q_\lambda(x_j; j \in K_\lambda)$. Thus, $h = \alpha(f^i; i \in I)(x) = (f^i(x); i \in I)$. Since $x \in q_\lambda(x_j; j \in K_\lambda)$ and $f^i \in Hom(\mathbf{H}, \mathbf{G}_i)$ for every $i \in I$, we have $f^i(x) \in p^i_\lambda(f^i(x_j); j \in K_\lambda)$ for every $i \in I$. Thus, $h = (f^i(x); i \in I) \in r_\lambda((f^i(x_j); i \in I); j \in K_\lambda) = r_\lambda(\alpha(f^i;\ i \in I)(x_j); j \in K_\lambda)$. Consequently, $\alpha(f^i;\ i \in I)(q_\lambda(x_j; j \in K_\lambda)) \subseteq r_\lambda(\alpha(f^i;\ i \in I)(x_j); j \in K_\lambda)$.

Let $h \in r_\lambda(\alpha(f^i;\ i \in I)(x_j); j \in K_\lambda) = r_\lambda((f^i(x_j); i \in I); j \in K_\lambda)$. Then $h = (h^i; i \in I)$ such that $h^i \in p_\lambda(f^i(x_j); j \in K_\lambda)$ for every $i \in I$. Since $f^i \in GHom(\mathbf{H}, \mathbf{G}_i)$, we have $p_\lambda(f^i(x_j); j \in K_\lambda) = f^i(q_\lambda(x_j; j \in K_\lambda))$ for every $i \in I$. Therefore $h = (h^i; i \in I) \in (f^i(q_\lambda(x_j; j \in K_\lambda); i \in I) = \alpha(f^i;\ i \in I)(q_\lambda(x_j; j \in K_\lambda))$. Thus $r_\lambda(\alpha(f^i;\ i \in I)(x_j); j \in K_\lambda) \subseteq \alpha(f^i;\ i \in I)(q_\lambda(x_j; j \in K_\lambda))$. Therefore, $\alpha(f^i;\ i \in I) \in GHom(\mathbf{H}, \prod_{i\in I}\mathbf{G}_i)$. We have shown that α maps $\prod_{i\in I}GHom(\mathbf{H}, \mathbf{G}_i)$ into $GHom(\mathbf{H}, \prod_{i\in I}\mathbf{G}_i)$.

Further, we define a map $\varphi\colon GHom(\mathbf{H}, \prod_{i\in I}\mathbf{G}_i) \to \prod_{i\in I}GHom(\mathbf{H}, \mathbf{G}_i)$ by $\varphi(f) = (pr_i \cdot f; i \in I)$ whenever $f \in GHom(\mathbf{H}, \prod_{i\in I}\mathbf{G}_i)$ where $pr_i\colon \prod_{i\in I}\mathbf{G}_i \to \mathbf{G}_i$ is the i-th projection for every $i \in I$. It may easily be seen that $\varphi(f) \in \prod_{i\in I}GHom(\mathbf{H}, \mathbf{G}_i)$ and $\varphi^{-1} = \alpha$.

Finally, we will show that $\alpha(s_\lambda((f^i_j;\ i \in I); j \in K_\lambda)) = t_\lambda(\alpha(f^i_j;\ i \in I); j \in K_\lambda)$ whenever $\lambda \in \Omega$ and $(f^i_j;\ i \in I) \in \prod_{i\in I}GHom(\mathbf{H}, \mathbf{G}_i)$. It is easy to see that the following conditions are equivalent:

(a) $f \in \alpha(s_\lambda((f^i_j;\ i \in I); j \in K_\lambda))$;
(b) $f = \alpha(f^i; i \in I)$ where $(f^i; i \in I) \in s_\lambda((f^i_j; i \in I); j \in K_\lambda)$;
(c) $f(x) = (f^i(x); i \in I)$ for every $x \in H$ and $f^i(x) \in p^i_\lambda(f^i_j(x); j \in K_\lambda)$ for every $i \in I$ and for every $x \in H$;
(e) $f(x) = (f^i(x); i \in I) \in r_\lambda((f^i_j(x);\ i \in I); j \in K_\lambda)$ for every $x \in H$;
(f) $f(x) \in r_\lambda(\alpha(f^i_j;\ i \in I)(x); j \in K_\lambda)$ for every $x \in H$;
(g) $f \in t_\lambda(\alpha(f^i_j;\ i \in I); j \in K_\lambda)$.

Consequently, $\prod_{i\in I}\mathbf{G}_i^{\mathbf{H}} \cong (\prod_{i\in I}\mathbf{G}_i)^{\mathbf{H}}$. ∎

Definition 3.14 ([9]). A hyperalgebra $\langle G, (p_\lambda; \lambda \in \Omega)\rangle$ of type τ is called *diagonal* if, for every $\lambda \in \Omega$ and every $K_\lambda \times K_\lambda$-matrix (a_{ij}) over G, we have

$$p_\lambda(p_\lambda(a_{ij}; j \in K_\lambda); i \in K_\lambda)) \cap p_\lambda(p_\lambda(a_{ij}; i \in K_\lambda); j \in K_\lambda) \subseteq p_\lambda(a_{ii}; i \in K_\lambda).$$

Example 3.15. (1) It may easily be seen that the hyperalgebra from Example 3.3 is diagonal.

(2) For any point (x_0, y_0) of the real plane $R \times R$, put $(x_0, y_0)^{\neg} = \{(x, y) \in R \times R \mid x \leq x_0, y \leq y_0\}$ (i.e., $(x_0, y_0)^{\neg}$ is the left lower quarter of $R \times R$ with the vertex (x_0, y_0)). Further, for any pair $(x_1, y_1), (x_2, y_2)$ of points of $R \times R$ put $(x_1, y_1) * (x_2, y_2) = (x_1, y_2)^{\neg}$. Then $(R \times R, *)$ is a diagonal hypergroupoid.

Theorem 3.16. *Let $\mathbf{G}, \mathbf{H}$ be hyperalgebras of the same type and $\mathbf{G}$ be powerable by $\mathbf{H}$. If $\mathbf{G}$ is medial and diagonal, then $\mathbf{G}^{\diamond\mathbf{H}}$ is diagonal.*

Proof. Let $\mathbf{H} = \langle H, (q_\lambda; \lambda \in \Omega)\rangle$, $\mathbf{G} = \langle G, (p_\lambda; \lambda \in \Omega)\rangle$, $\mathbf{G}^{\diamond\mathbf{H}} = \langle GHom(\mathbf{H}, \mathbf{G}), (r_\lambda; \lambda \in \Omega)\rangle, \lambda \in \Omega$ and let (f_{ij}) be a $K_\lambda \times K_\lambda$-matrix over $GHom(\mathbf{H}, \mathbf{G})$. Suppose that $f \in r_\lambda(r_\lambda(f_{ij}; j \in K_\lambda); i \in K_\lambda) \cap r_\lambda(r_\lambda(f_{ij}; i \in K_\lambda); j \in K_\lambda)$. Then $f(x) \in p_\lambda(p_\lambda(f_{ij}(x); j \in K_\lambda); i \in K_\lambda) \cap p_\lambda(p_\lambda(f_{ij}(x); i \in K_\lambda); j \in K_\lambda)$ for every $x \in H$. Since $\mathbf{G}$ is diagonal, we have $f(x) \in p_\lambda(f_{ii}(x); i \in K_\lambda)$ for every $x \in H$. Thus, $f \in r_\lambda(f_{ii}; i \in K_\lambda)$. Hence, $\mathbf{G}^{\diamond\mathbf{H}}$ is diagonal. ∎

Theorem 3.17. *Let $\mathbf{H} = \langle H, (q_\lambda; \lambda \in \Omega)\rangle$, $\mathbf{G} = \langle G, (p_\lambda; \lambda \in \Omega)\rangle$ be hyperalgebras of the same type $\tau = (K_\lambda; \lambda \in \Omega)$ and let $\mathbf{G}$ be powerable by $\mathbf{H}$ and $\mathbf{G}^{\diamond\mathbf{H}} = \langle GHom(\mathbf{H}, \mathbf{G}),$ $(r_\lambda; \lambda \in \Omega)\rangle$. If $\mathbf{G}$ be medial and diagonal, then for any $\lambda \in \Omega$, any $(f_i; i \in K_\lambda) \in (GHom(\mathbf{H}, \mathbf{G}))^{K_\lambda}$, and any $f \in GHom(\mathbf{H}, \mathbf{G})$, $f \in r_\lambda(f_i; i \in K_\lambda)$ implies $f(q_\lambda(x_i; i \in K_\lambda)) \subseteq p_\lambda(f_i(x_i); i \in K_\lambda)$ for every $(x_i; i \in K_\lambda) \in H^{K_\lambda}$.*

Proof. Let $\lambda \in \Omega$, $(f_i; i \in K_\lambda) \in (Hom(\mathbf{H}, \mathbf{G}))^{K_\lambda}$, $f \in Hom(\mathbf{H}, \mathbf{G})$, and let $f \in r_\lambda(f_i; i \in K_\lambda)$. Let $(x_i; i \in K_\lambda) \in H^{K_\lambda}$ and suppose that $y \in f(q_\lambda(x_i; i \in K_\lambda))$. Then there exists $x \in q_\lambda(x_i; i \in K_\lambda)$ such that $y = f(x)$. Thus, we have $f(x_i) \in p_\lambda(f_i(x_i); i \in K_\lambda)$ for each $i \in K_\lambda$ and $f(x) \in p_\lambda(f_i(x); i \in K_\lambda)$. Since f_j is a good homomorphism, we get $f_j(x) \in p_\lambda(f_j(x_i); i \in K_\lambda)$ for each $j \in K_\lambda$. Hence, the diagonality of $\mathbf{G}$ implies $f(x) \in p_\lambda(f_i(x_i); i \in K_\lambda)$. ∎

4. Direct powers of algebras

Next, we will restrict the concept of power discussed in the previous section for hyperalgebras to algebras. The hyperalgebras $\mathbf{G} = \langle G, (p_\lambda; \lambda \in \Omega)\rangle$ of type $\tau = (K_\lambda; \lambda \in \Omega)$ having the property that card $p_\lambda(x_i; i \in K_\lambda) = 1$ for every $\lambda \in \Omega$ and every $(x_i; i \in K_\lambda) \in G^{K_\lambda}$ are nothing but the usual algebras.

Let $\mathbf{H} = \langle H, (q_\lambda; \lambda \in \Omega)\rangle, \mathbf{G} = \langle G, (p_\lambda; \lambda \in \Omega)\rangle$ be a pair of algebras of type τ. Then $\mathbf{G}$ is called a *subalgebra* of $\mathbf{H}$ provided that it is a subhyperalgebra of $\mathbf{H}$. Since the good homomorphisms between algebras coincide with the usual homomorphism of algebras, we write $Hom(\mathbf{H}, \mathbf{G})$ instead of $GHom(\mathbf{H}, \mathbf{G})$.

Clearly, the direct product of a family $\mathbf{G}_i = \langle G_i, (p^i_\lambda; \lambda \in \Omega)\rangle, i \in I$, of algebras of type $\tau = (K_\lambda; \lambda \in \Omega)$ is the algebra $\prod_{i \in I} \mathbf{G}_i = \langle \prod_{i \in I} G_i, (q_\lambda; \lambda \in \Omega)\rangle$ of type τ where $\prod_{i \in I} G_i$ denotes the Cartesian product and, for any $\lambda \in \Omega$ and any $(f_k; k \in K_\lambda) \in (\prod_{i \in I} G_i)^{K_\lambda}$, $q_\lambda(f_k; k \in K_\lambda) = \prod_{i \in I} p^i_\lambda(f_k(i); k \in K_\lambda)$.

Let $\mathbf{G} = \langle G, (p_\lambda; \lambda \in \Omega)\rangle$ be an algebra of type $\tau = (K_\lambda; \lambda \in \Omega)$ and let $\lambda_0 \in \Omega$. An element $x \in G$ is called an *idempotent* with respect to the operation p_{λ_0} if $x = p_{\lambda_0}(x_i; i \in K_{\lambda_0})$ whenever $x_i = x$ for every $i \in K_{\lambda_0}$. If every element of G is an idempotent with respect to each operation $p_\lambda, \lambda \in \Omega$, then the algebra $\mathbf{G}$ is said to be *idempotent*.

Let $\mathbf{G} = \langle G, (p_\lambda; \lambda \in \Omega)\rangle$ be an algebra of type $\tau = (K_\lambda, \lambda \in \Omega)$. Cleary, $\mathbf{G}$ is medial if and only if for every $\mu, \nu \in \Omega$ and every $K_\mu \times K_\nu$-matrix (a_{ij}) over G, $p_\mu(p_\nu(a_{ij}; j \in K_\nu; i \in K_\mu) = p_\nu(p_\mu(a_{ij}; i \in K_\mu); j \in K_\nu)$.

G is called *diagonal* if, for every $\lambda \in \Omega$ and every $K_\lambda \times K_\lambda$-matrix (a_{ij}) over G and $x \in G$, we have $p_\lambda(p_\lambda(a_{ij}; j \in K_\lambda); i \in K_\lambda)) = p_\lambda(p_\lambda(a_{ij}; i \in K_\lambda); j \in K_\lambda) = x$ and it follows that $p_\lambda(a_{ii}; i \in K_\lambda) = x$.

The medial algebras are often called *commutative* or the algebras satisfying the interchange law and they were dealt with in [15]. For idempotent algebras with one finitary operation, the diagonality introduced coincides with the diagonality which was studied in [7].

Example 4.1. (a) Obviously, if G is a nonvoid set and if $p : G^n \to G$ is a constant map or projection, then (G, p) is a medial and diagonal n-ary algebra. Let us consider all the binary operations p on the set $\{1, 2, 3\}$ for which the groupoid $\langle\{1, 2, 3\}, p\rangle$ is medial and diagonal: these are the three constant operations, the two projections, the six binary operations given by the following tables

	1	2	3
1	1	1	1
2	2	2	2
3	2	2	2

	1	2	3
1	1	1	1
2	3	3	3
3	3	3	3

	1	2	3
1	1	1	1
2	2	2	2
3	1	1	1

	1	2	3
1	1	1	1
2	1	1	1
3	3	3	3

	1	2	3
1	2	2	2
2	3	3	3
3	3	3	3

	1	2	3
1	3	3	3
2	2	2	2
3	3	3	3

and the six dual operations.(Binary operations p, q on set G is said to be dual if $p(x, y) = q(y, x)$ for all $x, y \in G$.)

(b) Clearly, a groupoid $\mathbf{G} = (G, p)$ is a medial and diagonal if and only if it is a semigroup with $xyz = xz$ whenever $x, y, z \in G$. Idempotent and diagonal groupoids are called *rectangular bands.*

As a consequence of Theorem 3.7 and the definition of the direct power of hyperalgebras, we get that for any algebras **H** and **G** of the same type τ, **G** medial, there is a subalgebra of the direct product $\mathbf{G}^H$ whose carrier is $Hom(\mathbf{H}, \mathbf{G})$. Therefore, we get the following proposition:

Proposition 4.2. *Let* **H** *and* **G** *be algebras of the same type* τ *and* **G** *be medial. Then* $\mathbf{G}^{\diamond\mathbf{H}} = \langle Hom(\mathbf{H}, \mathbf{G}), (r_\lambda; \lambda \in \Omega)\rangle$ *where* $r_\lambda(f_i; i \in K_\lambda)(x) = p_\lambda(f_i(x); i \in K_\lambda)$ *whenever* $x \in H$*,* $\lambda \in \Omega$ *and* $(f_i; i \in K_\lambda) \in (Hom(\mathbf{H}, \mathbf{G}))^{K_\lambda}$*.*

By Theorem 3.8 and Theorem 3.16, we get the following theorem.

Theorem 4.3. *Let* **G**, **H** *be algebras of the same type. If* **G** *is medial and diagonal, then so is* $\mathbf{G}^{\mathbf{H}}$*.*

As a consequence of Theorem 3.13, we get

Theorem 4.4. *Let* $\mathbf{G}_i = \langle G_i, (p^i_\lambda; \lambda \in \Omega)\rangle, i \in I$*, be a family of algebras of type* $\tau = (K_\lambda; \lambda \in \Omega)$ *and let* H *be an algebra of type* τ*. If* $\mathbf{G}_i$ *is medial for every* $i \in I$*, then*

$$\prod_{i\in I} \mathbf{G}_i^{\diamond\mathbf{H}} \cong (\prod_{i\in I} \mathbf{G}_i)^{\diamond\mathbf{H}}.$$

By Theorem 3.7, we immediately have the following theorem.

Theorem 4.5. *Let* $\mathbf{G}, \mathbf{H}, \mathbf{K}$ *be algebras of the same type. If* $\mathbf{G}$ *is medial and* $\mathbf{H}$, $\mathbf{K}$ *are idempotent, then*

$$\mathbf{G}^{\diamond(\mathbf{H}\times\mathbf{K})} \preccurlyeq (\mathbf{G}^{\diamond\mathbf{H}})^{\diamond\mathbf{K}}.$$

By Theorem 3.17, we get the following theorem.

Theorem 4.6. *Let* $\mathbf{H} = \langle H, (q_\lambda; \lambda \in \Omega)\rangle$, $\mathbf{G} = \langle G, (p_\lambda; \lambda \in \Omega)\rangle$, $\mathbf{G}^{\diamond\mathbf{H}} = \langle Hom(\mathbf{H}, \mathbf{G}), (r_\lambda; \lambda \in \Omega)\rangle$ *be algebras of the same type* $\tau = (K_\lambda; \lambda \in \Omega)$ *and let* $\mathbf{G}$ *be medial and diagonal. Then any* $\lambda \in \Omega$, *any* $(f_i; i \in K_\lambda) \in (Hom(\mathbf{H}, \mathbf{G}))^{K_\lambda}$, *and any* $f \in Hom(\mathbf{H}, \mathbf{G})$, $f = r_\lambda(f_i; i \in K_\lambda)$ *implies* $f(q_\lambda(x_i; i \in K_\lambda)) = p_\lambda(f_i(x_i); i \in K_\lambda)$ *for any* $(x_i; i \in K_\lambda) \in H^{K_\lambda}$.

Theorem 4.7. *Let* $\mathbf{G}, \mathbf{H}, \mathbf{K}$ *be algebras of the same type. If* $\mathbf{G}$ *is both medial and diagonal and* $\mathbf{H}$,$\mathbf{K}$ *are idempotent, then*

$$\mathbf{G}^{\diamond(\mathbf{H}\times\mathbf{K})} \cong (\mathbf{G}^{\diamond\mathbf{H}})^{\diamond\mathbf{K}}.$$

Proof. Let $\mathbf{H} = \langle H, (q_\lambda; \lambda \in \Omega)\rangle, \mathbf{G} = \langle G, (p_\lambda; \lambda \in \Omega)\rangle, \mathbf{K} = \langle K, (s_\lambda; \lambda \in \Omega)\rangle$, $\mathbf{G}^{\mathbf{H}} = \langle Hom(\mathbf{H}, \mathbf{G}), (r_\lambda; \lambda \in \Omega)\rangle, \mathbf{H}\otimes\mathbf{K} = \langle H\times K, (v_\lambda; \lambda \in \Omega)\rangle, (\mathbf{G}^{\mathbf{H}})^{\mathbf{K}} = \langle Hom(\mathbf{K}, \mathbf{G}^{\mathbf{H}}), (t_\lambda; \lambda \in \Omega)\rangle$, $\mathbf{G}^{\mathbf{H}\times\mathbf{K}} = \langle Hom(\mathbf{H}\times\mathbf{K}, \mathbf{G}), (u_\lambda; \lambda \in \Omega)\rangle$ and let $\mathbf{G}$ be diagonal and medial.

We will show that the canonical bijection $\alpha\colon (G^H)^K \to G^{H\times K}$ restricted to $Hom(\mathbf{K}, \mathbf{G}^{\mathbf{H}})$ is a bijection of $Hom(\mathbf{K}, \mathbf{G}^{\mathbf{H}})$ onto $Hom(\mathbf{H}\times\mathbf{K}, \mathbf{G})$. Let $\lambda \in \Omega$, $h \in Hom(\mathbf{K}, \mathbf{G}^{\mathbf{H}})$ and let $(y_i, z_i) \in H \times K$ for every $i \in K_\lambda$. To show that $\alpha(h)(v_\lambda((y_i, z_i); i \in K_\lambda)) = p_\lambda(\alpha(h)(y_i, z_i); i \in K_\lambda)$, let $\alpha(h)(y, z) = \varphi(h)(v_\lambda((y_i, z_i); i \in K_\lambda))$ where $(y, z) = v_\lambda((y_i, z_i); i \in K_\lambda)$. Thus, $y = q_\lambda(y_i; i \in K_\lambda)$ and $z = s_\lambda(z_i; i \in K_\lambda)$. Since $h \in Hom(\mathbf{K}, \mathbf{G}^{\mathbf{H}})$, we have $h(z) = r_\lambda(h(z_i); i \in K_\lambda)$. By Theorem 4.6, we get $h(z)(y) = p_\lambda(h(z_i)(y_i); i \in K_\lambda)$ because $y = q_\lambda(y_i; i \in K_\lambda)$ and $\mathbf{G}$ is diagonal. Thus $\alpha(h)(y, z) = p_\lambda(\alpha(h(y_i, z_i); i \in K_\lambda))$. We have shown that $\alpha(h) \in Hom(\mathbf{H}\times\mathbf{K}, \mathbf{G})$.

We will show that α is a homomorphism. Let $\alpha(h) = \alpha(t_\lambda(h_i; i \in K_\lambda))$ such that $h = t_\lambda(h_i; i \in K_\lambda)$. Since $h = t_\lambda(h_i; i \in K_\lambda)$, we have $h(z) = r_\lambda(h_i(z); i \in K_\lambda))$ for every $z \in K$. Therefore, $h(z)(y) = p_\lambda(h_i(z)(y); i \in K_\lambda))$ for every $y \in H$. Then $\alpha(h)(y, z) \in p_\lambda(\alpha(h_i)(y, z); i \in K_\lambda)$ for every $(y, z) \in H \times K$. Hence, $\alpha(h) = u_\lambda(\alpha(h_i); i \in K_\lambda)$. Thus, $\alpha(t_\lambda(h_i; i \in K_\lambda)) = u_\lambda(\alpha(h_i); i \in K_\lambda)$. We have shown that $\alpha(t_\lambda(h_i; i \in K_\lambda)) = u_\lambda(\alpha(h_i); i \in K_\lambda)$.

By Theorem 3.11 and Theorem 4.5, it may easily be seen that α is an isomorphism with $\varphi^{-1} = \alpha$, i.e.,$\mathbf{G}^{\diamond(\mathbf{H}\times\mathbf{K})} \cong (\mathbf{G}^{\diamond\mathbf{H}})^{\diamond\mathbf{K}}$. ∎

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