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Direct Product of Finite Intuitionistic Fuzzy Normal Subrings over Non-Associative Rings

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Abstract Shal et al. [21], introduced the concept of intuitionistic fuzzy normal subrings over a non-associative ring. In this note, we extend the concept of [21]. Spcefically we prove that, $X = A \times B$ and $Y = C \times D$ be two LA-subrings of an LA-ring $R_1 \times R_2$. Then $X \cap Y$ is an LA-subring of an LA-ring $R_1 \times R_2$ if and only if the intuitionistic characteristic function $\chi_Z = \langle \mu_{\chi_Z}, \gamma_{\chi_Z} \rangle$ of $Z = X \cap Y$ is an intuitionistic fuzzy normal LA-subring of an LA-ring $R_1 \times R_2$.

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1. Introduction

In 1972, a generalization of commutative semigroups has been established by Kazim et al. [1]. In ternary commutative law: $abc = cba$, they introduced braces on the left side of this law and explored a new pseudo associative law, that is $(ab)c = (cb)a$. This law $(ab)c = (cb)a$ is called the left invertive law. A groupoid S is said to be a left almost semigroup (abbreviated as LA-semigroup) if it satisfies the left invertive law.

In [2] (resp. [3]), a groupoid S is said to be medial (resp. paramedial) if $(ab)(cd) = (ac)(bd)$ (resp. $(ab)(cd) = (db)(ca)$). In [1], an LA-semigroup is medial, but in general an LA-semigroup needs not to be paramedial. Every LA-semigroup with left identity is paramedial in [4] and also satisfies $a(bc) = b(ac)$, $(ab)(cd) = (dc)(ba)$.

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S. Kamran [5], extended the notion of LA-semigroup to the left almost group (LA-group). An LA-semigroup G is said to be a left almost group, if there exists left identity $e \in G$ such that $ea = a$ for all $a \in G$ and for every $a \in G$ there exists $b \in G$ such that $ba = e$.

Shah et al [6], discussed the left almost ring (LA-ring) of finitely nonzero functions which is a generalization of commutative semigroup ring. By a left almost ring, we mean a non-empty set R with at least two elements such that $(R, +)$ is an LA-group, $(R, \cdot)$ is an LA-semigroup, both left and right distributive laws hold. For example, from a commutative ring $(R, +, \cdot)$, we can always obtain an LA-ring $(R, \oplus, \cdot)$ by defining for all $a, b \in R$, $a \oplus b = b - a$ and $a \cdot b$ is same as in the ring. Although the structure is non-associative and noncommutative, nevertheless, it possesses many interesting properties which we usually find in associative and commutative algebraic structures.

A non-empty subset A of an LA-ring R is called an LA-subring of an LA-ring R if $a - b$ and $ab \in A$ for all $a, b \in A$. A is called a left (resp. right) ideal of R if $(A, +)$ is an LA-group and $RA \subseteq A$ (resp. $AR \subseteq A$). A is called an ideal of R if it is both a left ideal and a right ideal of R.

First time the concept of fuzzy set was introduced by Zadeh in his classical paper [7]. This concept has provided a useful mathematical tool for describing the behaviour of systems that are too complex to admit precise mathematical analysis by classical methods and tools. Extensive applications of fuzzy set theory have been found in various fields such as artificial intelligence, computer science, management science, expert systems, finite state machines, Languages, robotics, coding theory and others.

It soon invoked a natural question concerning a possible connection between fuzzy sets and algebraic systems like (set, group, semigroup, ring, near-ring, semiring, measure) theory, groupoids, real analysis, topology, differential equations and so forth.

After the introduction of fuzzy set by Zadeh [7], several researchers explored on the generalization of the notion of fuzzy set.

The concept of intuitionistic fuzzy set was introduced by Atanassov [8, 9], as a generalization of the notion of fuzzy set.

Sherwood [10], introduced the concept of product of fuzzy subgroups. After this, further study on this concept continued by Osman [11, 12] and Ray [13]. Zaid [14], gave the idea of normal fuzzy subgroups.

An intuitionistic fuzzy set (briefly, IFS) A in a non-empty set X is an object having the form $A = \{(x, \mu_A(x), \gamma_A(x)) : x \in X\}$, where the functions $\mu_A \colon X \to [0, 1]$ and $\gamma_A \colon X \to [0, 1]$ denote the degree of membership and the degree of nonmembership, respectively and $0 \leq \mu_A(x) + \gamma_A(x) \leq 1$ for all $x \in X$ [8, 9].

An intuitionistic fuzzy set $A = \{(x, \mu_A(x), \gamma_A(x)) : x \in X\}$ in X can be identified to be an ordered pair (μ_A, γ_A) in $I^X \times I^X$, where I^X is the set of all functions from X to $[0, 1]$. For the sake of simplicity, we shall use the symbol $A = (\mu_A, \gamma_A)$ for the IFS $A = \{(x, \mu_A(x), \gamma_A(x)) : x \in X\}$.

Intuitionistic fuzzy subrings and intuitionistic fuzzy ideals of a ring have been defined in [15–17]. Palaniappan et al [18, 19], explored the notions of homomorphism, antihomomorphism of intuitionistic fuzzy normal subrings and also discussed some properties of intuitionistic fuzzy normal subrings. Moreover intuitionistic fuzzy ring and its homomorphism image have been investigated by Yan [20].

Shal et al [21], coined the concept of intuitionistic fuzzy normal subrings over a non-associative ring (LA-ring). Kausar et al [22], characterized the non-associative rings by the properties of their fuzzy ideals. Islam et al [23], explored the intuitionistics fuzzy ideals with thresholds $(\alpha,\ \beta]$ in LA-rings. Kausar et al [24], studied the left almost-rings by anti-fuzzy bi-ideals. Munir et al [25] dicussed the direct product of finite anti-fuzzy normal subrings over non-associative rings.

In this paper, we will extend the concept of [21] and will define two sections. In the first section, we will investigate the some basic properties of intuitionistic fuzzy normal LA-subrings of an LA-ring $R_1 \times R_2$. In the second section, we will define the direct product of intuitionistic fuzzy subsets $A_1, A_2, ..., A_n$ of LA-rings $R_1, R_2, ..., R_n$, respectively and examine the some fundamental properties of intuitionistic fuzzy normal LA-subrings of an LA-ring $R_1 \times R_2 \times ... \times R_n$. Specifically we will show that:

(1) Let $X = A \times B$ and $Y = C \times D$ be two LA-subrings of an LA-ring $R_1 \times R_2$. Then $X \cap Y$ is an LA-subring of an LA-ring $R_1 \times R_2$ if and only if the intuitionistic characteristic function $\chi_Z = \langle \mu_{\chi_Z}, \gamma_{\chi_Z} \rangle$ of $Z = X \cap Y$ is an intuitionistic fuzzy normal LA-subring of an LA-ring $R_1 \times R_2$.

(2) Let $A = A_1 \times A_2 \times ... \times A_n$ and $B = B \times B_2 \times ... \times B_n$ be two LA-subrings of an LA-ring $R_1 \times R_2 \times ... \times R_n$. Then $A \cap B$ is an LA-subring of an LA-ring $R_1 \times R_2 \times ... \times R_n$ if and only if the intuitionistic characteristic function $\chi_Z = \langle \mu_{\chi_Z}, \gamma_{\chi_Z} \rangle$ of $Z = A \cap B$ is an intuitionistic fuzzy normal LA-subring of an LA-ring $R_1 \times R_2 \times ... \times R_n$.

(3) Let $A = A_1 \times A_2 \times ... \times A_n$ and $B = B_1 \times B_2 \times ... \times B_n$ be intuitionistic fuzzy subsets of LA-rings $R = R_1 \times R_2 \times ... \times R_n$ and $R' = R_1' \times R_2' \times ... \times R_n'$ with left identities $e = (e_1, e_2, ..., e_n)$ and $e' = (e_1\prime, e_2\prime, ..., e_n\prime)$, respectively and $A \times B$ is an intuitionistic fuzzy normal LA-subring of an LA-ring $R \times R'$. Then the following conditions are true.

(i) If $\mu_A(x) \leq \mu_B(e')$ and $\gamma_A(x) \geq \gamma_B(e')$, for all $x \in R$, then A is an intuitionistic fuzzy normal LA-subring of R.

(ii) If $\mu_B(x) \leq \mu_A(e)$ and $\gamma_B(x) \geq \gamma_A(e)$, for all $x \in R'$, then B is an intuitionistic fuzzy normal LA-subring of R'.

2. Intuitionistic Fuzzy Normal LA-subrings

In this section, we extend the idea of second section of [21] and investigate the some basic properties of intuitionistic fuzzy normal LA-subrings of an LA-ring $R_1 \times R_2$.

Let μ_1 and μ_2 be fuzzy subsets of LA-rings R_1 and R_2, respectively. The direct product of fuzzy subsets μ_1 and μ_2 is denoted by $\mu_1 \times \mu_2$ and defined by $(\mu_1 \times \mu_2)(x_1, x_2) = min\{\mu_1(x_1), \mu_2(x_2)\}$.

A fuzzy subset $\mu_1 \times \mu_2$ of an LA-ring $R_1 \times R_2$ is said to be a fuzzy LA-subring of $R_1 \times R_2$ if

(1) $(\mu_1 \times \mu_2)(x - y) \geq min\{\mu_1(x), \mu_2(y)\}$,

(2) $(\mu_1 \times \mu_2)(xy) \geq min\{\mu_1(x), \mu_2(y)\}$ for all $x = (x_1, x_2), y = (y_1, y_2) \in R_1 \times R_2$.

A fuzzy subset $\mu_1 \times \mu_2$ of an LA-ring $R_1 \times R_2$ is said to be an anti fuzzy LA-subring of $R_1 \times R_2$ if

(1) $(\mu_1 \times \mu_2)(x - y) \leq max\{\mu_1(x), \mu_2(y)\}$

(2) $(\mu_1 \times \mu_2)(xy) \leq max\{\mu_1(x), \mu_2(y)\}$ for all $x = (x_1, x_2), y = (y_1, y_2) \in R_1 \times R_2$.

A fuzzy LA-subring of an LA-ring $R_1 \times R_2$ is said to be a fuzzy normal LA-subring of $R_1 \times R_2$ if $(\mu_1 \times \mu_2)(xy) = (\mu_1 \times \mu_2)(yx)$ for all $x = (x_1, x_2), y = (y_1, y_2) \in R_1 \times R_2$. Similarly for anti fuzzy normal LA-subring.

Let A and B be intuitionistic fuzzy sets of LA-rings R_1 and R_2, respectively. The direct product of A and B, is denoted by $A \times B$ and defined by

$$A \times B = \{((x, y), \mu_{A \times B}(x, y), \gamma_{A \times B}(x, y)) \mid \text{ for all } x \in R_1 \text{ and } y \in R_2\},$$

where $\mu_{A\times B}(x,y) = min\{\mu_A(x), \mu_B(y)\}$ and $\gamma_{A\times B}(x,y) = max\{\gamma_A(x), \gamma_B(y)\}$.

An intuitionistic fuzzy set (IFS) $A \times B = (\mu_{A\times B}, \gamma_{A\times B})$ of an LA-ring $R_1 \times R_2$ is said to be an intuitionistic fuzzy LA-subring (IFLSR) of $R_1 \times R_2$ if

(1) $\mu_{A\times B}(x-y) \geq min\{\mu_{A\times B}(x), \mu_{A\times B}(y)\}$,
(2) $\mu_{A\times B}(xy) \geq min\{\mu_{A\times B}(x), \mu_{A\times B}(y)\}$,
(3) $\gamma_{A\times B}(x-y) \leq max\{\gamma_{A\times B}(x), \gamma_{A\times B}(y)\}$,
(4) $\gamma_{A\times B}(xy) \leq max\{\gamma_{A\times B}(x), \gamma_{A\times B}(y)\}$, for all $x = (x_1, x_2), y = (y_1, y_2) \in R_1 \times R_2$.

An intuitionistic fuzzy LA-subring $A \times B = (\mu_{A\times B}, \gamma_{A\times B})$ of an LA-ring $R_1 \times R_2$ is said to be an intuitionistic fuzzy normal LA-subring (IFNLSR) of $R_1 \times R_2$ if $\mu_{A\times B}(xy) = \mu_{A\times B}(yx)$ and $\gamma_{A\times B}(xy) = \gamma_{A\times B}(yx)$ for all $x = (x_1, x_2), y = (y_1, y_2) \in R_1 \times R_2$.

Let $A \times B$ be a non-empty subset of an LA-ring $R_1 \times R_2$. The intuitionistic characteristic function of $A \times B$ is denoted by $\chi_{A\times B} = \langle \mu_{\chi_{A\times B}}, \gamma_{\chi_{A\times B}} \rangle$ and defined by

$$\mu_{\chi_{A\times B}}(x) = \begin{cases} 1 \text{ if } x \in A \times B \\ 0 \text{ if } x \notin A \times B \end{cases} \quad \text{and } \gamma_{\chi_{A\times B}}(x) = \begin{cases} 0 \text{ if } x \in A \times B \\ 1 \text{ if } x \notin A \times B \end{cases}$$

Lemma 2.1. [21, Lemma 4.2] *If A and B are LA-subrings of LA-rings R_1 and R_2, respectively, then $A \times B$ is an LA-subring of an LA-ring $R_1 \times R_2$ under the same operations defined as in $R_1 \times R_2$.*

Example 2.2. $R = \{0, 1, 2, 3, 4, 5, 6, 7, 8\}$ is an LA-ring, with left identity 7, such that $(xe)R \neq xR$, for all $x \in R$.

+	0	1	2	3	4	5	6	7	8
0	3	4	6	8	7	2	5	1	0
1	2	3	7	6	8	4	1	0	5
2	1	5	3	4	2	0	8	6	7
3	0	1	2	3	4	5	6	7	8
4	5	0	4	2	3	1	7	8	6
5	4	2	8	7	6	3	0	5	1
6	7	6	0	1	5	8	3	2	4
7	6	8	1	5	0	7	4	3	2
8	8	7	5	0	1	6	2	4	3

and

·	0	1	2	3	4	5	6	7	8
0	3	1	6	3	1	6	6	1	3
1	0	3	0	3	8	8	3	0	8
2	8	1	5	3	7	2	6	4	0
3	3	3	3	3	3	3	3	3	3
4	0	6	7	3	5	4	1	2	8
5	8	6	4	3	2	7	1	5	0
6	8	3	8	3	0	0	3	8	0
7	0	1	2	3	4	5	6	7	8
8	3	6	1	3	6	1	1	6	3

Example 2.3. $R' = \{0, 1, 2, 3, 4, 5, 6, 7, 8\}$ is an LA-ring, with left identity 5, such that $(xe)R \neq xR$, for all $x \in R$.

+	0	1	2	3	4	5	6	7	8
0	3	0	5	1	8	7	4	2	6
1	1	3	7	0	6	2	8	5	4
2	4	6	3	8	7	1	5	0	2
3	0	1	2	3	4	5	6	7	8
4	2	7	6	5	3	4	0	8	1
5	6	8	0	4	5	3	2	1	7
6	5	2	4	7	1	8	3	6	0
7	8	4	1	6	2	0	7	3	5
8	7	5	8	2	0	6	1	4	3

and

·	0	1	2	3	4	5	6	7	8
0	4	5	1	3	6	7	2	8	0
1	5	4	0	3	7	6	8	2	1
2	6	7	5	3	2	8	1	0	4
3	3	3	3	3	3	3	3	3	3
4	1	0	8	3	5	4	7	6	2
5	0	1	2	3	4	5	6	7	8
6	8	2	6	3	0	1	4	5	7
7	2	8	7	3	1	0	5	4	6
8	7	6	4	3	8	2	0	1	5

Example 2.4. Using Example 2.2 and 2.3, we write

$$\begin{aligned} R \quad = \quad R \times R' = \{&(0,0),(0,1),(0,2),(0,3),(0,4),(0,5),(0,6),(0,7),(0,8),\\ &(1,0),(1,1),(1,2),(1,3),(1,4),(1,5),(1,6),(1,7),(1,8)\\ &(2,0),(2,1),(2,2),(2,3),(2,4),(2,5),(2,6),(2,7),(2,8)\\ &(3,0),(3,1),(3,2),(3,3),(3,4),(3,5),(3,6),(3,7),(3,8)\\ &(4,0),(4,1),(4,2),(4,3),(4,4),(4,5),(4,6),(4,7),(4,8)\\ &(5,0),(5,1),(5,2),(5,3),(5,4),(5,5),(5,6),(5,7),(5,8)\\ &(6,0),(6,1),(6,2),(6,3),(6,4),(6,5),(6,6),(6,7),(6,8)\\ &(7,0),(7,1),(7,2),(7,3),(7,4),(7,5),(7,6),(7,7),(7,8)\\ &(8,0),(8,1),(8,2),(8,3),(8,4),(8,5),(8,6),(8,7),(8,8)\} \end{aligned}$$

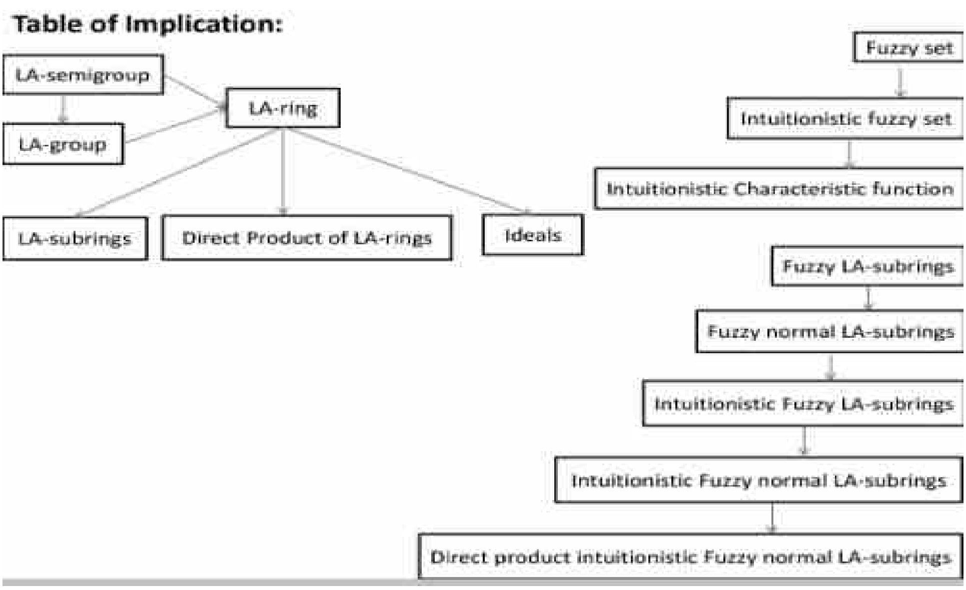
is an LA-ring under the addition and multiplication defined as

$$\begin{aligned} (a,b)\oplus(c,d) &= (a+c, b+d)\\ (a,b)\odot(c,d) &= (a\cdot c, b\cdot d) \end{aligned}$$

for all $(a,b),(c,d) \in R \times R'$.
For example $(4,2)\oplus(5,2) = (1,3)$ and $(4,2)\odot(5,2) = (4,5)$, $4,5 \in R$ and $2,2 \in R'$.

Noted: this ring is called direct product LA-ring of LA-ring R and LA-ring R'. And $S = \{(3,0),(3,1),(3,2),(3,3),(3,4),(3,5),(3,6),(3,7),(3,8)\}$ is a subring of $R \times R'$, this subring is called LA-subring of $L \times L'$.

Note the figure for implication:



Theorem 2.5. [21, Theorem 4.3] *Let A and B be LA-subrings of LA-rings R_1 and R_2, respectively. Then $A\times B$ is an LA-subring of an LA-ring $R_1\times R_2$ if and only if the intuitionistic characteristic function $\chi_C = \langle \mu_{\chi_C}, \gamma_{\chi_C}\rangle$ of $C = A \times B$ is an intuitionistic fuzzy normal LA-subring of an LA-ring $R_1 \times R_2$.*

Lemma 2.6. *If $X = A \times B$ and $Y = C \times D$ are two LA-subrings of an LA-ring $R_1 \times R_2$, then their intersection $X \cap Y$ is also an LA-subring of an LA-ring $R_1 \times R_2$.*

Proof. Straight forward. ∎

Theorem 2.7. *Let $X = A \times B$ and $Y = C \times D$ be two LA-subrings of an LA-ring $R_1 \times R_2$. Then $X \cap Y$ is an LA-subring of an LA-ring $R_1 \times R_2$ if and only if the intuitionistic characteristic function $\chi_Z = \langle \mu_{\chi_Z}, \gamma_{\chi_Z} \rangle$ of $Z = X \cap Y$ is an intuitionistic fuzzy normal LA-subring of an LA-ring $R_1 \times R_2$.*

Proof. Let $Z = X \cap Y$ be an LA-subring of an LA-ring $R_1 \times R_2$ and $a = (a_1, a_2), b = (b_1, b_2) \in R_1 \times R_2$. If $a, b \in Z = X \cap Y$, then by definition of intuitionistic characteristic function $\mu_{\chi_Z}(a) = 1 = \mu_{\chi_Z}(b)$ and $\gamma_{\chi_Z}(a) = 0 = \gamma_{\chi_Z}(b)$. Since $a - b$ and $ab \in Z$, Z being an LA-subring of $R_1 \times R_2$. This implies that

$$\begin{aligned}
\mu_{\chi_Z}(a-b) &= 1 = 1 \wedge 1 = \mu_{\chi_Z}(a) \wedge \mu_{\chi_Z}(b), \\
\mu_{\chi_Z}(ab) &= 1 = 1 \wedge 1 = \mu_{\chi_Z}(a) \wedge \mu_{\chi_Z}(b), \\
\gamma_{\chi_Z}(a-b) &= 0 = 0 \vee 0 = \gamma_{\chi_Z}(a) \vee \gamma_{\chi_Z}(b), \\
\gamma_{\chi_Z}(ab) &= 0 = 0 \vee 0 = \gamma_{\chi_Z}(a) \vee \gamma_{\chi_Z}(b).
\end{aligned}$$

Thus

$$\begin{aligned}
\mu_{\chi_Z}(a-b) &\geq min\{\mu_{\chi_Z}(a), \mu_{\chi_Z}(b)\}, \\
\mu_{\chi_Z}(ab) &\geq min\{\mu_{\chi_Z}(a), \mu_{\chi_Z}(b)\}, \\
\gamma_{\chi_Z}(a-b) &\leq max\{\gamma_{\chi_Z}(a), \gamma_{\chi_Z}(b)\}, \\
\gamma_{\chi_Z}(a-b) &\leq max\{\gamma_{\chi_Z}(a), \gamma_{\chi_Z}(b)\}.
\end{aligned}$$

As ab and $ba \in Z$, by definition $\mu_{\chi_Z}(ab) = 1 = \mu_{\chi_Z}(ba)$ and $\gamma_{\chi_Z}(ab) = 0 = \gamma_{\chi_Z}(ba)$, i.e., $\mu_{\chi_Z}(ab) = \mu_{\chi_Z}(ba)$ and $\gamma_{\chi_Z}(ab) = \gamma_{\chi_Z}(ba)$. Similarly, we have

$$\begin{aligned}
\mu_{\chi_Z}(a-b) &\geq min\{\mu_{\chi_Z}(a), \mu_{\chi_Z}(b)\},\ \mu_{\chi_Z}(ab) \geq min\{\mu_{\chi_Z}(a), \mu_{\chi_Z}(b)\}, \\
\gamma_{\chi_Z}(a-b) &\leq max\{\gamma_{\chi_Z}(a), \gamma_{\chi_Z}(b)\},\ \gamma_{\chi_Z}(ab) \leq max\{\gamma_{\chi_Z}(a), \gamma_{\chi_Z}(b)\}, \\
\gamma_{\chi_Z}(ab) &= \gamma_{\chi_Z}(ba),\ \gamma_{\chi_Z}(ab) = \gamma_{\chi_Z}(ba),
\end{aligned}$$

when $a, b \notin Z$. Hence the intuitionistic characteristic function $\chi_Z = \langle \mu_{\chi_Z}, \gamma_{\chi_Z} \rangle$ of Z is an intuitionistic fuzzy normal LA-subring of an LA-ring $R_1 \times R_2$.

Conversely, suppose that the intuitionistic characteristic function $\chi_Z = \langle \mu_{\chi_Z}, \gamma_{\chi_Z} \rangle$ of $Z = X \cap Y$ is an intuitionistic fuzzy normal LA-subring of an LA-ring $R_1 \times R_2$. Let $a, b \in Z = X \cap Y$, this means that $\mu_{\chi_Z}(a) = 1 = \mu_{\chi_Z}(b)$ and $\gamma_{\chi_Z}(a) = 0 = \gamma_{\chi_Z}(b)$. By our supposition

$$\begin{aligned}
\mu_{\chi_Z}(a-b) &\geq \mu_{\chi_Z}(a) \wedge \mu_{\chi_Z}(b) = 1 \wedge 1 = 1, \\
\mu_{\chi_Z}(ab) &\geq \mu_{\chi_Z}(a) \wedge \mu_{\chi_Z}(b) = 1 \wedge 1 = 1, \\
\gamma_{\chi_Z}(a-b) &\leq \gamma_{\chi_Z}(a) \vee \gamma_{\chi_Z}(b) = 0 \vee 0 = 0, \\
\gamma_{\chi_Z}(ab) &\leq \gamma_{\chi_Z}(a) \vee \gamma_{\chi_Z}(b) = 0 \vee 0 = 0.
\end{aligned}$$

Thus $\mu_{\chi_Z}(a-b) = 1 = \mu_{\chi_Z}(ab)$ and $\gamma_{\chi_Z}(a-b) = 0 = \gamma_{\chi_Z}(ab)$, i.e., $a - b$ and $ab \in Z$. Hence Z is an LA-subring of an LA-ring $R_1 \times R_2$. ∎

Corollary 2.8. *Let $\{C_i\}_{i \in I} = \{A_i \times B_i\}_{i \in I}$ be a family of LA-subrings of an LA-ring $R_1 \times R_2$. Then $C = \cap C_i$ is an LA-subring of an LA-ring $R_1 \times R_2$ if and only if the intuitionistic characteristic function $\chi_C = \langle \mu_{\chi_c}, \gamma_{\chi_c} \rangle$ of $C = \cap C_i$ is an intuitionistic fuzzy normal LA-subring of an LA-ring $R_1 \times R_2$.*

Theorem 2.9. [21, 4.4] *If A and B are intuitionistic fuzzy normal LA-subrings of LA-rings R_1 and R_2, respectively, then $A \times B$ is an intuitionistic fuzzy normal LA-subring of an LA-ring $R_1 \times R_2$.*

Theorem 2.10. *If $X = A \times B$ and $Y = C \times D$ are two intuitionistic fuzzy normal LA-subrings of an LA-ring $R_1 \times R_2$, then their intersection $X \cap Y$ is also an intuitionistic fuzzy normal LA-subring of an LA-ring $R_1 \times R_2$.*

Proof. Let $X = A \times B = \{((x_1, x_2), \mu_{A\times B}(x_1, x_2), \gamma_{A\times B}(x_1, x_2)) \mid \text{for all } (x_1, x_2) \in R_1 \times R_2\}$ and $Y = C \times D = \{((y_1, y_2), \mu_{C\times D}(y_1, y_2), \gamma_{C\times D}(y_1, y_2)) \mid \text{for all } (y_1, y_2) \in R_1 \times R_2\}$ be two intuitionistic fuzzy normal LA-subrings of an LA-ring $R_1 \times R_2$. Let $Z = X \cap Y$ and $Z = \{((z_1, z_2), \mu_Z(z_1, z_2), \gamma_Z(z_1, z_2)) \mid (z_1, z_2) \in R_1 \times R_2\}$, where $\mu_Z(z_1, z_2) = \mu_{X\cap Y}(z_1, z_2) = min\{\mu_X(z_1, z_2), \mu_Y(z_1, z_2)\}$ and $\gamma_Z(z_1, z_2) = \gamma_{X\cap Y}(z_1, z_2) = max\{\gamma_X(z_1, z_2), \gamma_Y(z_1, z_2)\}$. Now

$$\begin{aligned}
\mu_Z((z_1, z_2) - (z_3, z_4)) &= \mu_{X\cap Y}((z_1, z_2) - (z_3, z_4)) \\
&= min\{\mu_X((z_1, z_2) - (z_3, z_4)), \mu_Y((z_1, z_2) - (z_3, z_4))\} \\
&\geq \{\mu_X(z_1, z_2) \wedge \mu_X(z_3, z_4)\} \wedge \{\mu_Y(z_1, z_2) \wedge \mu_Y(z_3, z_4)\} \\
&= \mu_X(z_1, z_2) \wedge \{\mu_X(z_3, z_4) \wedge \mu_Y(z_1, z_2)\} \wedge \mu_Y(z_3, z_4) \\
&= \mu_X(z_1, z_2) \wedge \{\mu_Y(z_1, z_2) \wedge \mu_X(z_3, z_4)\} \wedge \mu_Y(z_3, z_4) \\
&= \{\mu_X(z_1, z_2) \wedge \mu_Y(z_1, z_2)\} \wedge \{\mu_X(z_3, z_4) \wedge \mu_Y(z_3, z_4)\} \\
&= min\{\mu_{X\cap Y}(z_1, z_2), \mu_{X\cap Y}(z_3, z_4)\} \\
&= min\{\mu_Z(z_1, z_2), \mu_Z(z_3, z_4)\}.
\end{aligned}$$

and

$$\begin{aligned}
\mu_Z((z_1, z_2) \circ (z_3, z_4)) &= \mu_{X\cap Y}((z_1, z_2) \circ (z_3, z_4)) \\
&= min\{\mu_X((z_1, z_2) \circ (z_3, z_4)), \mu_Y((z_1, z_2) \circ (z_3, z_4))\} \\
&\geq \{\mu_X(z_1, z_2) \wedge \mu_X(z_3, z_4)\} \wedge \{\mu_Y(z_1, z_2) \wedge \mu_Y(z_3, z_4)\} \\
&= \mu_X(z_1, z_2) \wedge \{\mu_X(z_3, z_4) \wedge \mu_Y(z_1, z_2)\} \wedge \mu_Y(z_3, z_4) \\
&= \mu_X(z_1, z_2) \wedge \{\mu_Y(z_1, z_2) \wedge \mu_X(z_3, z_4)\} \wedge \mu_Y(z_3, z_4) \\
&= \{\mu_X(z_1, z_2) \wedge \mu_Y(z_1, z_2)\} \wedge \{\mu_X(z_3, z_4) \wedge \mu_Y(z_3, z_4)\} \\
&= min\{\mu_{X\cap Y}(z_1, z_2), \mu_{X\cap Y}(z_3, z_4)\} \\
&= min\{\mu_Z(z_1, z_2), \mu_Z(z_3, z_4)\}.
\end{aligned}$$

Similarly

$$\begin{aligned}
\gamma_Z((z_1, z_2) - (z_3, z_4)) &\leq max\{\gamma_Z(z_1, z_2), \gamma_Z(z_3, z_4)\} \\
\text{and } \gamma_Z((z_1, z_2) \circ (z_3, z_4)) &\leq max\{\gamma_Z(z_1, z_2), \gamma_Z(z_3, z_4)\}.
\end{aligned}$$

Thus $Z = (\mu_Z, \gamma_Z)$ is an intuitionistic fuzzy LA-subring of an LA-ring $R_1 \times R_2$. Now

$$\begin{aligned}
\mu_Z((z_1, z_2) \circ (z_3, z_4)) &= \mu_{X\cap Y}(z_1z_3, z_2z_4) \\
&= min\{\mu_X(z_1z_3, z_2z_4), \mu_Y(z_1z_3, z_2z_4)\} \\
&= min\{\mu_X(z_3z_1, z_4z_2), \mu_Y(z_3z_1, z_4z_2)\} \\
&= \mu_{X\cap Y}(z_3z_1, z_4z_2) \\
&= \mu_Z((z_3, z_4) \circ (z_1, z_2)).
\end{aligned}$$

Similarly $\gamma_Z((z_1, z_2)\circ(z_3, z_4)) = \gamma_Z((z_3, z_4)\circ(z_1, z_2))$. Hence $Z = X\cap Y$ is an intuitionistic fuzzy normal LA-subring of an LA-ring $R_1 \times R_2$. ■

Corollary 2.11. *If $\{C_i\}_{i\in I} = \{A_i \times B_i\}_{i\in I}$ is a family of intuitionistic fuzzy normal LA-subrings of an LA-ring $R_1\times R_2$, then $C = \cap C_i$ is also an intuitionistic fuzzy normal LA-subring of an LA-ring $R_1 \times R_2$.*

Theorem 2.12. *If $X = A\times B$ and $Y = C\times D$ are intuitionistic fuzzy normal LA-subrings of LA-rings $R' = R_1 \times R_2$ and $R'' = R_3 \times R_4$, respectively, then $Z = X \times Y$ is an intuitionistic fuzzy normal LA-subring of an LA-ring $R' \times R'' = (R_1 \times R_2) \times (R_3 \times R_4)$.*

Proof. Let $X = A \times B = \{((x_1, x_2), \mu_{A\times B}(x_1, x_2), \gamma_{A\times B}(x_1, x_2)) \mid \text{for all } (x_1, x_2) \in R_1 \times R_2\}$ and $Y = C \times D = \{((y_1, y_2), \mu_{C\times D}(y_1, y_2), \gamma_{C\times D}(y_1, y_2)) \mid \text{for all } (y_1, y_2) \in R_3 \times R_4\}$ be intuitionistic fuzzy normal LA-subrings of LA-rings $R' = R_1 \times R_2$ and $R'' = R_3 \times R_4$, respectively. Let $Z = X \times Y$ and $Z = \{((z', z''), \mu_Z(z', z''), \gamma_Z(z', z'')) \mid (z', z'') = ((z_1, z_2), (z_3, z_4)) \in R' \times R''\}$, where

$$\begin{aligned} \mu_Z(z', z'') &= \mu_{X\times Y}((z_1, z_2), (z_3, z_4)) = min\{\mu_X(z_1, z_2), \mu_Y(z_3, z_4)\} \\ \text{and } \gamma_Z(z', z'') &= \gamma_{X\times Y}((z_1, z_2), (z_3, z_4)) = max\{\gamma_X(z_1, z_2), \gamma_Y(z_3, z_4)\}. \end{aligned}$$

Now

$$\begin{aligned} & \mu_Z(((z_1, z_2), (z_3, z_4)) - ((z_5, z_6), (z_7, z_8))) \\ = {} & \mu_{X\times Y}(((z_1, z_2), (z_3, z_4)) - ((z_5, z_6), (z_7, z_8))) \\ = {} & \mu_{X\times Y}(((z_1, z_2) - (z_5, z_6)), ((z_3, z_4) - (z_7, z_8))) \\ = {} & min\{\mu_X((z_1, z_2) - (z_5, z_6)), \mu_Y((z_3, z_4) - (z_7, z_8))\} \\ \geq {} & min\{(\mu_X(z_1, z_2) \wedge \mu_X(z_5, z_6)), (\mu_Y(z_3, z_4) \wedge \mu_Y(z_7, z_8))\} \\ = {} & \{(\mu_X(z_1, z_2) \wedge \mu_Y(z_5, z_6)) \wedge (\mu_X(z_3, z_4) \wedge \mu_Y(z_7, z_8))\} \\ = {} & \{(\mu_X(z_1, z_2) \wedge \mu_Y(z_3, z_4)) \wedge (\mu_X(z_5, z_6) \wedge \mu_Y(z_7, z_8))\} \\ = {} & min\{(\mu_X(z_1, z_2) \wedge \mu_Y(z_3, z_4)), (\mu_X(z_5, z_6) \wedge \mu_Y(z_7, z_8))\} \\ = {} & min\{\mu_{X\times Y}((z_1, z_2), (z_3, z_4)), \mu_{X\times Y}((z_5, z_6), (z_7, z_8))\} \\ = {} & min\{\mu_Z((z_1, z_2), (z_3, z_4)), \mu_Z((z_5, z_6), (z_7, z_8))\}. \end{aligned}$$

and

$$\begin{aligned} & \mu_Z(((z_1, z_2), (z_3, z_4)) \circ ((z_5, z_6), (z_7, z_8))) \\ = {} & \mu_{X\times Y}(((z_1, z_2), (z_3, z_4)) \circ ((z_5, z_6), (z_7, z_8))) \\ = {} & \mu_{X\times Y}(((z_1, z_2) \circ (z_5, z_6)), ((z_3, z_4) \circ (z_7, z_8))) \\ = {} & min\{\mu_X((z_1, z_2) \circ (z_5, z_6)), \mu_Y((z_3, z_4) \circ (z_7, z_8))\} \\ \geq {} & min\{(\mu_X(z_1, z_2) \wedge \mu_X(z_5, z_6)), (\mu_Y(z_3, z_4) \wedge \mu_Y(z_7, z_8))\} \\ = {} & \{(\mu_X(z_1, z_2) \wedge \mu_Y(z_5, z_6)) \wedge (\mu_X(z_3, z_4) \wedge \mu_Y(z_7, z_8))\} \\ = {} & \{(\mu_X(z_1, z_2) \wedge \mu_Y(z_3, z_4)) \wedge (\mu_X(z_5, z_6) \wedge \mu_Y(z_7, z_8))\} \\ = {} & min\{(\mu_X(z_1, z_2) \wedge \mu_Y(z_3, z_4)), (\mu_X(z_5, z_6) \wedge \mu_Y(z_7, z_8))\} \\ = {} & min\{\mu_{X\times Y}((z_1, z_2), (z_3, z_4)), \mu_{X\times Y}((z_5, z_6), (z_7, z_8))\} \\ = {} & min\{\mu_Z((z_1, z_2), (z_3, z_4)), \mu_Z((z_5, z_6), (z_7, z_8))\}. \end{aligned}$$

Similarly

$$\begin{aligned} & \gamma_Z(((z_1, z_2), (z_3, z_4)) - ((z_5, z_6), (z_7, z_8))) \\ \leq\ & max\{\gamma_Z((z_1, z_2), (z_3, z_4)), \gamma_Z((z_5, z_6), (z_7, z_8))\} \\ & \text{and } \gamma_Z(((z_1, z_2), (z_3, z_4)) \circ ((z_5, z_6), (z_7, z_8))) \\ \leq\ & max\{\gamma_Z((z_1, z_2), (z_3, z_4)), \gamma_Z((z_5, z_6), (z_7, z_8))\}. \end{aligned}$$

Thus $Z = (\mu_Z, \gamma_Z)$ is an intuitionistic fuzzy LA-subring of an LA-ring $R\prime \times R''$. Now

$$\begin{aligned} & \mu_Z(((z_1, z_2), (z_3, z_4)) \circ ((z_5, z_6), (z_7, z_8))) \\ =\ & \mu_{X\times Y}(((z_1, z_2), (z_3, z_4)) \circ ((z_5, z_6), (z_7, z_8))) \\ =\ & \mu_{X\times Y}(((z_1, z_2) \circ (z_5, z_6)), ((z_3, z_4) \circ (z_7, z_8))) \\ =\ & min\{\mu_X((z_1, z_2) \circ (z_5, z_6)), \mu_Y((z_3, z_4) \circ (z_7, z_8))\} \\ =\ & min\{\mu_X((z_5, z_6) \circ (z_1, z_2)), \mu_Y((z_7, z_8) \circ (z_3, z_4))\} \\ =\ & \mu_{X\times Y}(((z_5, z_6) \circ (z_1, z_2)), ((z_7, z_8) \circ (z_3, z_4))) \\ =\ & \mu_{X\times Y}(((z_5, z_6), (z_7, z_8)) \circ ((z_1, z_2), (z_3, z_4))) \\ =\ & \mu_Z(((z_5, z_6), (z_7, z_8)) \circ ((z_1, z_2), (z_3, z_4))). \end{aligned}$$

Similarly

$$\begin{aligned} & \gamma_Z(((z_1, z_2), (z_3, z_4)) \circ ((z_5, z_6), (z_7, z_8))) \\ =\ & \gamma_Z(((z_5, z_6), (z_7, z_8)) \circ ((z_1, z_2), (z_3, z_4))). \end{aligned}$$

Hence $Z = X \times Y$ is an intuitionistic fuzzy normal LA-subring of an LA-ring $R' \times R''$. ■

Proposition 2.13. *If an IFS $A \times B$ is an intuitionistic fuzzy normal LA-subring of an LA-ring $R_1 \times R_2$, then $\Box A \times B = \left(\mu_{A\times B}, \overline{\mu}_{A\times B}\right)$ (resp. $\Diamond A \times B = \left(\overline{\gamma}_{A\times B}, \gamma_{A\times B}\right)$) is also an intuitionistic fuzzy normal LA-subring of an LA-ring $R_1 \times R_2$.*

Proof. Let $A \times B$ be an intuitionistic fuzzy normal LA-subring of an LA-ring $R_1 \times R_2$. We have to show that $\Box A \times B = \left(\mu_{A\times B}, \overline{\mu}_{A\times B}\right)$ is also an intuitionistic fuzzy normal LA-subring of an LA-ring $R_1 \times R_2$. Now

$$\begin{aligned} \overline{\mu}_{A\times B}((x_1, x_2) - (y_1, y_2)) &= 1 - \mu_{A\times B}((x_1, x_2) - (y_1, y_2)) \\ &\leq 1 - min\{\mu_{A\times B}(x_1, x_2), \mu_{A\times B}(y_1, y_2)\} \\ &= max\{1 - \mu_{A\times B}(x_1, x_2), 1 - \mu_{A\times B}(y_1, y_2)\} \\ &= max\{\overline{\mu}_{A\times B}(x_1, x_2), \overline{\mu}_{A\times B}(y_1, y_2)\}. \\ \overline{\mu}_{A\times B}((x_1, x_2) \circ (y_1, y_2)) &= 1 - \mu_{A\times B}((x_1, x_2) \circ (y_1, y_2)) \\ &\leq 1 - min\{\mu_{A\times B}(x_1, x_2), \mu_{A\times B}(y_1, y_2)\} \\ &= max\{1 - \mu_{A\times B}(x_1, x_2), 1 - \mu_{A\times B}(y_1, y_2)\} \\ &= max\{\overline{\mu}_{A\times B}(x_1, x_2), \overline{\mu}_{A\times B}(y_1, y_2)\}. \end{aligned}$$

Thus $\Box A \times B = \left(\mu_{A\times B}, \overline{\mu}_{A\times B}\right)$ is an intuitionistic fuzzy LA-subring of an LA-ring $R_1 \times R_2$. Now

$$\begin{aligned} \overline{\mu}_{A\times B}((x_1, x_2) \circ (y_1, y_2)) &= 1 - \mu_{A\times B}((x_1, x_2) \circ (y_1, y_2)) \\ &= 1 - \mu_{A\times B}((y_1, y_2) \circ (x_1, x_2)) \\ &= \overline{\mu}_{A\times B}((y_1, y_2) \circ (x_1, x_2)). \end{aligned}$$

Hence $\square A\times B=\left(\mu_{A\times B},\overline{\mu}_{A\times B}\right)$ is an intuitionistic fuzzy normal LA-subring of an LA-ring $R_1\times R_2$. Similarly, $\Diamond A\times B=\left(\overline{\gamma}_{A\times B},\gamma_{A\times B}\right)$ is an intuitionistic fuzzy normal LA-subring of an LA-ring $R_1\times R_2$. ∎

Corollary 2.14. *An IFS $A\times B$ is an intuitionistic fuzzy normal LA-subring of an LA-ring $R_1\times R_2$ if and only if $\square A\times B=\left(\mu_{A\times B},\overline{\mu}_{A\times B}\right)$ and (resp. $\Diamond A\times B=\left(\overline{\gamma}_{A\times B},\gamma_{A\times B}\right)$) is an intuitionistic fuzzy normal LA-subring of an LA-ring $R_1\times R_2$.*

Theorem 2.15. *An IFS $A\times B=(\mu_{A\times B},\gamma_{A\times B})$ is an intuitionistic fuzzy normal LA-subring of an LA-ring $R_1\times R_2$ if and only if the fuzzy subsets $\mu_{A\times B}$ and $\overline{\gamma}_{A\times B}$ are fuzzy normal LA-subrings of an LA-ring $R_1\times R_2$.*

Proof. Let $A\times B=(\mu_{A\times B},\gamma_{A\times B})$ be an intuitionistic fuzzy normal LA-subring of an LA-ring $R_1\times R_2$, this implies that $\mu_{A\times B}$ is a fuzzy normal LA-subring of an LA-ring $R_1\times R_2$. We have to show that $\overline{\gamma}_{A\times B}$ is also a fuzzy normal LA-subring of $R_1\times R_2$. Now

$$\begin{aligned}
\overline{\gamma}_{A\times B}((x_1,x_2)-(y_1,y_2)) &= 1-\gamma_{A\times B}((x_1,x_2)-(y_1,y_2))\\
&\geq 1-max\{\gamma_{A\times B}(x_1,x_2),\gamma_{A\times B}(y_1,y_2)\}\\
&= min\{1-\gamma_{A\times B}(x_1,x_2),1-\gamma_{A\times B}(y_1,y_2)\}\\
&= min\{\overline{\gamma}_{A\times B}(x_1,x_2),\overline{\gamma}_{A\times B}(y_1,y_2)\}.\\
\overline{\gamma}_{A\times B}((x_1,x_2)\circ(y_1,y_2)) &= 1-\gamma_{A\times B}((x_1,x_2)\circ(y_1,y_2))\\
&\geq 1-max\{\gamma_{A\times B}(x_1,x_2),\gamma_{A\times B}(y_1,y_2)\}\\
&= min\{1-\gamma_{A\times B}(x_1,x_2),1-\gamma_{A\times B}(y_1,y_2)\}\\
&= min\{\overline{\gamma}_{A\times B}(x_1,x_2),\overline{\gamma}_{A\times B}(y_1,y_2)\}.
\end{aligned}$$

Thus $\overline{\gamma}_{A\times B}$ is a fuzzy LA-subring of an LA-ring $R_1\times R_2$. Now

$$\begin{aligned}
\overline{\gamma}_{A\times B}((x_1,x_2)\circ(y_1,y_2)) &= 1-\gamma_{A\times B}((x_1,x_2)\circ(y_1,y_2))\\
&= 1-\gamma_{A\times B}((y_1,y_2)\circ(x_1,x_2))\\
&= \overline{\gamma}_{A\times B}((y_1,y_2)\circ(x_1,x_2)).
\end{aligned}$$

Hence $\overline{\gamma}_{A\times B}$ is a fuzzy normal LA-subring of an LA-ring $R_1\times R_2$.

Conversely, suppose that $\mu_{A\times B}$ and $\overline{\gamma}_{A\times B}$ are fuzzy normal LA-subrings of an LA-ring $R_1\times R_2$. We have to show that $A\times B=(\mu_{A\times B},\gamma_{A\times B})$ is an intuitionistic fuzzy normal LA-subring of an LA-ring $R_1\times R_2$. Now

$$\begin{aligned}
1-\gamma_{A\times B}((x_1,x_2)-(y_1,y_2)) &= \overline{\gamma}_{A\times B}((x_1,x_2)-(y_1,y_2))\\
&\geq min\{\overline{\gamma}_{A\times B}(x_1,x_2),\overline{\gamma}_{A\times B}(y_1,y_2)\}\\
&= min\{1-\gamma_{A\times B}(x_1,x_2),1-\gamma_{A\times B}(y_1,y_2)\}\\
&= 1-max\{\gamma_{A\times B}(x_1,x_2),\gamma_{A\times B}(y_1,y_2)\}.\\
1-\gamma_{A\times B}((x_1,x_2)\circ(y_1,y_2)) &= \overline{\gamma}_{A\times B}((x_1,x_2)\circ(y_1,y_2))\\
&\geq min\{\overline{\gamma}_{A\times B}(x_1,x_2),\overline{\gamma}_{A\times B}(y_1,y_2)\}\\
&= min\{1-\gamma_{A}(x_1,x_2),1-\gamma_{A}(y_1,y_2)\}\\
&= 1-max\{\gamma_{A}(x_1,x_2),\gamma_{A}(y_1,y_2)\}.
\end{aligned}$$

Thus $A \times B = (\mu_{A\times B}, \gamma_{A\times B})$ is an intuitionistic fuzzy LA-subring of an LA-ring $R_1 \times R_2$. Now

$$\begin{aligned} 1 - \gamma_{A\times B}((x_1, x_2) \circ (y_1, y_2)) &= \overline{\gamma}_{A\times B}((x_1, x_2) \circ (y_1, y_2)) \\ &= \overline{\gamma}_{A\times B}((y_1, y_2) \circ (x_1, x_2)) \\ &= 1 - \gamma_A((y_1, y_2) \circ (x_1, x_2)). \end{aligned}$$

Hence $A \times B = (\mu_{A\times B}, \gamma_{A\times B})$ is an intuitionistic fuzzy normal LA-subring of an LA-ring $R_1 \times R_2$. ∎

Theorem 2.16. *An IFS $A\times B = (\mu_{A\times B}, \gamma_{A\times B})$ is an intuitionistic fuzzy normal LA-subring of an LA-ring $R_1 \times R_2$ if and only if the fuzzy subsets $\overline{\mu}_{A\times B}$ and $\gamma_{A\times B}$ are anti fuzzy normal LA-subrings of an LA-ring $R_1 \times R_2$.*

Proof. Let $A\times B = (\mu_{A\times B}, \gamma_{A\times B})$ be an intuitionistic fuzzy normal LA-subring of an LA-ring $R_1 \times R_2$, this means that $\gamma_{A\times B}$ is an anti fuzzy normal LA-subring of an LA-ring $R_1 \times R_2$. We have to show that $\overline{\mu}_{A\times B}$ is also an anti fuzzy normal LA-subring of $R_1 \times R_2$. Now

$$\begin{aligned} \overline{\mu}_{A\times B}((x_1, x_2) - (y_1, y_2)) &= 1 - \mu_{A\times B}((x_1, x_2) - (y_1, y_2)) \\ &\leq 1 - min\{\mu_{A\times B}(x_1, x_2), \mu_{A\times B}(y_1, y_2)\} \\ &= max\{1 - \mu_{A\times B}(x_1, x_2), 1 - \mu_{A\times B}(y_1, y_2)\} \\ &= max\{\overline{\mu}_{A\times B}(x_1, x_2), \overline{\mu}_{A\times B}(y_1, y_2)\}. \\ \overline{\mu}_{A\times B}((x_1, x_2) \circ (y_1, y_2)) &= 1 - \mu_{A\times B}((x_1, x_2) \circ (y_1, y_2)) \\ &\leq 1 - min\{\mu_A(x_1, x_2), \mu_A(y_1, y_2)\} \\ &= max\{1 - \mu_A(x_1, x_2), 1 - \mu_A(y_1, y_2)\} \\ &= max\{\overline{\mu}_{A\times B}(x_1, x_2), \overline{\mu}_{A\times B}(y_1, y_2)\}. \end{aligned}$$

Thus $\overline{\mu}_{A\times B}$ is an anti fuzzy LA-subring of an LA-ring $R_1 \times R_2$. Now

$$\begin{aligned} \overline{\mu}_{A\times B}((x_1, x_2) \circ (y_1, y_2)) &= 1 - \mu_A((x_1, x_2) \circ (y_1, y_2)) \\ &= 1 - \mu_A((y_1, y_2) \circ (x_1, x_2)) \\ &= \overline{\mu}_{A\times B}((y_1, y_2) \circ (x_1, x_2)). \end{aligned}$$

Hence $\overline{\mu}_{A\times B}$ is an anti fuzzy normal LA-subring of an LA-ring $R_1 \times R_2$.

Conversely, assume that $\overline{\mu}_{A\times B}$ and $\gamma_{A\times B}$ are anti fuzzy normal LA-subrings of an LA-ring $R_1 \times R_2$. We have to show that $A \times B = (\mu_{A\times B}, \gamma_{A\times B})$ is an intuitionistic fuzzy normal LA-subring of an LA-ring $R_1 \times R_2$. Now

$$\begin{aligned} 1 - \mu_{A\times B}((x_1, x_2) - (y_1, y_2)) &= \overline{\mu}_{A\times B}((x_1, x_2) - (y_1, y_2)) \\ &\leq max\{\overline{\mu}_{A\times B}(x_1, x_2), \overline{\mu}_{A\times B}(y_1, y_2)\} \\ &= max\{1 - \mu_{A\times B}(x_1, x_2), 1 - \mu_{A\times B}(y_1, y_2)\} \\ &= 1 - min\{\mu_{A\times B}(x_1, x_2), \mu_{A\times B}(y_1, y_2)\}. \\ 1 - \mu_{A\times B}((x_1, x_2) \circ (y_1, y_2)) &= \overline{\mu}_{A\times B}((x_1, x_2) \circ (y_1, y_2)) \\ &\leq max\{\overline{\mu}_{A\times B}(x_1, x_2), \overline{\mu}_{A\times B}(y_1, y_2)\} \\ &= max\{1 - \mu_{A\times B}(x_1, x_2), 1 - \mu_{A\times B}(y_1, y_2)\} \\ &= 1 - min\{\mu_{A\times B}(x_1, x_2), \mu_{A\times B}(y_1, y_2)\}. \end{aligned}$$

Thus $A \times B = (\mu_{A\times B}, \gamma_{A\times B})$ is an intuitionistic fuzzy LA-subring of an LA-ring $R_1 \times R_2$. Now

$$\begin{aligned} 1-\mu_{A\times B}((x_1,x_2)\circ(y_1,y_2)) &= \overline{\mu}_{A\times B}((x_1,x_2)\circ(y_1,y_2)) \\ &= \overline{\mu}_{A\times B}((y_1,y_2)\circ(x_1,x_2)) \\ &= 1-\mu_{A\times B}((y_1,y_2)\circ(x_1,x_2)). \end{aligned}$$

Hence $A \times B = (\mu_{A\times B}, \gamma_{A\times B})$ is an intuitionistic fuzzy normal LA-subring of an LA-ring $R_1 \times R_2$. ■

Lemma 2.17. *Let A and B be intuitionistic fuzzy sets of LA-rings R_1 and R_2 with left identities e_1 and e_2, respectively. If $A \times B$ is an intuitionistic fuzzy LA-subring of an LA-ring $R_1 \times R_2$, then at least one of the following two statements must hold.*
(i) $\mu_A(x) \le \mu_B(e_2)$ and $\gamma_A(x) \ge \gamma_B(e_2)$ for all $x \in R_1$.
(ii) $\mu_B(x) \le \mu_A(e_1)$ and $\gamma_B(x) \ge \gamma_A(e_1)$ for all $x \in R_2$.

Proof. Let $A \times B$ be an intuitionistic fuzzy LA-subring of an LA-ring $R_1 \times R_2$. By contraposition, suppose that none of the statements (i) and (ii) holds. Then we can find a and b in R_1 and R_2, respectively such that

$$\begin{aligned} \mu_A(a) &\ge \mu_B(e_2) \text{ and } \gamma_A(a) \le \gamma_B(e_2), \\ \mu_B(b) &\ge \mu_A(e_1) \text{ and } \gamma_B(b) \le \gamma_A(e_1). \end{aligned}$$

Thus we have

$$\begin{aligned} \mu_{A\times B}(a,b) &= min\{\mu_A(a), \mu_B(b)\} \\ &\ge min\{\mu_A(e_1), \mu_B(e_2)\} \\ &= \mu_{A\times B}(e_1,e_2). \\ \text{and } \gamma_{A\times B}(a,b) &= max\{\gamma_A(a), \gamma_B(b)\} \\ &\le max\{\gamma_A(e_1), \gamma_B(e_2)\} \\ &= \gamma_{A\times B}(e_1,e_2). \end{aligned}$$

This implies that $A \times B$ is not an intuitionistic fuzzy LA-subring of an LA-ring $R_1 \times R_2$. Hence either $\mu_A(x) \le \mu_B(e_2)$ and $\gamma_A(x) \ge \gamma_B(e_2)$, for all $x \in R_1$ or $\mu_B(x) \le \mu_A(e_1)$ and $\gamma_B(x) \ge \gamma_A(e_1)$, for all $x \in R_2$. ■

3. Direct Product of Finite Intuitionistic Fuzzy Normal LA-subrings

In this section, we define the direct product of intuitionistic fuzzy sets $A_1, A_2, ..., A_n$ of LA-rings $R_1, R_2, ..., R_n$, respectively and examine the some fundamental properties of direct product of intuitionistic fuzzy normal LA-subrings of an LA-ring $R_1 \times R_2 \times ... \times R_n$.

Let $\mu_1, \mu_2, ..., \mu_n$ be fuzzy subsets of LA-rings $R_1, R_2, ..., R_n$, respectively. The direct product of fuzzy subsets $\mu_1, \mu_2, ..., \mu_n$ of LA-rings $R_1, R_2, ..., R_n$, respectively, is denoted by $\mu_1 \times \mu_2 \times ... \times \mu_n$ and defined by $(\mu_1 \times \mu_2 \times ... \times \mu_n)(x_1, x_2, ..., x_n) = min\{\mu_1(x_1), \mu_2(x_2), ..., \mu_n(x_n)\}$.

A fuzzy subset $\mu_1 \times \mu_2 \times ... \times \mu_n$ of an LA-ring $R_1 \times R_2 \times ... \times R_n$ is said to be a fuzzy LA-subring of $R_1 \times R_2 \times ... \times R_n$ if

(1) $(\mu_1 \times \mu_2 \times ... \times \mu_n)(x-y) \ge min\{(\mu_1 \times \mu_2 \times ... \times \mu_n)(x), (\mu_1 \times \mu_2 \times ... \times \mu_n)(y)\}$,

(2) $(\mu_1 \times \mu_2 \times ... \times \mu_n)(xy) \geq min\{(\mu_1 \times \mu_2 \times ... \times \mu_n)(x), (\mu_1 \times \mu_2 \times ... \times \mu_n)(y)\}$ for all $x = (x_1, x_2, ..., x_n), y = (y_1, y_2, ..., y_n) \in R_1 \times R_2 \times ... \times R_n$.

A fuzzy subset $\mu_1 \times \mu_2 \times ... \times \mu_n$ of an LA-ring $R_1 \times R_2 \times ... \times R_n$ is said to be an anti fuzzy LA-subring of $R_1 \times R_2 \times ... \times R_n$ if

(1) $(\mu_1 \times \mu_2 \times ... \times \mu_n)(x - y) \leq max\{(\mu_1 \times \mu_2 \times ... \times \mu_n)(x), (\mu_1 \times \mu_2 \times ... \times \mu_n)(y)\}$,

(2) $(\mu_1 \times \mu_2 \times ... \times \mu_n)(xy) \leq max\{(\mu_1 \times \mu_2 \times ... \times \mu_n)(x), (\mu_1 \times \mu_2 \times ... \times \mu_n)(y)\}$ for all $x = (x_1, x_2, ..., x_n), y = (y_1, y_2, ..., y_n) \in R_1 \times R_2 \times ... \times R_n$.

A fuzzy LA-subring of an LA-ring $R_1 \times R_2 \times ... \times R_n$ is said to be a fuzzy normal LA-subring of $R_1 \times R_2 \times ... \times R_n$ if $(\mu_1 \times \mu_2 \times ... \times \mu_n)(xy) = (\mu_1 \times \mu_2 \times ... \times \mu_n)(yx)$ for all $x = (x_1, x_2, ..., x_n), y = (y_1, y_2, ..., y_n) \in R_1 \times R_2 \times ... \times R_n$. Similarly for anti fuzzy normal LA-subring.

Let $A_1, A_2, ..., A_n$ be intuitionistic fuzzy sets of LA-rings $R_1, R_2, ..., R_n$, respectively. The direct product of intuitionistic fuzzy sets $A_1, A_2, ..., A_n$ is denoted by $A = A_1 \times A_2 \times ... \times A_n$ and defined by $A_1 \times A_2 \times ... \times A_n = \{(x, \mu_A(x), \gamma_A(x)) \mid \text{for all } x = (x_1, x_2, ..., x_n) \in R_1 \times R_2 \times ... \times R_n\}$, where

$$\begin{aligned} \mu_{A_1 \times A_2 \times ... \times A_n}(x_1, x_2, ..., x_n) &= min\{\mu_{A_1}(x_1), \mu_{A_2}(x_2), ..., \mu_{A_n}(x_n)\} \\ \text{and } \gamma_{A_1 \times A_2 \times ... \times A_n}(x_1, x_2, ..., x_n) &= max\{\gamma_{A_1}(x_1), \gamma_{A_2}(x_2), ..., \gamma_{A_n}(x_n)\}. \end{aligned}$$

An intuitionistic fuzzy set (IFS) $A_1 \times A_2 \times ... \times A_n = (\mu_{A_1 \times A_2 \times ... \times A_n}, \gamma_{A_1 \times A_2 \times ... \times A_n})$ of an LA-ring $R_1 \times R_2 \times ... \times R_n$ is said to be an intuitionistic fuzzy LA-subring (IFLSR) of $R_1 \times R_2 \times ... \times R_n$ if

(1) $\mu_{A_1 \times A_2 \times ... \times A_n}(x - y) \geq min\{\mu_{A_1 \times A_2 \times ... \times A_n}(x), \mu_{A_1 \times A_2 \times ... \times A_n}(y)\}$,

(2) $\mu_{A_1 \times A_2 \times ... \times A_n}(xy) \geq min\{\mu_{A_1 \times A_2 \times ... \times A_n}(x), \mu_{A_1 \times A_2 \times ... \times A_n}(y)\}$,

(3) $\gamma_{A_1 \times A_2 \times ... \times A_n}(x - y) \leq max\{\gamma_{A_1 \times A_2 \times ... \times A_n}(x), \gamma_{A_1 \times A_2 \times ... \times A_n}(y)\}$,

(4) $\gamma_{A_1 \times A_2 \times ... \times A_n}(xy) \leq max\{\gamma_{A_1 \times A_2 \times ... \times A_n}(x), \gamma_{A_1 \times A_2 \times ... \times A_n}(y)\}$,

for all $x = (x_1, x_2, ..., x_n), y = (y_1, y_2, ..., y_n) \in R_1 \times R_2 \times ... \times R_n$.

An intuitionistic fuzzy LA-subring $A_1 \times A_2 \times ... \times A_n = (\mu_{A_1 \times A_2 \times ... \times A_n}, \gamma_{A_1 \times A_2 \times ... \times A_n})$ of an LA-ring $R_1 \times R_2 \times ... \times R_n$ is said to be an intuitionistic fuzzy normal LA-subring (IFNLSR) of $R_1 \times R_2 \times ... \times R_n$ if

(1) $\mu_{A_1 \times A_2 \times ... \times A_n}(xy) = \mu_{A_1 \times A_2 \times ... \times A_n}(yx)$

(2) $\gamma_{A_1 \times A_2 \times ... \times A_n}(xy) = \gamma_{A_1 \times A_2 \times ... \times A_n}(yx)$ for all $x = (x_1, x_2, ..., x_n), y = (y_1, y_2, ..., y_n) \in R_1 \times R_2 \times ... \times R_n$.

Let $A_1 \times A_2 \times ... \times A_n$ be a non-empty subset of an LA-ring $R = R_1 \times R_2 \times ... \times R_n$. The intuitionistic characteristic function of $A = A_1 \times A_2 \times ... \times A_n$ is denoted by $\chi_A = \langle \mu_{\chi_A}, \gamma_{\chi_A} \rangle$ and defined by

$$\mu_{\chi_A}(x) = \begin{cases} 1 \text{ if } x \in A \\ 0 \text{ if } x \notin A \end{cases} \quad \text{and } \gamma_{\chi_A}(x) = \begin{cases} 0 \text{ if } x \in A \\ 1 \text{ if } x \notin A \end{cases}$$

In [21], if R_1 and R_2 are LA-rings, then the direct product $R_1 \times R_2$ of R_1 and R_2 is an LA-ring with pointwise addition '$+$' and multiplication '$\circ$' defined as $(a, b) + (c, d) = (a + c, b + d)$ and $(a, b) \circ (c, d) = (ac, bd)$, respectively for every $(a, b), (c, d) \in R_1 \times R_2$. Likewise the direct product $R = \times_{i \in \Omega} R_i$ of a family of LA-rings $\{R_i : i \in \Omega\}$ has the structure of an LA-ring with the operations of addition and multiplication defined as

$$\begin{aligned} a+b &= (a_1,a_2,a_3,...)+(b_1,b_2,b_3,...) \\ &= (a_1+b_1,a_2+b_2,a_3+b_3,...) \\ \text{and } a\circ b &= (a_1,a_2,a_3,...)\circ(b_1,b_2,b_3,...) \\ &= (a_1b_1,a_2b_2,a_3b_3,...). \end{aligned}$$

for all $a=(a_1,a_2,...,a_n), b=(b_1,b_2,...,b_n)\in R$.

Lemma 3.1. *If $A_1,A_2,...,A_n$ are LA-subrings of LA-rings $R_1,R_2,...,R_n$, respectively, then $A_1\times A_2\times...\times A_n$ is an LA-subring of an LA-ring $R_1\times R_2\times...\times R_n$ under the same operations defined as in [21].*

Proof. Straight forward. ■

Theorem 3.2. *Let $A_1,A_2,...,A_n$ be LA-subrings of LA-rings $R_1,R_2,...,R_n$, respectively. Then $A_1\times A_2\times...\times A_n$ is an LA-subring of an LA-ring $R_1\times R_2\times...\times R_n$ if and only if the intuitionistic characteristic function $\chi_A=\langle\mu_{\chi_A},\gamma_{\chi_A}\rangle$ of $A=A_1\times A_2\times...\times A_n$ is an intuitionistic fuzzy normal LA-subring of an LA-ring $R_1\times R_2\times...\times R_n$.*

Proof. Let $A=A_1\times A_2\times...\times A_n$ be an LA-subring of an LA-ring $R_1\times R_2\times...\times R_n$ and $a=(a_1,a_2,...,a_n), b=(b_1,b_2,...,b_n)\in R_1\times R_2\times...\times R_n$. If $a,b\in A=A_1\times A_2\times...\times A_n$, then by definition of intuitionistic characteristic function $\mu_{\chi_A}(a)=1=\mu_{\chi_A}(b)$ and $\gamma_{\chi_A}(a)=0=\gamma_{\chi_A}(b)$. Since $a-b$ and $ab\in A$, A being an LA-subring. This implies that

$$\begin{aligned} \mu_{\chi_A}(a-b) &= 1=1\wedge 1=\mu_{\chi_A}(a)\wedge\mu_{\chi_A}(b), \\ \mu_{\chi_A}(ab) &= 1=1\wedge 1=\mu_{\chi_{A+}}(a)\wedge\mu_{\chi_A}(b), \\ \gamma_{\chi_A}(a-b) &= 0=0\vee 0=\gamma_{\chi_A}(a)\vee\gamma_{\chi_A}(b), \\ \gamma_{\chi_A}(ab) &= 0=0\vee 0=\gamma_{\chi_A}(a)\vee\gamma_{\chi_A}(b). \end{aligned}$$

Thus

$$\begin{aligned} \mu_{\chi_A}(a-b) &\geq min\{\mu_{\chi_A}(a),\mu_{\chi_A}(b)\}, \\ \mu_{\chi_A}(ab) &\geq min\{\mu_{\chi_A}(a),\mu_{\chi_A}(b)\}, \\ \gamma_{\chi_A}(a-b) &\leq max\{\gamma_{\chi_A}(a),\gamma_{\chi_A}(b)\}, \\ \gamma_{\chi_A}(ab) &\leq max\{\gamma_{\chi_A}(a),\gamma_{\chi_A}(b)\}. \end{aligned}$$

As ab and $ba\in A$, so $\mu_{\chi_A}(ab)=1=\mu_{\chi_A}(ba)$ and $\gamma_{\chi_A}(ab)=0=\gamma_{\chi_A}(ba)$, i.e., $\mu_{\chi_A}(ab)=\mu_{\chi_A}(ba)$ and $\gamma_{\chi_A}(ab)=\gamma_{\chi_A}(ba)$. Similarly, we have

$$\begin{aligned} \mu_{\chi_A}(a-b) &\geq min\{\mu_{\chi_A}(a),\mu_{\chi_A}(b)\},\ \mu_{\chi_A}(ab)\geq min\{\mu_{\chi_A}(a),\mu_{\chi_A}(b)\}, \\ \gamma_{\chi_A}(a-b) &\leq max\{\gamma_{\chi_A}(a),\gamma_{\chi_A}(b)\},\ \gamma_{\chi_A}(ab)\leq max\{\gamma_{\chi_A}(a),\gamma_{\chi_A}(b)\}, \\ \mu_{\chi_A}(ab) &= \mu_{\chi_A}(ba),\ \gamma_{\chi_A}(ab)=\gamma_{\chi_A}(ba), \end{aligned}$$

when $a,b\notin A$. Hence the intuitionistic characteristic function $\chi_A=\langle\mu_{\chi_A},\gamma_{\chi_A}\rangle$ of $A=A_1\times A_2\times...\times A_n$ is an intuitionistic fuzzy normal LA-subring of an LA-ring $R_1\times R_2\times...\times R_n$.

Conversely, suppose that the intuitionistic characteristic function $\chi_A=\langle\mu_{\chi_A},\gamma_{\chi_A}\rangle$ of $A=A_1\times A_2\times...\times A_n$ is an intuitionistic fuzzy normal LA-subring of an LA-ring $R_1\times R_2\times...\times R_n$. We have to show that
$A=A_1\times A_2\times...\times A_n$ is an LA-subring of an LA-ring $R_1\times R_2\times...\times R_n$. Let $a,b\in A$,

where $a = (a_1, a, ..., a_n)$ and $b = (b_1, b_2, ..., b_n)$, then by definition $\mu_{\chi_A}(a) = 1 = \mu_{\chi_A}(b)$ and $\gamma_{\chi_A}(a) = 0 = \gamma_{\chi_A}(b)$. By our supposition

$$\begin{aligned}\mu_{\chi_A}(a-b) &\geq \mu_{\chi_A}(a) \wedge \mu_{\chi_A}(b) = 1 \wedge 1 = 1,\\ \mu_{\chi_A}(ab) &\geq \mu_{\chi_A}(a) \wedge \mu_{\chi_A}(b) = 1 \wedge 1 = 1,\\ \gamma_{\chi_A}(a-b) &\leq \gamma_{\chi_A}(a) \vee \gamma_{\chi_A}(b) = 0 \vee 0 = 0,\\ \gamma_{\chi_A}(ab) &\leq \gamma_{\chi_A}(a) \vee \gamma_{\chi_A}(b) = 0 \vee 0 = 0.\end{aligned}$$

Thus $\mu_{\chi_A}(a-b) = 1 = \mu_{\chi_A}(ab)$ and $\gamma_{\chi_A}(a-b) = 0 = \gamma_{\chi_A}(ab)$, i.e., $a-b$ and $ab \in A$. Hence $A = A_1 \times A_2 \times ... \times A_n$ is an LA-subring of an LA-ring $R_1 \times R_2 \times ... \times R_n$. ■

Lemma 3.3. *If $A = A_1 \times A_2 \times ... \times A_n$ and $B = B_1 \times B_2 \times ... \times B_n$ are two LA-subrings of an LA-ring $R_1 \times R_2 \times ... \times R_n$, then their intersection $A \cap B$ is also an LA-subring of an LA-ring $R_1 \times R_2 \times ... \times R_n$.*

Proof. Straight forward. ■

Theorem 3.4. *Let $A = A_1 \times A_2 \times ... \times A_n$ and $B = B \times B_2 \times ... \times B_n$ be two LA-subrings of an LA-ring $R_1 \times R_2 \times ... \times R_n$. Then $A \cap B$ is an LA-subring of $R_1 \times R_2 \times ... \times R_n$ if and only if the intuitionistic characteristic function $\chi_Z = \langle \mu_{\chi_Z}, \gamma_{\chi_Z} \rangle$ of $Z = A \cap B$ is an intuitionistic fuzzy normal LA-subring of an LA-ring $R_1 \times R_2 \times ... \times R_n$.*

Proof. Let $Z = A \cap B$ be an LA-subring of an LA-ring $R_1 \times R_2 \times ... \times R_n$ and $a = (a_1, a_2, ..., a_n), b = (b_1, b_1, ..., b_n) \in R_1 \times R_2 \times ... \times R_n$. If $a, b \in Z = A \cap B$, then by definition of intuitionistic characteristic function $\mu_{\chi_Z}(a) = 1 = \mu_{\chi_Z}(b)$ and $\gamma_{\chi_Z}(a) = 0 = \gamma_{\chi_Z}(b)$. Since $a-b$ and $ab \in Z$, Z being an LA-subring. This implies that

$$\begin{aligned}\mu_{\chi_Z}(a-b) &= 1 = 1 \wedge 1 = \mu_{\chi_Z}(a) \wedge \mu_{\chi_Z}(b),\\ \mu_{\chi_Z}(ab) &= 1 = 1 \wedge 1 = \mu_{\chi_Z}(a) \wedge \mu_{\chi_Z}(b),\\ \gamma_{\chi_Z}(a-b) &= 0 = 0 \vee 0 = \gamma_{\chi_Z}(a) \vee \gamma_{\chi_Z}(b),\\ \gamma_{\chi_Z}(ab) &= 0 = 0 \vee 0 = \gamma_{\chi_Z}(a) \vee \gamma_{\chi_Z}(b).\end{aligned}$$

Thus

$$\begin{aligned}\mu_{\chi_Z}(a-b) &\geq min\{\mu_{\chi_Z}(a), \mu_{\chi_Z}(b)\},\\ \mu_{\chi_Z}(ab) &\geq min\{\mu_{\chi_Z}(a), \mu_{\chi_Z}(b)\},\\ \gamma_{\chi_Z}(a-b) &\leq max\{\gamma_{\chi_Z}(a), \gamma_{\chi_Z}(b)\},\\ \gamma_{\chi_Z}(ab) &\leq max\{\gamma_{\chi_Z}(a), \gamma_{\chi_Z}(b)\}.\end{aligned}$$

As ab and $ba \in Z$, this means that $\mu_{\chi_Z}(ab) = 1 = \mu_{\chi_Z}(ba)$ and $\gamma_{\chi_Z}(ab) = 0 = \gamma_{\chi_Z}(ba)$, i.e., $\mu_{\chi_Z}(ab) = \mu_{\chi_Z}(ba)$ and $\gamma_{\chi_Z}(ab) = \gamma_{\chi_Z}(ba)$. Similarly, we have

$$\begin{aligned}\mu_{\chi_Z}(a-b) &\geq min\{\mu_{\chi_Z}(a), \mu_{\chi_Z}(b)\},\ \mu_{\chi_Z}(ab) \geq min\{\mu_{\chi_Z}(a), \mu_{\chi_Z}(b)\},\\ \gamma_{\chi_Z}(a-b) &\leq max\{\gamma_{\chi_Z}(a), \gamma_{\chi_Z}(b)\},\ \gamma_{\chi_Z}(ab) \leq max\{\gamma_{\chi_Z}(a), \gamma_{\chi_Z}(b)\},\\ \gamma_{\chi_Z}(ab) &= \gamma_{\chi_Z}(ba),\ \gamma_{\chi_Z}(ab) = \gamma_{\chi_Z}(ba),\end{aligned}$$

when $a, b \notin Z$. Hence the intuitionistic characteristic function $\chi_Z = \langle \mu_{\chi_Z}, \gamma_{\chi_Z} \rangle$ of Z is an intuitionistic fuzzy normal LA-subring of an LA-ring $R_1 \times R_2 \times ... \times R_n$.

Conversely, assume that the intuitionistic characteristic function $\chi_Z = \langle \mu_{\chi_Z}, \gamma_{\chi_Z} \rangle$ of $Z = A \cap B$ is an intuitionistic fuzzy normal LA-subring of an LA-ring $R_1 \times R_2 \times ... \times R_n$. Let

$a, b \in Z = A \cap B$, then by definition $\mu_{\chi_Z}(a) = 1 = \mu_{\chi_Z}(b)$ and $\gamma_{\chi_Z}(a) = 0 = \gamma_{\chi_Z}(b)$. By our assumption

$$\begin{aligned}
\mu_{\chi_Z}(a-b) &\geq \mu_{\chi_Z}(a) \wedge \mu_{\chi_Z}(b) = 1 \wedge 1 = 1, \\
\mu_{\chi_Z}(ab) &\geq \mu_{\chi_Z}(a) \wedge \mu_{\chi_Z}(b) = 1 \wedge 1 = 1, \\
\gamma_{\chi_Z}(a-b) &\leq \gamma_{\chi_Z}(a) \vee \gamma_{\chi_Z}(b) = 0 \vee 0 = 0, \\
\gamma_{\chi_Z}(ab) &\leq \gamma_{\chi_Z}(a) \vee \gamma_{\chi_Z}(b) = 0 \vee 0 = 0.
\end{aligned}$$

Thus $\mu_{\chi_Z}(a-b) = 1 = \mu_{\chi_Z}(ab)$ and $\gamma_{\chi_Z}(a-b) = 0 = \gamma_{\chi_Z}(ab)$, i.e., $a-b$ and $ab \in Z$. Hence Z is an LA-subring of an LA-ring $R_1 \times R_2 \times ... \times R_n$. ■

Corollary 3.5. *Let $\{B_i\}_{i \in I} = \{A_{i1} \times A_{i2} \times ... \times A_{in}\}_{i \in I}$ be a family of LA-subrings of an LA-ring $R_1 \times R_2 \times ... \times R_n$. Then $B = \cap B_i$ is an LA-subring of an LA-ring $R_1 \times R_2 \times ... \times R_n$ if and only if the intuitionistic characteristic function $\chi_B = \langle \mu_{\chi_B}, \gamma_{\chi_B} \rangle$ of $B = \cap B_i$ is an intuitionistic fuzzy normal LA-subring of an LA-ring $R_1 \times R_2 \times ... \times R_n$.*

Theorem 3.6. *If $A = A_1 \times A_2 \times ... \times A_n$ and $B = B_1 \times B_2 \times ... \times B_n$ are two intuitionistic fuzzy normal LA-subrings of an LA-ring $R_1 \times R_2 \times ... \times R_n$, then their intersection $A \cap B$ is also an intuitionistic fuzzy normal LA-subring of an LA-ring $R_1 \times R_2 \times ... \times R_n$.*

Proof. Let $A = A_1 \times A_2 \times ... \times A_n = \{((a), \mu_{A_1 \times A_2 \times ... \times A_n}(a), \gamma_{A_1 \times A_2 \times ... \times A}(a)) \mid$ for all $a = (a_1, a_2, ..., a_n) \in R_1 \times R_2 \times ... \times R_n\}$ and
$B = B_1 \times B_2 \times ... \times B_n = \{((b), \mu_{B_1 \times B_2 \times ... \times B_n}(b), \gamma_{B_1 \times B_2 \times ... \times B_n}(b)) \mid$ for all $b = (b_1, b_2, ..., b_n) \in R_1 \times R_2 \times ... \times R_n\}$ be two intuitionistic fuzzy normal LA-subrings of an LA-ring $R_1 \times R_2 \times ... \times R_n$.

Let $Z = A \cap B$ and $Z = \{((z), \mu_Z(z), \gamma_Z(z)) \mid$ for all $z = (z_1, z_2, ..., z_n) \in R_1 \times R_2 \times ... \times R_n\}$, where

$$\begin{aligned}
\mu_Z(z_1, z_2, ..., z_n) &= \mu_{A \cap B}(z_1, z_2, ..., z_n) \\
&= min\{\mu_A(z_1, z_2, ..., z_n), \mu_B(z_1, z_2, ..., z_n)\} \\
\text{and } \gamma_Z(z_1, z_2, ..., z_n) &= \gamma_{A \cap B}(z_1, z_2, ..., z_n) \\
&= max\{\gamma_A(z_1, z_2, ..., z_n), \gamma_B(z_1, z_2, ..., z_n)\}.
\end{aligned}$$

Let $z = (z_1, z_n, ..., z_n), w = (w_1, w_2, ..., w_n) \in R_1 \times R_2 \times ... \times R_n$. Now

$$\begin{aligned}
\mu_Z(z-w) &= \mu_Z(z-w) = min\{\mu_A(z-w), \mu_B(z-w)\} \\
&\geq \{\mu_A(z) \wedge \mu_A(w)\} \wedge \{\mu_B(z) \wedge \mu_B(w)\} \\
&= \mu_A(z) \wedge \{\mu_A(w) \wedge \mu_B(z)\} \wedge \mu_B(w) \\
&= \mu_A(z) \wedge \{\mu_B(z) \wedge \mu_A(w)\} \wedge \mu_B(w) \\
&= \{\mu_A(z) \wedge \mu_B(z)\} \wedge \{\mu_A(w) \wedge \mu_B(w)\} \\
&= min\{\mu_{A \cap B}(z), \mu_{A \cap B}(w)\} \\
&= min\{\mu_Z(z), \mu_Z(w)\}
\end{aligned}$$

and

$$\begin{aligned}
\mu_Z(z\circ w) &= \mu_Z(z\circ w) = min\{\mu_A(z\circ w), \mu_B(z\circ w)\}\\
&\geq \{\mu_A(z)\wedge\mu_A(w)\}\wedge\{\mu_B(z)\wedge\mu_B(w)\}\\
&= \mu_A(z)\wedge\{\mu_A(w)\wedge\mu_B(z)\}\wedge\mu_B(w)\\
&= \mu_A(z)\wedge\{\mu_B(z)\wedge\mu_A(w)\}\wedge\mu_B(w)\\
&= \{\mu_A(z)\wedge\mu_B(z)\}\wedge\{\mu_A(w)\wedge\mu_B(w)\}\\
&= min\{\mu_{A\cap B}(z), \mu_{A\cap B}(w)\}\\
&= min\{\mu_Z(z), \mu_Z(w)\}.
\end{aligned}$$

Thus

$$\begin{aligned}
&\mu_Z((z_1, z_2, ..., z_n) - (w_1, w_2, ..., w_n))\\
\geq\ & min\{\mu_Z(z_1, z_2, ..., z_n), \mu_Z(w_1, w_2, ..., w_n)\}\\
&\text{and } \mu_Z((z_1, z_2, ..., z_n)\circ(w_1, w_2, ..., w_n))\\
\geq\ & min\{\mu_Z(z_1, z_2, ..., z_n), \mu_Z(w_1, w_2, ..., w_n)\}.
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
&\gamma_Z((z_1, z_2, ..., z_n) - (w_1, w_2, ..., w_n))\\
\leq\ & max\{\gamma_Z(z_1, z_2, ..., z_n), \gamma_Z(w_1, w_2, ..., w_n)\}\\
&\text{and } \gamma_Z((z_1, z_2, ..., z_n)\circ(w_1, w_2, ..., w_n))\\
\leq\ & max\{\gamma_Z(z_1, z_2, ..., z_n), \gamma_Z(w_1, w_2, ..., w_n)\}.
\end{aligned}$$

Thus $Z = (\mu_Z, \gamma_Z)$ is an intuitionistic fuzzy LA-subring of an LA-ring $R_1 \times R_2 \times ... \times R_n$. Now

$$\begin{aligned}
&\mu_Z((z_1, z_2, ..., z_n)\circ(w_1, w_2, ..., w_n))\\
=\ & \mu_{A\cap B}(z_1w_1, z_2w_2, ..., z_nw_n)\\
=\ & min\{\mu_A(z_1w_1, z_2w_2, ..., z_nw_n), \mu_B(z_1w_1, z_2w_2, ..., z_nw_n)\}\\
=\ & min\{\mu_A(w_1z_1, w_2z_2, ..., w_nz_n), \mu_B(w_1z_1, w_2z_2, ..., w_nz_n)\}\\
=\ & \mu_{A\cap B}(w_1z_1, w_2z_2, ..., w_nz_n)\\
=\ & \mu_Z((w_1, w_2, ..., w_n)\circ(z_1, z_2, ..., z_n)).
\end{aligned}$$

Similarly,

$$\begin{aligned}
&\gamma_Z((z_1, z_2, ..., z_n)\circ(w_1, w_2, ..., w_n))\\
=\ & \gamma_Z((w_1, w_2, ..., w_n)\circ(z_1, z_2, ..., z_n)).
\end{aligned}$$

Hence $Z = A \cap B$ is an intuitionistic fuzzy normal LA-subring of an LA-ring $R_1 \times R_2 \times ... \times R_n$. ■

Corollary 3.7. *If $\{B_i\}_{i\in I} = \{A_{i1} \times A_{i2} \times ... \times A_{in}\}_{i\in I}$ is a family of intuitionistic fuzzy normal LA-subrings of an LA-ring $R_1 \times R_2 \times ... \times R_n$, then $B = \cap B_i$ is also an intuitionistic fuzzy normal LA-subring of an LA-ring $R_1 \times R_2 \times ... \times R_n$.*

Proposition 3.8. *If an IFS $A_1 \times A_2 \times ... \times A_n$ is an intuitionistic fuzzy normal LA-subring of an LA-ring $R_1 \times R_2 \times ... \times R_n$, then $\Box A_1 \times A_2 \times ... \times A_n = (\mu_{A_1\times A_2\times...\times A_n}, \overline{\mu}_{A_1\times A_2\times...\times A_n})$ and $\Diamond A_1 \times A_2 \times ... \times A_n = (\overline{\gamma}_{A_1\times A_2\times...\times A_n}, \gamma_{A_1\times A_2\times...\times A_n})$ are also intuitionistic fuzzy normal LA-subrings of an LA-ring $R_1 \times R_2 \times ... \times R_n$.*

Proof. Let $A_1 \times A_2 \times ... \times A_n$ be an intuitionistic fuzzy normal LA-subring of an LA-ring $R_1 \times R_2 \times ... \times R_n$. We have to show that
$\Box A_1 \times A_2 \times ... \times A_n$ is also an intuitionistic fuzzy normal LA-subring of an LA-ring $R_1 \times R_2 \times ... \times R_n$. Now

$$\begin{aligned}
& \overline{\mu}_{A_1\times A_2\times...\times A_n}((x_1,x_2,...,x_n)-(y_1,y_2,...,y_n)) \\
= \; & 1-\mu_{A_1\times A_2\times...\times A_n}((x_1,x_2,...,x_n)-(y_1,y_2,...,y_n)) \\
\leq \; & 1-min\{\mu_{A_1\times A_2\times...\times A_n}(x_1,x_2,...,x_n),\mu_{A_1\times A_2\times...\times A_n}(y_1,y_2,...,y_n)\} \\
= \; & max\{1-\mu_{A_1\times A_2\times...\times A_n}(x_1,x_2,...,x_n),1-\mu_{A_1\times A_2\times...\times A_n}(y_1,y_2,...,y_n)\} \\
= \; & max\{\overline{\mu}_{A_1\times A_2\times...\times A_n}(x_1,x_2,...,x_n),\overline{\mu}_{A_1\times A_2\times...\times A_n}(y_1,y_2,...,y_n)\}. \\
& \text{and } \overline{\mu}_{A_1\times A_2\times...\times A_n}((x_1,x_2,...,x_n)\circ(y_1,y_2,...,y_n)) \\
= \; & 1-\mu_{A_1\times A_2\times...\times A_n}((x_1,x_2,...,x_n)\circ(y_1,y_2,...,y_n)) \\
\leq \; & 1-min\{\mu_{A_1\times A_2\times...\times A_n}(x_1,x_2,...,x_n),\mu_{A_1\times A_2\times...\times A_n}(y_1,y_2,...,y_n)\} \\
= \; & max\{1-\mu_{A_1\times A_2\times...\times A_n}(x_1,x_2,...,x_n),1-\mu_{A_1\times A_2\times...\times A_n}(y_1,y_2,...,y_n)\} \\
= \; & max\{\overline{\mu}_{A_1\times A_2\times...\times A_n}(x_1,x_2,...,x_n),\overline{\mu}_{A_1\times A_2\times...\times A_n}(y_1,y_2,...,y_n)\}.
\end{aligned}$$

Thus $\Box A_1 \times A_2 \times ... \times A_n = (\mu_{A_1\times A_2\times...\times A_n}, \overline{\mu}_{A_1\times A_2\times...\times A_n})$ is an intuitionistic fuzzy LA-subring of an LA-ring $R_1 \times R_2 \times ... \times R_n$. Now

$$\begin{aligned}
= \; & \overline{\mu}_{A_1\times A_2\times...\times A_n}((x_1,x_2,...,x_n)\circ(y_1,y_2,...,y_n)) \\
= \; & 1-\mu_{A_1\times A_2\times...\times A_n}((x_1,x_2,...,x_n)\circ(y_1,y_2,...,y_n)) \\
= \; & 1-\mu_{A_1\times A_2\times...\times A_n}((y_1,y_2,...,y_n)\circ(x_1,x_2,...,x_n)) \\
= \; & \overline{\mu}_{A_1\times A_2\times...\times A_n}((y_1,y_2,...,y_n)\circ(x_1,x_2,...,x_n)).
\end{aligned}$$

Hence $\Box A_1 \times A_2 \times ... \times A_n = (\mu_{A_1\times A_2\times...\times A_n}, \overline{\mu}_{A_1\times A_2\times...\times A_n})$ is an intuitionistic fuzzy normal LA-subring of an LA-ring $R_1 \times R_2 \times ... \times R_n$. ■

Corollary 3.9. *An IFS $A_1 \times A_2 \times ... \times A_n$ is an intuitionistic fuzzy normal LA-subring of an LA-ring $R_1 \times R_2 \times ... \times R_n$ if and only if $\Box A_1 \times A_2 \times ... \times A_n$ and $\Diamond A_1 \times A_2 \times ... \times A_n$ are intuitionistic fuzzy normal LA-subrings of an LA-ring $R_1 \times R_2 \times ... \times R_n$.*

Theorem 3.10. *An IFS $A_1 \times A_2 \times ... \times A_n = (\mu_{A_1\times A_2\times...\times A_n}, \gamma_{A_1\times A_2\times...\times A_n})$ is an intuitionistic fuzzy normal LA-subring of an LA-ring $R_1 \times R_2 \times ... \times R_n$ if and only if the fuzzy subsets $\mu_{A_1\times A_2\times...\times A_n}$ and $\overline{\gamma}_{A_1\times A_2\times...\times A_n}$ are fuzzy normal LA-subrings of an LA-ring $R_1 \times R_2 \times ... \times R_n$.*

Proof. Let $A_1 \times A_2 \times ... \times A_n = (\mu_{A_1\times A_2\times...\times A_n}, \gamma_{A_1\times A_2\times...\times A_n})$ be an intuitionistic fuzzy normal LA-subring of an LA-ring $R_1 \times R_2 \times ... \times R_n$, this implies that $\mu_{A_1\times A_2\times...\times A_n}$ is fuzzy normal LA-subring of an LA-ring $R_1 \times R_2 \times ... \times R_n$. We have to show that

$\overline{\gamma}_{A_1\times A_2\times...\times A_n}$ is also a fuzzy normal LA-subring of an LA-ring $R_1 \times R_2 \times ... \times R_n$. Now

$$\begin{aligned}
& \overline{\gamma}_{A_1\times A_2\times...\times A_n}((x_1,x_2,...,x_n)-(y_1,y_2,...,y_n)) \\
= \ & 1-\gamma_{A_1\times A_2\times...\times A_n}((x_1,x_2,...,x_n)-(y_1,y_2,...,y_n)) \\
\geq \ & 1-max\{\gamma_{A_1\times A_2\times...\times A_n}(x_1,x_2,...,x_n),\gamma_{A_1\times A_2\times...\times A_n}(y_1,y_2,...,y_n)\} \\
= \ & min\{1-\gamma_{A_1\times A_2\times...\times A_n}(x_1,x_2,...,x_n),1-\gamma_{A_1\times A_2\times...\times A_n}(y_1,y_2,...,y_n)\} \\
= \ & min\{\overline{\gamma}_{A_1\times A_2\times...\times A_n}(x_1,x_2,...,x_n),\overline{\gamma}_{A_1\times A_2\times...\times A_n}(y_1,y_2,...,y_n)\}. \\
& \text{and } \overline{\gamma}_{A_1\times A_2\times...\times A_n}((x_1,x_2,...,x_n)\circ(y_1,y_2,...,y_n)) \\
= \ & 1-\gamma_{A_1\times A_2\times...\times A_n}((x_1,x_2,...,x_n)\circ(y_1,y_2,...,y_n)) \\
\geq \ & 1-max\{\gamma_{A_1\times A_2\times...\times A_n}(x_1,x_2,...,x_n),\gamma_{A_1\times A_2\times...\times A_n}(y_1,y_2,...,y_n)\} \\
= \ & min\{1-\gamma_{A_1\times A_2\times...\times A_n}(x_1,x_2,...,x_n),1-\gamma_{A_1\times A_2\times...\times A_n}(y_1,y_2,...,y_n)\} \\
= \ & min\{\overline{\gamma}_{A_1\times A_2\times...\times A_n}(x_1,x_2,...,x_n),\overline{\gamma}_{A_1\times A_2\times...\times A_n}(y_1,y_2,...,y_n)\}.
\end{aligned}$$

Thus $\overline{\gamma}_{A_1\times A_2\times...\times A_n}$ is a fuzzy LA-subring of an LA-ring $R_1 \times R_2 \times ... \times R_n$. Now

$$\begin{aligned}
& \overline{\gamma}_{A_1\times A_2\times...\times A_n}((x_1,x_2,...,x_n)\circ(y_1,y_2,...,y_n)) \\
= \ & 1-\gamma_{A_1\times A_2\times...\times A_n}((x_1,x_2,...,x_n)\circ(y_1,y_2,...,y_n)) \\
= \ & 1-\gamma_{A_1\times A_2\times...\times A_n}((y_1,y_2,...,y_n)\circ(x_1,x_2,...,x_n)) \\
= \ & \overline{\gamma}_{A_1\times A_2\times...\times A_n}((y_1,y_2,...,y_n)\circ(x_1,x_2,...,x_n)).
\end{aligned}$$

Hence $\overline{\gamma}_{A_1\times A_2\times...\times A_n}$ is a fuzzy normal LA-subring of an LA-ring $R_1 \times R_2 \times ... \times R_n$.

Conversely, suppose that $\mu_{A_1\times A_2\times...\times A_n}$ and $\overline{\gamma}_{A_1\times A_2\times...\times A_n}$ are fuzzy normal LA-subrings of an LA-ring $R_1 \times R_2 \times ... \times R_n$. We have to show that $A_1 \times A_2 \times ... \times A_n = (\mu_{A_1\times A_2\times...\times A_n}, \gamma_{A_1\times A_2\times...\times A_n})$ is an intuitionistic fuzzy normal LA-subring of an LA-ring $R_1 \times R_2 \times ... \times R_n$. Now

$$\begin{aligned}
& 1-\gamma_{A_1\times A_2\times...\times A_n}((x_1,x_2,...,x_n)-(y_1,y_2,...,y_n)) \\
= \ & \overline{\gamma}_{A_1\times A_2\times...\times A_n}((x_1,x_2,...,x_n)-(y_1,y_2,...,y_n)) \\
\geq \ & min\{\overline{\gamma}_{A_1\times A_2\times...\times A_n}(x_1,x_2,...,x_n),\overline{\gamma}_{A_1\times A_2\times...\times A_n}(y_1,y_2,...,y_n)\} \\
= \ & min\{1-\gamma_{A_1\times A_2\times...\times A_n}(x_1,x_2,...,x_n),1-\gamma_{A_1\times A_2\times...\times A_n}(y_1,y_2,...,y_n)\} \\
= \ & 1-max\{\gamma_{A_1\times A_2\times...\times A_n}(x_1,x_2,...,x_n),\gamma_{A_1\times A_2\times...\times A_n}(y_1,y_2,...,y_n)\}. \\
& \text{and } 1-\gamma_{A_1\times A_2\times...\times A_n}((x_1,x_2,...,x_n)\circ(y_1,y_2,...,y_n)) \\
= \ & \overline{\gamma}_{A_1\times A_2\times...\times A_n}((x_1,x_2,...,x_n)\circ(y_1,y_2,...,y_n)) \\
\geq \ & min\{\overline{\gamma}_{A_1\times A_2\times...\times A_n}(x_1,x_2,...,x_n),\overline{\gamma}_{A_1\times A_2\times...\times A_n}(y_1,y_2,...,y_n)\} \\
= \ & min\{1-\gamma_{A_1\times A_2\times...\times A_n}(x_1,x_2,...,x_n),1-\gamma_{A_1\times A_2\times...\times A_n}(y_1,y_2,...,y_n)\} \\
= \ & 1-max\{\gamma_{A_1\times A_2\times...\times A_n}(x_1,x_2,...,x_n),\gamma_{A_1\times A_2\times...\times A_n}(y_1,y_2,...,y_n)\}.
\end{aligned}$$

Thus $A_1 \times A_2 \times ... \times A_n = (\mu_{A_1\times A_2\times...\times A_n}, \gamma_{A_1\times A_2\times...\times A_n})$ is an intuitionistic fuzzy LA-subring of an LA-ring $R_1 \times R_2 \times ... \times R_n$. Now

$$\begin{aligned}
& 1-\gamma_{A_1\times A_2\times...\times A_n}((x_1,x_2,...,x_n)\circ(y_1,y_2,...,y_n)) \\
= \ & \overline{\gamma}_{A_1\times A_2\times...\times A_n}((x_1,x_2,...,x_n)\circ(y_1,y_2,...,y_n)) \\
= \ & \overline{\gamma}_{A_1\times A_2\times...\times A_n}((y_1,y_2,...,y_n)\circ(x_1,x_2,...,x_n)) \\
= \ & 1-\gamma_{A_1\times A_2\times...\times A_n}((y_1,y_2,...,y_n)\circ(x_1,x_2,...,x_n)).
\end{aligned}$$

Hence $A_1 \times A_2 \times ... \times A_n = (\mu_{A_1\times A_2\times ...\times A_n}, \gamma_{A_1\times A_2\times ...\times A_n})$ is an intuitionistic fuzzy normal LA-subring of an LA-ring $R_1 \times R_2 \times ... \times R_n$. ∎

Theorem 3.11. *An IFS* $A_1 \times A_2 \times ... \times A_n = (\mu_{A_1\times A_2\times ...\times A_n}, \gamma_{A_1\times A_2\times ...\times A_n})$ *is an intuitionistic fuzzy normal LA-subring of an LA-ring* $R_1 \times R_2 \times ... \times R_n$ *if and only if the fuzzy subsets* $\overline{\mu}_{A_1\times A_2\times ...\times A_n}$ *and* $\gamma_{A_1\times A_2\times ...\times A_n}$ *are anti fuzzy normal LA-subrings of an LA-ring* $R_1 \times R_2 \times ... \times R_n$.

Proof. Let $A_1 \times A_2 \times ... \times A_n = (\mu_{A_1\times A_2\times ...\times A_n}, \gamma_{A_1\times A_2\times ...\times A_n})$ be an intuitionistic fuzzy normal LA-subring of an LA-ring $R_1 \times R_2 \times ... \times R_n$, this means that $\gamma_{A_1\times A_2\times ...\times A_n}$ is an anti fuzzy normal LA-subring of an LA-ring $R_1 \times R_2 \times ... \times R_n$. We have to show that $\overline{\mu}_{A_1\times A_2\times ...\times A_n}$ is also an anti fuzzy normal LA-subring of an LA-ring $R_1 \times R_2 \times ... \times R_n$. Now

$$\begin{aligned}
& \overline{\mu}_{A_1\times A_2\times ...\times A_n}((x_1, x_2, ..., x_n) - (y_1, y_2, ..., y_n)) \\
= \; & 1 - \mu_{A_1\times A_2\times ...\times A_n}((x_1, x_2, ..., x_n) - (y_1, y_2, ..., y_n)) \\
\leq \; & 1 - min\{\mu_{A_1\times A_2\times ...\times A_n}(x_1, x_2, ..., x_n), \mu_{A_1\times A_2\times ...\times A_n}(y_1, y_2, ..., y_n)\} \\
= \; & max\{1 - \mu_{A_1\times A_2\times ...\times A_n}(x_1, x_2, ..., x_n), 1 - \mu_{A_1\times A_2\times ...\times A_n}(y_1, y_2, ..., y_n)\} \\
= \; & max\{\overline{\mu}_{A_1\times A_2\times ...\times A_n}(x_1, x_2, ..., x_n), \overline{\mu}_{A_1\times A_2\times ...\times A_n}(y_1, y_2, ..., y_n)\}. \\
& \text{and } \overline{\mu}_{A_1\times A_2\times ...\times A_n}((x_1, x_2, ..., x_n) \circ (y_1, y_2, ..., y_n)) \\
= \; & 1 - \mu_{A_1\times A_2\times ...\times A_n}((x_1, x_2, ..., x_n) \circ (y_1, y_2, ..., y_n)) \\
\leq \; & 1 - min\{\mu_{A_1\times A_2\times ...\times A_n}(x_1, x_2, ..., x_n), \mu_{A_1\times A_2\times ...\times A_n}(y_1, y_2, ..., y_n)\} \\
= \; & max\{1 - \mu_{A_1\times A_2\times ...\times A_n}(x_1, x_2, ..., x_n), 1 - \mu_{A_1\times A_2\times ...\times A_n}(y_1, y_2, ..., y_n)\} \\
= \; & max\{\overline{\mu}_{A_1\times A_2\times ...\times A_n}(x_1, x_2, ..., x_n), \overline{\mu}_{A_1\times A_2\times ...\times A_n}(y_1, y_2, ..., y_n)\}.
\end{aligned}$$

Thus $\overline{\mu}_{A_1\times A_2\times ...\times A_n}$ is an anti fuzzy LA-subring of an LA-ring $R_1 \times R_2 \times ... \times R_n$. Now

$$\begin{aligned}
& \overline{\mu}_{A_1\times A_2\times ...\times A_n}((x_1, x_2, ..., x_n) \circ (y_1, y_2, ..., y_n)) \\
= \; & 1 - \mu_{A_1\times A_2\times ...\times A_n}((x_1, x_2, ..., x_n) \circ (y_1, y_2, ..., y_n)) \\
= \; & 1 - \mu_{A_1\times A_2\times ...\times A_n}((y_1, y_2, ..., y_n) \circ (x_1, x_2, ..., x_n)) \\
= \; & \overline{\mu}_{A_1\times A_2\times ...\times A_n}((y_1, y_2, ..., y_n) \circ (x_1, x_2, ..., x_n)).
\end{aligned}$$

Hence $\overline{\mu}_{A_1\times A_2\times ...\times A_n}$ is an anti fuzzy normal LA-subring of an LA-ring $R_1 \times R_2 \times ... \times R_n$.

Conversely, assume that $\overline{\mu}_{A_1\times A_2\times ...\times A_n}$ and $\gamma_{A_1\times A_2\times ...\times A_n}$ are anti fuzzy normal LA-subrings of an LA-ring $R_1 \times R_2 \times ... \times R_n$. We have to show that $A_1 \times A_2 \times ... \times A_n = (\mu_{A_1\times A_2\times ...\times A_n}, \gamma_{A_1\times A_2\times ...\times A_n})$ is an intuitionistic fuzzy normal LA-subring of an LA-ring $R_1 \times R_2 \times ... \times R_n$. Now

$$\begin{aligned}
& 1 - \mu_{A_1\times A_2\times ...\times A_n}((x_1, x_2, ..., x_n) - (y_1, y_2, ..., y_n)) \\
= \; & \overline{\mu}_{A_1\times A_2\times ...\times A_n}((x_1, x_2, ..., x_n) - (y_1, y_2, ..., y_n)) \\
\leq \; & max\{\overline{\mu}_{A_1\times A_2\times ...\times A_n}(x_1, x_2, ..., x_n), \overline{\mu}_{A_1\times A_2\times ...\times A_n}(y_1, y_2, ..., y_n)\} \\
= \; & max\{1 - \mu_{A_1\times A_2\times ...\times A_n}(x_1, x_2, ..., x_n), 1 - \mu_{A_1\times A_2\times ...\times A_n}(y_1, y_2, ..., y_n)\} \\
= \; & 1 - min\{\mu_{A_1\times A_2\times ...\times A_n}(x_1, x_2, ..., x_n), \mu_{A_1\times A_2\times ...\times A_n}(y_1, y_2, ..., y_n)\}.
\end{aligned}$$

and

$$\begin{aligned} & 1-\mu_{A_1\times A_2\times...\times A_n}((x_1,x_2,...,x_n)\circ(y_1,y_2,...,y_n)) \\ = & \overline{\mu}_{A_1\times A_2\times...\times A_n}((x_1,x_2,...,x_n)\circ(y_1,y_2,...,y_n)) \\ \leq & max\{\overline{\mu}_{A_1\times A_2\times...\times A_n}(x_1,x_2,...,x_n),\overline{\mu}_{A_1\times A_2\times...\times A_n}(y_1,y_2,...,y_n)\} \\ = & max\{1-\mu_{A_1\times A_2\times...\times A_n}(x_1,x_2,...,x_n),1-\mu_{A_1\times A_2\times...\times A_n}(y_1,y_2,...,y_n)\} \\ = & 1-min\{\mu_{A_1\times A_2\times...\times A_n}(x_1,x_2,...,x_n),\mu_{A_1\times A_2\times...\times A_n}(y_1,y_2,...,y_n)\}. \end{aligned}$$

Thus $A_1\times A_2\times...\times A_n=(\mu_{A_1\times A_2\times...\times A_n},\gamma_{A_1\times A_2\times...\times A_n})$ is an intuitionistic fuzzy LA-subring of an LA-ring $R_1\times R_2\times...\times R_n$. Now

$$\begin{aligned} & 1-\mu_{A_1\times A_2\times...\times A_n}((x_1,x_2,...,x_n)\circ(y_1,y_2,...,y_n)) \\ = & \overline{\mu}_{A_1\times A_2\times...\times A_n}((x_1,x_2,...,x_n)\circ(y_1,y_2,...,y_n)) \\ = & \overline{\mu}_{A_1\times A_2\times...\times A_n}((y_1,y_2,...,y_n)\circ(x_1,x_2,...,x_n)) \\ = & 1-\mu_{A_1\times A_2\times...\times A_n}((y_1,y_2,...,y_n)\circ(x_1,x_2,...,x_n)). \end{aligned}$$

Hence $A_1\times A_2\times...\times A_n=(\mu_{A_1\times A_2\times...\times A_n},\gamma_{A_1\times A_2\times...\times A_n})$ is an intuitionistic fuzzy normal LA-subring of an LA-ring $R_1\times R_2\times...\times R_n$. ∎

Proposition 3.12. *Let $A=A_1\times A_2\times...\times A_n$ and $B=B_1\times B_2\times...\times B_n$ be intuitionistic fuzzy sets of LA-rings $R=R_1\times R_2\times...\times R_n$ and $R'=R'_1\times R'_2\times...\times R'_n$ with left identities $e=(e_1,e_2,...,e_n)$ and $e'=(e_1\prime,e_2\prime,...,e_n\prime)$, respectively. If $A\times B$ is an intuitionistic fuzzy LA-subring of an LA-ring $R\times R'$, then at least one of the following two statements must hold.*
(i) $\mu_A(x)\leq\mu_B(e')$ and $\gamma_A(x)\geq\gamma_B(e')$, for all $x\in R$.
(ii) $\mu_B(x)\leq\mu_A(e)$ and $\gamma_B(x)\geq\gamma_A(e)$, for all $x\in R'$.

Proof. Let $A\times B$ be an intuitionistic fuzzy LA-subring of an LA-ring $R\times R\prime$. By contraposition, suppose that none of the statements (i) and (ii) holds. Then we can find a and b in R and R', respectively such that

$$\begin{aligned} \mu_A(a) & \geq \mu_B(e') \text{ and } \gamma_A(a)\leq\gamma_B(e'), \\ \mu_B(b) & \geq \mu_A(e) \text{ and } \gamma_B(b)\leq\gamma_A(e). \end{aligned}$$

Thus

$$\begin{aligned} \mu_{A\times B}(a,b) & = min\{\mu_A(a),\mu_B(b)\} \\ & \geq min\{\mu_A(e),\mu_B(e')\} \\ & = \mu_{A\times B}(e,e') \\ \text{and } \gamma_{A\times B}(a,b) & = max\{\gamma_A(a),\gamma_B(b)\} \\ & \leq max\{\gamma_A(e),\gamma_B(e')\} \\ & = \gamma_{A\times B}(e,e'). \end{aligned}$$

This implies that $A\times B$ is not an intuitionistic fuzzy LA-subring of an LA-ring $R\times R'$. Hence either $\mu_A(x)\leq\mu_B(e')$ and $\gamma_A(x)\geq\gamma_B(e')$, for all $x\in R_1$ or $\mu_B(x)\leq\mu_A(e)$ and $\gamma_B(x)\geq\gamma_A(e)$, for all $x\in R_2$. ∎

Theorem 3.13. *Let $A=A_1\times A_2\times...\times A_n$ and $B=B_1\times B_2\times...\times B_n$ be intuitionistic fuzzy sets of LA-rings $R=R_1\times R_2\times...\times R_n$ and $R'=R'_1\times R'_2\times...\times R'_n$ with left identities $e=(e_1,e_2,...,e_n)$ and $e'=(e_1\prime,e_2\prime,...,e_n\prime)$, respectively and $A\times B$ is an intuitionistic fuzzy normal LA-subring of an LA-ring $R\times R'$. Then the following conditions are true.*

(i) If $\mu_A(x) \leq \mu_B(e')$ and $\gamma_A(x) \geq \gamma_B(e')$, for all $x \in R$, then A is an intuitionistic fuzzy normal LA-subring of R.

(ii) If $\mu_B(x') \leq \mu_A(e)$ and $\gamma_B(x') \geq \gamma_A(e)$, for all $x' \in R'$, then B is an intuitionistic fuzzy normal LA-subring of R'.

Proof. Let $\mu_A(x) \leq \mu_B(e')$ and $\gamma_A(x) \geq \gamma_B(e')$ for all $x \in R$, and $y \in R$. We have to show that A is an intuitionistic fuzzy normal LA-subring of an LA-ring R. Now

$$\begin{aligned} \mu_A(x-y) &= \mu_A(x+(-y)) \\ &= min\{\mu_A(x+(-y)), \mu_B(e'+(-e'))\} \\ &= \mu_{A\times B}(x+(-y), e'+(-e')) \\ &= \mu_{A\times B}((x,e')+(-y,-e')) \\ &= \mu_{A\times B}((x,e')-(y,e')) \\ &\geq \mu_{A\times B}(x,e') \wedge \mu_{A\times B}(y,e') \\ &= min\{min\{\mu_A(x), \mu_B(e')\}, min\{\mu_A(y), \mu_B(e')\}\} \\ &= \mu_A(x) \wedge \mu_A(y) \end{aligned}$$

and

$$\begin{aligned} \mu_A(xy) &= min\{\mu_A(xy), \mu_B(e'e')\} \\ &= \mu_{A\times B}(xy, e'e') \\ &= \mu_{A\times B}((x,e')\circ(y,e')) \\ &\geq \mu_{A\times B}(x,e') \wedge \mu_{A\times B}(y,e') \\ &= min\{min\{\mu_A(x), \mu_B(e')\}, min\{\mu_A(y), \mu_B(e')\}\} \\ &= \mu_A(x) \wedge \mu_A(y). \end{aligned}$$

Similarly, we have

$$\gamma_A(x-y) \leq max\{\gamma_A(x), \gamma_A(y)\} \text{ and } \gamma_A(xy) \leq max\{\gamma_A(x), \gamma_A(y)\}.$$

Thus A is an intuitionistic fuzzy LA-subring of an LA-ring R. Now

$$\begin{aligned} \mu_A(xy) &= min\{\mu_A(xy), \mu_B(e'e')\} \\ &= \mu_{A\times B}(xy, e'e') \\ &= \mu_{A\times B}((x,e')\circ(y,e')) \\ &= \mu_{A\times B}((y,e')\circ(x,e')) \\ &= \mu_{A\times B}(yx, e'e') \\ &= min\{\mu_A(yx), \mu_B(e'e')\} \\ &= \mu_A(yx). \end{aligned}$$

Similarly, $\gamma_B(xy) = \gamma_B(yx)$. Hence A is an intuitionistic fuzzy normal LA-subring of an LA-ring R. (ii), is same as (i). ∎

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