Thai Journal of Mathematics
Volume 20 Number 3 (2022)
Pages 1031–1039

http://thaijmath.in.cmu.ac.th

ISSN 1686-0209

On Statistical $\mathcal{A}^{\mathcal{I}}$ and Statistical $\mathcal{A}^{\mathcal{I}^*}-$ Summability

Osama H. H. Edely

Department of Mathematics, Tafila Technical University, P.O.Box 179, Tafila (66110) - Jordan
e-mail : osamaedely@yahoo.com

Abstract In this paper we define statistical $\mathcal{A}^{\mathcal{I}}-$ summability of $\mathfrak{x}$, that is, $\mathfrak{x}$ is said to be statistically $\mathcal{A}^{\mathcal{I}}-$ summable if $\mathcal{A}\mathfrak{x}$ is $\mathcal{I}-$ statistically convergent. Moreover, we also introduce the concept of statistical $\mathcal{A}^{\mathcal{I}^*}-$ summability and find its relationship with statistical $\mathcal{A}^{\mathcal{I}}-$ summability. Also we prove that under what conditions $\mathcal{A}^{\mathcal{I}}-$ statistical convergence implies statistical $\mathcal{A}^{\mathcal{I}}-$ summability.

MSC: 40A35; 40G15; 40D25

Keywords: $\mathcal{A}^{\mathcal{I}}-$ summability; $\mathcal{A}^{\mathcal{I}^*}-$ summability; statistical $\mathcal{A}^{\mathcal{I}}-$ summability; statistical $\mathcal{A}^{\mathcal{I}^*}-$ summability

Submission date: 31.05.2020 / Acceptance date: 16.09.2020

1. Introduction and Preliminaries

The concept of statistical convergence based on the notion of natural density of subset of natural number $\mathbb{N}$ was first introduced by Fast [1], which is a natural generalization of the usual convergence of sequences. In 1953 the concept arises as an example of convergence in density as introduced by Buck [2]. Schoenberg [3] studied statistical convergence as a summability method and Zygmund [4] established a relation between it and strong summability. Fridy [5], Salat [6], Connor [7], Kolk [8], Mursaleen and Edely [9] and many others studied it as a summability method. Let $\mathcal{K} \subseteq \mathbb{N}$, then the natural density of $\mathcal{K}$ [10] is defined by

$$\delta(\mathcal{K}) = \lim_n \frac{1}{n} \mid \{k \leq n : k \in \mathcal{K}\} \mid = \lim_n (\mathfrak{C}_1 \chi_{\mathcal{K}})_n,$$

if the limit exists, where the vertical bars denote the cardinality of the enclosed set. Recall that $\mathfrak{C}_1 = (\mathfrak{C}, 1)$ is the Cesàro matrix of order 1 and $\chi_{\mathcal{K}}$ denotes the characteristic sequence of $\mathcal{K}$. A sequence $\mathfrak{x} = (\mathfrak{x}_k)$ of real numbers $\mathbb{R}$ is said to be statistically convergent to the number ℓ provided that for every $\epsilon > 0$, the set $\mathcal{K}(\epsilon) = \{k \in \mathbb{N} : |\mathfrak{x}_k - \ell| \geq \epsilon\}$ has natural density zero [1], and we write $st - \lim \mathfrak{x} = \ell$.

Let $\mathcal{A} = (\mathfrak{a}_{nk})_{n,k=1}^{\infty}$ be an infinite matrix and $\mathfrak{x} = (\mathfrak{x}_k)_{k=1}^{\infty}$ be a number sequence. By $\mathcal{A}\mathfrak{x} = (\mathcal{A}_n(\mathfrak{x}))$, we denote the $\mathcal{A}-$ transform of the sequence $\mathfrak{x} = (\mathfrak{x}_k)$, where $\mathcal{A}_n(\mathfrak{x}) = \sum_{k=1}^{\infty} \mathfrak{a}_{nk}\mathfrak{x}_k$. Thus we say that $\mathfrak{x}$ is $\mathcal{A}-$summable to ℓ if $\lim_n \mathcal{A}_n(\mathfrak{x}) = \ell$. A matrix $\mathcal{A}$ is called regular i.e. $\mathcal{A} \in (c,c)_{reg}$ if $\mathcal{A} \in (c,c)$ and $\lim_n \mathcal{A}_n(\mathfrak{x}) = \lim_k \mathfrak{x}_k$ for all

Published by The Mathematical Association of Thailand.

$\mathfrak{x} \in c$; the space of all convergent sequences. The well-known necessary and sufficient conditions (Silverman-Toeplitz) for $\mathcal{A}$ to be regular are (i) $||\mathcal{A}|| = \sup_n \sum_k |\mathfrak{a}_{nk}| < \infty$, (ii) $\lim_n \mathfrak{a}_{nk} = 0$, for each k, (iii) $\lim_n \sum_k \mathfrak{a}_{nk} = 1$.

Let Ψ denote the class of all non-negative regular matrices. Freedmann and Sember [11] generalized the natural density by replacing $\mathcal{C}_1$ with $\mathcal{A} \in \Psi$. A subset $\mathcal{K}$ of $\mathbb{N}$ has $\mathcal{A}-$density if $\delta_{\mathcal{A}}(\mathcal{K}) = \lim_n \sum_{k\in\mathcal{K}} \mathfrak{a}_{nk}$ exists. Kolk [8] and Connor [7] extended the idea of statistical convergence to $\mathcal{A}-$statistical convergence. A sequence $\mathfrak{x}$ is said to be $\mathcal{A}-$statistically convergent to ℓ if $\delta_{\mathcal{A}}(\mathcal{K}(\epsilon)) = 0$ for every $\epsilon > 0$, which we write $st_{\mathcal{A}} - \lim \mathfrak{x}_k = \ell$.

The idea of $\mathcal{I}-$convergence based on the notion of ideals of $\mathbb{N}$ was introduced by Kostyrko et al. [12] as a generalization of statistical convergence. More generalization and recent work can be found in ([13], [14], [15], [16], [17], [18], [19], [20], [21]).

A non-empty class $\mathcal{I} \subseteq \mathcal{P}(\mathcal{S})$ of subsets of $\mathcal{S} \neq \varnothing$ is said to be an ideal in $\mathcal{S}$ if (i) $\varnothing \in \mathcal{I}$, (ii) $\mathcal{G}, \mathcal{H} \in \mathcal{I} \Longrightarrow \mathcal{G} \cup \mathcal{H} \in \mathcal{I}$, (iii) $\mathcal{G} \in \mathcal{I}, \mathcal{H} \subseteq \mathcal{G} \Longrightarrow \mathcal{H} \in \mathcal{I}$. An ideal $\mathcal{I}$ is called a non-trivial if $\mathcal{I} \neq \varnothing$ and $\mathcal{S} \notin \mathcal{I}$. A non-trivial ideal $\mathcal{I}$ in $\mathcal{S}$ is called admissible if $\{\mathfrak{g}\} \in \mathcal{I}$, for each $\mathfrak{g} \in \mathcal{S}$. We denote the set of all non-trivial admissible ideal in $\mathbb{N}$ by $\Im$

A non-empty class $\mathcal{F} \subseteq \mathcal{P}(\mathcal{S})$ of subsets of $\mathcal{S}$ is said to be a filter in $\mathcal{S} \neq \varnothing$ if (i) $\varnothing \notin \mathcal{F}$, (ii) $\mathcal{G}, \mathcal{H} \in \mathcal{F} \Longrightarrow \mathcal{G} \cap \mathcal{H} \in \mathcal{F}$, (iii) $\mathcal{G} \in \mathcal{F}, \mathcal{H} \supseteq \mathcal{G} \Longrightarrow \mathcal{H} \in \mathcal{F}$. Let $\mathcal{I}$ be a non-trivial ideal in $\mathcal{S}$, the filter $\mathcal{F}(\mathcal{I}) = \{\mathcal{M} = \mathcal{S} \setminus \mathcal{H} : \mathcal{H} \in \mathcal{I}\}$ is called the filter associated with the ideal $\mathcal{I}$.

In [12] defined $\mathcal{I}-$convergence and $\mathcal{I}^*-$convergence and gave a necessary and sufficient condition for the equivalency of both definitions.

Remark 1.1. Throughout the paper, $\mathcal{I} \in \Im$ and $\mathcal{A} \in \Psi$.

Definition 1.2. ([12]). A real sequence $\mathfrak{x} = (\mathfrak{x}_k)$ is said to be $\mathcal{I}-$convergent to $\ell \in \mathbb{R}$ if for every $\epsilon > 0$, the set $\mathcal{K}(\epsilon) = \{k : |\mathfrak{x}_k - \ell| \geq \epsilon\} \in \mathcal{I}$. In this case we write $\mathcal{I} - \lim \mathfrak{x}_k = \ell$.

Remark 1.3. (a) If $\mathcal{I} = \mathcal{I}_{fin} = \{\mathcal{K} \subseteq \mathbb{N} : \mathcal{K}$ is finite$\}$, then $\mathcal{I}-$convergence coincide with the usual convergence.

(b) If $\mathcal{I} = \mathcal{I}_\delta = \{\mathcal{K} \subseteq \mathbb{N} : \delta(\mathcal{K}) = 0\}$, then $\mathcal{I}-$convergence coincide with the statistical convergence [1].

Definition 1.4. ([12]). A real sequence $\mathfrak{x} = (\mathfrak{x}_k)$ is said to be $\mathcal{I}^*-$convergent to $\ell \in \mathbb{R}$ if there is a set $\mathcal{H} \in \mathcal{I}$ such that for $\mathcal{M} = \mathbb{N} \setminus \mathcal{H} = \{\mathfrak{m}_1, \mathfrak{m}_2, ...\} \in \mathcal{F}(\mathcal{I})$, where $\mathfrak{m}_1 < \mathfrak{m}_2 < .....$, we have $\lim\limits_i \mathfrak{x}_{\mathfrak{m}_i} = \ell$. In this case we write $\mathcal{I}^* - \lim \mathfrak{x}_k = \ell$.

In [22], Savas et al. introduced the following definition.

Definition 1.5. ([22]). A real sequence $\mathfrak{x} = (\mathfrak{x}_k)$ is said to be $\mathcal{A}^{\mathcal{I}}-$summable to $\ell \in \mathbb{R}$ if the sequence $(\mathcal{A}_n(\mathfrak{x}))$ is $\mathcal{I}-$convergent to ℓ, and we write $\mathcal{A}^{\mathcal{I}} - \lim \mathfrak{x}_k = \ell$.

Remark 1.6. If $\mathcal{I} = \mathcal{I}_\delta$, then $\mathcal{A}^{\mathcal{I}}-$ summability reduces to statistical $\mathcal{A}-$ summability due to [23].

Recently, Edely [24] introduced the notion of $\mathcal{A}^{\mathcal{I}^*}-$ summability and gave some relations with $\mathcal{A}^{\mathcal{I}}-$summability.

Definition 1.7. ([24]). A real sequence $\mathfrak{x} = (\mathfrak{x}_k)$ is $\mathcal{A}^{\mathcal{I}^*}-$ summable to ℓ if there is a set $\mathcal{H} \in \mathcal{I}$ such that $\mathcal{M} = \mathbb{N} \setminus \mathcal{H} = \{\mathfrak{m}_1, \mathfrak{m}_2, ...\} \in F(\mathcal{I})$, and $\lim\limits_i \sum\limits_k \mathfrak{a}_{\mathfrak{m}_i k}\mathfrak{x}_k = \lim\limits_i y_{\mathfrak{m}_i} = \ell$. In this case we write $\mathcal{A}^{\mathcal{I}^*} - \lim \mathfrak{x}_k = \ell$.

Definition 1.8. ([12]). A set $\mathcal{I} \in \Im$ satisfies the condition (AP), if for every sequence $(\mathcal{C}_n)$ of pairwise disjoint sets from $\mathcal{I}$ there are sets $\mathcal{D}_n \subset \mathbb{N}$, $n \in \mathbb{N}$ such that the symmetric difference $\mathcal{C}_n \Delta \mathcal{D}_n$ is finite for every n and $\bigcup_n \mathcal{D}_n \in \mathcal{I}$.

Theorem 1.9 ([24]). *(a) If* $\mathcal{A}^{\mathcal{I}^*} - \lim \mathfrak{x}_k = \ell$ *then* $\mathcal{A}^{\mathcal{I}} - \lim \mathfrak{x}_k = \ell$.

(b) If $\mathcal{I}$ *satisfies the condition* (AP)*, then whenever* $\mathcal{A}^{\mathcal{I}} - \lim \mathfrak{x}_k = \ell$ *we have* $\mathcal{A}^{\mathcal{I}^*} - \lim \mathfrak{x}_k = \ell$.

In [21], Savas at el. introduced the notion of $\mathcal{I}-$statistical convergence which is a natural generalization of the concept of statistical convergence.

Definition 1.10. ([21]). A real sequence $\mathfrak{x} = (\mathfrak{x}_k)$ is $\mathcal{I}-$statistically convergent to $\ell \in \mathbb{R}$ if for each $\epsilon > 0$ and $\nu > 0$, the set

$$\left\{ n : \frac{1}{n} \left| \{ k \leq n : |\mathfrak{x}_k - \ell| \geq \epsilon \} \right| \geq \nu \right\} \in \mathcal{I}.$$

In this case we write $\mathcal{I} - st \lim \mathfrak{x}_k = \ell$.

Remark 1.11. If $\mathcal{I} = \mathcal{I}_{fin}$, then $\mathcal{I}-$statistically convergent coincide with the statistical convergence due to Fast [1].

The notion of $\mathcal{A}^{\mathcal{I}}-$statistical convergence introduced by [22], and gave a relation with $\mathcal{A}^{\mathcal{I}}-$ summability.

Definition 1.12. ([22]). A real sequence $\mathfrak{x} = (\mathfrak{x}_k)$ is said to be $\mathcal{A}^{\mathcal{I}}-$statistically convergent to $\ell \in \mathbb{R}$ if for every $\epsilon > 0$ and $\nu > 0$, the set $\left\{ n : \sum_{k \in \mathcal{K}(\epsilon)} \mathfrak{a}_{nk} \geq \nu \right\} \in \mathcal{I}$, where $\mathcal{K}(\epsilon) = \{ k \leq n : |\mathfrak{x}_k - \ell| \geq \epsilon \}$. In this case we write $\mathcal{I} - st_{\mathcal{A}} \lim \mathfrak{x}_k = \ell$.

Remark 1.13. (a) If $\mathcal{I} = \mathcal{I}_{fin}$, then $\mathcal{A}^{\mathcal{I}}-$statistical convergence becomes $\mathcal{A}-$statistical convergence due to Kolk [8].

(b) If $\mathcal{A} = (\mathfrak{C}, 1)$, then $\mathcal{A}^{\mathcal{I}}-$statistically convergent becomes $\mathcal{I}-$statistical convergence due to [21].

2. Statistical $\mathcal{A}^{\mathcal{I}}$ and statistical $\mathcal{A}^{\mathcal{I}^*}-$ summability

In this section we introduce the following concepts of statistically $\mathcal{A}^{\mathcal{I}}-$ summability and statistically $\mathcal{A}^{\mathcal{I}^*}-$ summability and find some relations.

Definition 2.1. A real sequence $\mathfrak{x} = (\mathfrak{x}_k)$ is said to be statistically $\mathcal{A}^{\mathcal{I}}-$ summable to ℓ if for each $\epsilon > 0$ and every $\nu > 0$, the set

$$\left\{ n \in \mathbb{N} : \frac{1}{n} \left| \{ j \leq n : |y_j - \ell| \geq \epsilon \} \right| \geq \nu \right\} \in \mathcal{I},$$

where $y_j = \mathcal{A}_j(\mathfrak{x})$. Thus $\mathfrak{x}$ is statistically $\mathcal{A}^{\mathcal{I}}-$ summable to ℓ iff the sequence (y_j) is $\mathcal{I}-$statistically convergent to ℓ. In this case we write $(\mathcal{A}^{\mathcal{I}})_{st} - \lim \mathfrak{x} = \mathcal{I} - st \lim \mathcal{A}\mathfrak{x}$.

Remark 2.2. (a) If $\mathcal{A}$ is the identity matrix, then Definition 2.1 becomes $\mathcal{I}-$statistical convergence due to [21].

(b) If $\mathcal{I} = \mathcal{I}_{fin}$, then statistical $\mathcal{A}^{\mathcal{I}}-$ summable coincide with the statistical $\mathcal{A}-$summable due to Edely and Mursaleen [23].

(*c*) Let $\mathcal{A} = (\mathfrak{C}, 1) = (\mathfrak{a}_{jk})$ be a Cesàro matrix defined as

$$\mathfrak{a}_{jk} = \begin{cases} \frac{1}{j} & , \text{ if } j \geq k \\ 0 & , \text{ otherwise }, \end{cases}$$

then we say $\mathfrak{x}$ is statistically $(\mathfrak{C}, 1)^{\mathcal{I}}-$summable. In case $\mathcal{I} = \mathcal{I}_{fin}$ it is reduced to statistical $(\mathfrak{C}, 1)-$summable due to Moricz [25].

Definition 2.3. A real sequence $\mathfrak{x} = (\mathfrak{x}_k)$ is said to be statistically $\mathcal{A}^{\mathcal{I}^*}-$ summable to ℓ if there is a set $\mathcal{M} = \{\mathfrak{m}_i\}$, where $\mathfrak{m}_1 < \mathfrak{m}_2 < \ldots$, and $\mathcal{M} \in F(\mathcal{I})$, $\delta(\mathcal{M}) = 1$, such that

$$st - \lim_i \mathcal{A}_{\mathfrak{m}_i}\mathfrak{x} = st - \lim_i y_{\mathfrak{m}_i} = \ell,$$

where $y_{\mathfrak{m}_i} = \sum\limits_k \mathfrak{a}_{\mathfrak{m}_i k}\mathfrak{x}_k$ i.e. $(\mathcal{A}_{\mathfrak{m}_i}\mathfrak{x})$ is statistically convergent to ℓ. In this case we write $(\mathcal{A}^{\mathcal{I}^*})_{st} - \lim \mathfrak{x} = \mathcal{I}^* - st \lim \mathcal{A}\mathfrak{x} = \ell$.

Remark 2.4. (*a*) If $(\mathcal{A}^{\mathcal{I}^*})_{st} - \lim \mathfrak{x}$ exists, then it is unique.
(*b*) If $\mathcal{A}$ is the identity matrix, then we say that $\mathfrak{x}$ is $\mathcal{I}^*-$statistical convergence.

We give analogue results for statistically $\mathcal{A}^{\mathcal{I}}-$ summability and statistically $\mathcal{A}^{\mathcal{I}^*}-$summability as in [12] . For this we define (APO) condition which is given by [12] and [26] .

Definition 2.5. We say that $\mathcal{I}$ satisfies (APO) condition, if for every sequence $(\mathcal{C}_n)$ of (pairwise disjoint) sets from $\mathcal{I}$ such that $\delta(\mathcal{C}_n) = 0$ for each n, then there exist sets $\mathcal{D}_n \in \mathcal{I}$, $n \in \mathbb{N}$ such that the symmetric difference $\mathcal{C}_n \Delta \mathcal{D}_n$ is finite for every n, $\bigcup\limits_n \mathcal{D}_n \in \mathcal{I}$, $\delta(\bigcup\limits_n \mathcal{D}_n) = 0$.

The following Proposition can be directly obtained by Proposition 1 of [13] and properties of density.

Proposition 2.6. *$\mathcal{I}$ satisfies (APO) iff for every sequence $(\mathcal{C}_n)$ of (pairwise disjoint) sets from $\mathcal{I}$ such that $\delta(\mathcal{C}_n) = 0$ for each n, then there exists $\mathcal{C} \in \mathcal{I}$ with $\mathcal{C}_n \setminus \mathcal{C}$ is finite for every n, $\delta(\mathcal{C}) = 0$.*

Theorem 2.7. (*a*) *If $(\mathcal{A}^{\mathcal{I}^*})_{st} - \lim \mathfrak{x}_k = \ell$ then $(\mathcal{A}^{\mathcal{I}})_{st} - \lim \mathfrak{x}_k = \ell$.*
(*b*) *If $\mathcal{I}$ satisfies the condition (APO), then whenever $\left(\mathcal{A}^{\mathcal{I}}\right)_{st} - \lim \mathfrak{x}_k = \ell$ we have $\left(\mathcal{A}^{\mathcal{I}^*}\right)_{st} - \lim \mathfrak{x}_k = \ell$.*

Proof. (*a*) Let $(\mathcal{A}^{\mathcal{I}^*})_{st}- \lim \mathfrak{x}_k = \ell$, then there exists $\mathcal{H} \in \mathcal{I}$ such that $\mathcal{M} = \{\mathfrak{m}_i\} = \mathbb{N} \setminus \mathcal{H} \in F(\mathcal{I})$, $\delta(\mathcal{M}) = 1$, and

$$st - \lim_i \sum_k \mathfrak{a}_{\mathfrak{m}_i k}\mathfrak{x}_k = \ell,$$

i.e. for every $\epsilon > 0$, we have

$$\lim_n \frac{1}{n} \left|\{\mathfrak{m}_i \leq n : |y_{\mathfrak{m}_i} - \ell| \geq \epsilon\}\right| = 0.$$

Therefore for each $\nu > 0$, there exists N such that $\frac{1}{N} < \nu$, so the set

$$\mathcal{G} = \left\{n : \frac{1}{n} \left|\{j \leq n : |y_j - \ell| \geq \epsilon\}\right| \geq \nu\right\} \subseteq \mathcal{H} \cup \{\mathfrak{m}_1, \mathfrak{m}_2, ..., \mathfrak{m}_N\}.$$

Since $\mathcal{H} \in \mathcal{I}$ and $\{\mathfrak{m}_1, \mathfrak{m}_2, ..., \mathfrak{m}_N\} \in \mathcal{I}$, we have $\mathcal{G} \in \mathcal{I}$. Hence $(\mathcal{A}^{\mathcal{I}})_{st} - \lim \mathfrak{x}_k = \ell$.

(b) Let $\left(\mathcal{A}^{\mathcal{I}}\right)_{st} - \lim \mathfrak{x}_k = \ell$, then for every $\epsilon > 0$ and for each $\nu > 0$, we have

$$\left\{n : \frac{1}{n}\left|\mathcal{K}(\epsilon)\right| \geq \nu\right\} \in \mathcal{I},$$

where $\mathcal{K}(\epsilon) = \{j \leq n : |y_j - \ell| \geq \epsilon\}$. Therefore for every i, define the sequence $(\mathcal{C}_i)$ of sets as

$$\mathcal{C}_1 = \left\{n : \frac{1}{n}\left|\mathcal{K}(\epsilon)\right| \geq 1\right\},\ \mathcal{C}_i = \left\{n : \frac{1}{i-1} > \frac{1}{n}\left|\mathcal{K}(\epsilon)\right| \geq \frac{1}{i}\right\},\ \forall i > 1, i \in \mathbb{N}.$$

It is easy to see that each $\mathcal{C}_i \in \mathcal{I}$ and $\delta\left(\mathcal{C}_i\right) = 0, \forall i$. Since $\mathcal{I}$ satisfies the condition (APO) by Proposition 2.6, there exists a set $\mathcal{C} \in \mathcal{I}$ such that $\delta(\mathcal{C}) = 0$ and $\mathcal{C}_i \setminus \mathcal{C}$ is finite for each i. Let $\mathcal{M} = \mathbb{N} \setminus \mathcal{C} = \{\mathfrak{m}_1, \mathfrak{m}_2, ...\}$, so $\delta(\mathcal{M}) = 1$. Now for any $\eta > 0$, there is $N \in \mathbb{N}$ such that $\frac{1}{N} < \eta$, therefore the set

$$\mathcal{C}_N = \left\{n : \frac{1}{N-1} > \frac{1}{n}\left|\mathcal{K}(\epsilon)\right| \geq \frac{1}{N}\right\} \in \mathcal{I}.$$

Then

$$\mathcal{D} = \left\{n : \frac{1}{n}\left|\mathcal{K}(\epsilon)\right| < \frac{1}{N}\right\} \bigcup \left\{n : \frac{1}{n}\left|\mathcal{K}(\epsilon)\right| \geq \frac{1}{N-1}\right\} \setminus \mathcal{C} \in F(\mathcal{I}), \delta(\mathcal{D}) = 1.$$

Hence we have

$$\frac{1}{n}\left|\mathcal{K}(\epsilon)\right| < \eta,\ \forall n > N,\ n \in \mathcal{D},$$

i.e.

$$\lim_n \frac{1}{n}\left|\{j \leq n : |y_j - \ell| \geq \epsilon\}\right| = 0, \quad n \in \mathcal{D}.$$

Hence $st - \lim_j y_{\mathrm{j}} = \ell$, $j \in \mathcal{D}$, $\delta\left(\mathcal{D}\right) = 1$, i.e. $\left(\mathcal{A}^{\mathcal{I}^*}\right)_{st} - \lim \mathfrak{x}_k = \ell$. ■

Remark 2.8. The converse of Theorem 2.7 (a) is not true in general.

Example 2.9. Let $\mathcal{D}_i = \left\{2^{i-1}(2k-1) : k \in \mathbb{N}\right\}$ be mutually disjoint infinite sets such that $\mathbb{N} = \bigcup_{i=1}^{\infty} \mathcal{D}_i$. Let $\mathcal{I}$ be the class defined as

$$\mathcal{I} = \left\{\mathcal{D} \subset \mathbb{N} : \ \mathcal{D} \text{ intersects only finite numbers of } \mathcal{D}_i's\right\},$$

then $\mathcal{I} \in \Im$. Define $\mathfrak{x} = (\mathfrak{x}_k)$ as

$$\mathfrak{x}_k = \frac{1}{i},\ k \in \mathcal{D}_i,$$

and $\mathcal{A} = (\mathfrak{a}_{jk})$ defined as

$$\mathfrak{a}_{jk} = \begin{cases} 1 & , \text{ if } k = j^2 \\ 0 & , \text{ otherwise,} \end{cases}$$

so, we have

$$y_j = \sum_k \mathfrak{a}_{jk}\mathfrak{x}_k = \frac{1}{t},\ j^2 \in \mathcal{D}_t.$$

Here $\mathfrak{x}$ is statistically $\mathcal{A}^{\mathcal{I}}-$summable to zero, since for any $\epsilon > 0$ and for every $\nu > 0$, the set

$$\left\{n \in \mathbb{N} : \frac{1}{n}\left|\{j \leq n : |y_j| \geq \epsilon\}\right| \geq \nu\right\} \subseteq \left\{n \in \mathbb{N} : \frac{\sqrt{n}}{n} \geq \nu\right\} \in I.$$

Now to show that $\mathfrak{x}$ is not statistically $\mathcal{A}^{\mathcal{I}^*}-$summable to zero. Suppose if it is possible that $\mathfrak{x}$ is statistical $\mathcal{A}^{\mathcal{I}^*}-$summable to zero, then there exists a set $\mathcal{M} = \mathbb{N} \setminus \mathcal{H} = \{\mathfrak{m}_1, \mathfrak{m}_2, ...\}$, where $\mathcal{H} \in \mathcal{I}$, $\delta(\mathcal{M}) = 1$, and $st - \lim_i y_{\mathfrak{m}_i} = 0$. Since $\mathcal{H} \in \mathcal{I}$, then there exists $r \in \mathbb{N}$ such that r is odd and $\mathcal{H} \subseteq \mathcal{D}_1 \cup \mathcal{D}_2 \cup .... \cup \mathcal{D}_r$. So $\mathcal{D}_{r+1} \subseteq \mathbb{N} \setminus \mathcal{H} = \mathcal{M}$. Therefore $y_{\mathfrak{m}_i} = \frac{1}{r+1}$ for infinitely many i 's. Now let us choose $\eta > 0$ such that $\eta < \frac{1}{r+1}$. Hence the set

$$\delta\left\{\mathfrak{m}_i \in \mathcal{D}_{r+1} : |y_{\mathfrak{m}_i}| \geq \eta\right\} = \frac{1}{2^{r+1}} \neq 0,$$

i.e. $(\mathcal{A}^{\mathcal{I}^*})_{st}-\lim \mathfrak{x}_k \neq 0$, which is a contradiction, hence $\mathfrak{x}$ is not statistically $\mathcal{A}^{\mathcal{I}^*}-$summable to zero.

In [21], proved that if $\mathfrak{x}$ is bounded and $\mathcal{A}^{\mathcal{I}}-$statistically convergent to ℓ, then $\mathfrak{x}$ is $\mathcal{A}^{\mathcal{I}}-$summable to ℓ, and by Theorem 1.9 (b), if $\mathcal{I}$ satisfied the condition (AP), then x is $\mathcal{A}^{\mathcal{I}^*}-$summable to ℓ. Let $\mathcal{I}$ satisfied the condition (AP), and let us define the set

$$\Gamma = \left\{x \in \ell_\infty : \mathcal{I} - st_{\mathcal{A}} - \lim x = \ell, \exists\ \mathcal{M} = \{m_i\} \in \mathcal{F}(\mathcal{I}),\ \delta(\mathcal{M}) = 1, \lim_i y_{m_i} = \ell\right\}.$$

Then we have the following relation between $\mathcal{A}^{\mathcal{I}}-$statistical convergence and statistically $\mathcal{A}^{\mathcal{I}}-$summable.

Theorem 2.10. *If $x \in \Gamma$, then x is statistically $\mathcal{A}^{\mathcal{I}}-$summable.*

Proof. Let $x \in \Gamma$, so there exists $\mathcal{M} = \{\mathfrak{m}_i\}$, $\mathcal{M} \in \mathcal{F}(\mathcal{I})$ and $\delta(\mathcal{M}) = 1$, such that $\lim_i y_{\mathfrak{m}_i} = \ell$. Hence

$$st - \lim_i y_{\mathfrak{m}_i} = \ell,\ \delta(\mathcal{M}) = 1,$$

i.e. $\mathfrak{x}$ is statistically $\mathcal{A}^{\mathcal{I}^*}-$summable to ℓ. Now by Theorem 2.7 (a), we have $\mathfrak{x}$ is statistically $\mathcal{A}^{\mathcal{I}}-$summable to ℓ. ∎

Remark 2.11. The converse of Theorem 2.10 is not true in general.

Example 2.12. Let $\mathcal{I}$ be the class defined in Example 2.9. Define $\mathfrak{x} = (\mathfrak{x}_k)$ as

$$\mathfrak{x}_k = \begin{cases} 1 & , k \in \mathcal{D}_1, \\ 2 & , k \notin \mathcal{D}_1,\ k \text{ is square}, \\ 1 & , k \notin \mathcal{D}_1,\ k \text{ is nonsquare}, \end{cases}$$

and $\mathcal{A} = (\mathfrak{a}_{jk})$ be defined as

$$\mathfrak{a}_{jk} = \begin{cases} 1 & , j \in \mathcal{D}_1,\ j = k \\ 1 & , j \notin \mathcal{D}_1,\ j = k \text{ square}, \\ 1 & , j \notin \mathcal{D}_1,\ j = k \text{ nonsquare}, \\ 0 & , \text{otherwise}, \end{cases}$$

then

$$y_j = \sum_k \mathfrak{a}_{jk}\mathfrak{x}_k = \begin{cases} 1 & , j \in \mathcal{D}_1, \\ 2 & , j \notin \mathcal{D}_1,\ j \text{ is square}, \\ 1 & , j \notin \mathcal{D}_1,\ j \text{ is nonsquare}. \end{cases}$$

Therefore for $\epsilon = \frac{1}{2}$, and for any $\ell \in \mathbb{R}$, the set

$$\left\{j : |y_j - \ell| \geq \frac{1}{2}\right\} \notin \mathcal{I},$$

i.e. $\mathfrak{x}$ is not $\mathcal{A}^{\mathcal{I}}-$summable to any number and hence $\mathfrak{x}$ is not $\mathcal{A}^{\mathcal{I}}-$ statistically convergent to any number. Now for any choice of $\epsilon > 0$ and for every $\nu > 0$, there exists $N \in \mathbb{N}$, such that the set

$$\begin{aligned}\left\{n \in \mathbb{N} : \frac{1}{n} \left|\{j \leq n : |y_j - 1| \geq \epsilon\}\right| \geq \nu\right\} &\subseteq \left\{n \in \mathbb{N} : \frac{\sqrt{n}}{n} \geq \nu\right\} \\ &\subseteq \mathcal{D}_1 \cup \mathcal{D}_2 \cup ..... \cup \mathcal{D}_N \in \mathcal{I},\end{aligned}$$

i.e. $\mathfrak{x}$ is statistically $\mathcal{A}^{\mathcal{I}}-$ summable to 1.

Example 2.13. Let $\mathcal{I}$ be the class defined in Example 2.9 and $\mathfrak{x} = (\mathfrak{x}_k)$ defined as

$$\mathfrak{x}_k = \begin{cases} 2 & , \text{ if } k \in \mathcal{D}_1, \\ 0 & , \text{ otherwise,} \end{cases}$$

and let us defined $\mathcal{A} = (\mathfrak{C}, 1) = (\mathfrak{a}_{jk})$, then $y_j = \sum_k \mathfrak{a}_{jk}\mathfrak{x}_k = 1$. It is obvious that $\mathfrak{x}$ is $\mathcal{A}^{\mathcal{I}}-$ summable to 1 and $\mathfrak{x}$ is also statistically $\mathcal{A}^{\mathcal{I}}-$ summable to 1.

Now for $\epsilon = \frac{1}{4}$ and for any $\ell \in \mathbb{R}$, the set $\mathcal{K}(\frac{1}{4}) = \left\{k : |\mathfrak{x}_k - \ell| \geq \frac{1}{4}\right\}$ contains either $\mathcal{D}_1$ (the set of odd) or the set of even or both. So $\sum_{k \in \mathcal{K}(\frac{1}{4})} \mathfrak{a}_{jk} = \frac{1}{2}$ or 1. Therefore for $\nu = \frac{1}{3}$, the set

$$\left\{j : \sum_{k \in \mathcal{K}(\frac{1}{4})} \mathfrak{a}_{jk} \geq \frac{1}{3}\right\} = \mathbb{N} \notin \mathcal{I},$$

since $\mathcal{I} \in \Im$, we have $\mathfrak{x}$ is not $\mathcal{A}^{\mathcal{I}}-$statistically convergent to any number. Note that $\mathfrak{x}$ is $\mathcal{I}-$convergent to zero but not $I-$statistically convergent.

Remark 2.14. The notions of $\mathcal{I}-$convergence, $\mathcal{A}^{\mathcal{I}}-$summable, $\mathcal{I}-$statistical convergence, $\mathcal{A}^{\mathcal{I}}-$statistical convergence and statistical $\mathcal{A}^{\mathcal{I}}-$ summable are not comparable in general.

The next result of this section is an analogous result for continuity as in [27] and [12].

Theorem 2.15. *A real valued function $f : \mathbb{R} \to \mathbb{R}$ is continuous if and only if whenever $\left(\mathcal{A}^{\mathcal{I}}\right)_{st} - \lim \mathfrak{x}_k = \ell$, we have $\mathcal{I} - st \lim f(y_j) = f(\ell)$.*

Proof. Let $\left(\mathcal{A}^{\mathcal{I}}\right)_{st} - \lim \mathfrak{x}_k = \ell$, i.e. $\mathcal{I} - st \lim y_j = \ell$. Therefore for any $\epsilon > 0$ and for each $\nu > 0$, the set

$$\left\{n \in \mathbb{N} : \frac{1}{n} \left|\{j \leq n : |y_j - \ell| \geq \epsilon\}\right| \geq \nu\right\} \in \mathcal{I}.$$

Since f is continuous, then for each $\vartheta > 0$ there exists $\eta > 0$ such that if $|\mathfrak{x} - \ell| < \eta$ implies $|f(\mathfrak{x}) - f(\ell)| < \vartheta$. Therefore, we have

$$\{j : |y_j - \ell| \geq \eta\} \supseteq \{j : |f(y_j) - f(\ell)| \geq \vartheta\},$$

hence

$$\frac{1}{n} \left|\{j \leq n : |y_j - \ell| \geq \eta\}\right| \geq \frac{1}{n} \left|\{j \leq n : |f(y_j) - f(\ell)| \geq \vartheta\}\right|.$$

Therefore for each $\nu > 0$ and for any $\vartheta > 0$,

$$\begin{aligned}\mathcal{H} &= \left\{n : \frac{1}{n}\left|\{j \le n : |y_j - \ell| \ge \eta\}\right| \ge \nu\right\} \\ &\supseteq \left\{n : \frac{1}{n}\left|\{j \le n : |f(y_j) - f(\ell)| \ge \vartheta\}\right| \ge \nu\right\} = \mathcal{G}.\end{aligned}$$

Since $\mathcal{H} \in \mathcal{I}$, we have $\mathcal{G} \in \mathcal{I}$. Hence $\mathcal{I} - st \lim f(y_j) = f(\ell)$.

Conversely, let us assume that f is not continuous at $\ell \in \mathbb{R}$, then there exist a sequence $(\mathfrak{x}_k)$ converges to ℓ and $\eta > 0$ such that $|f(\mathfrak{x}_k) - f(\ell)| \ge \eta$ for $k \in \mathbb{N}$. So that the set

$$\{k : |f(\mathfrak{x}_k) - f(\ell)| \ge \eta\} = \mathbb{N}.$$

Hence for any $0 < \nu < 1$, the set

$$\left\{n : \frac{1}{n}\left|\{k \le n : |f(\mathfrak{x}_k) - f(\ell)| \ge \eta\}\right| \ge \nu\right\} = \mathbb{N}.$$

Since $\lim \mathfrak{x}_k = \ell$, and $\mathcal{A}$ is regular, we have $\mathcal{I} - st \lim y_j = \ell$. Now let $\mathcal{A} = (\mathfrak{a}_{jk})$ be the identity matrix, then the set

$$\left\{n : \frac{1}{n}\left|\{j \le n : |f(y_j) - f(\ell)| \ge \eta\}\right| \ge \nu\right\} = \mathbb{N} \notin \mathcal{I},$$

since $\mathcal{I} \in \Im$. Hence we have a contradiction, i.e. $\mathcal{I} - st \lim f(y_j) = \mathcal{I} - st \lim f(\mathfrak{x}_k) \ne f(\ell)$. Hence f is continuous. ■

References

[1] H. Fast, Sur la convergence statistique, Colloq. Math. 2 (1951) 241–244.

[2] R.C. Buck, Generalized asymptotic density, American Journal of Mathematics 75 (2) (1953) 335–346.

[3] I.J. Schoenberg, The integrability of certain functions and related summability methods, The American Mathematical Monthly 66 (5) (1959) 361–775.

[4] A. Zygmund, Trigonometric series, Cambridge University Press, 2002.

[5] J.A. Fridy, On statistical convergence, Analysis 5 (1985) 301-313.

[6] T.Šalát, On statistically convergent sequences of real numbers, Math. Slovaca 30 (1980) 139–150.

[7] J. Connor, On strong matrix summability with respect to a modulus and statistical convergence, Canad. Math. Bull. 32 (1989) 194–198.

[8] E. Kolk, Matrix summability of statistically convergent sequences, Analysis 13 (1-2) (1993) 77–84.

[9] M. Mursaleen, O.H.H. Edely, Generalized statistical convergence, Information Sciences 162 (3-4) (2004) 287–294.

[10] I. Niven, H.S. Zuckerman, An Introduction to the Theory of Numbers, John Wiley and Sons, New York, 1980.

[11] A.R. Freedman, I.J. Sember, Densities and summability, Pacific Journal of Mathematics 95 (2) (1981) 293–305.

[12] P. Kostyrko, T. Šalát, W. Wilczyńki, I-convergence, Real Anal. Exchange 26 (2) (2000/2001) 669–686.

[13] M. Balcerzak, K. Dems, A. Komisarski, Statistical convergence and ideal convergence for sequences of functions, J. Math. Anal. Appl. 328 (1) (2007) 715–729.

[14] K. Demirci, I-limit superior and limit inferior, Math. Commun, 6 (2) (2001) 165–172.

[15] H. Gümüş, Ö. Kişi, E. Savas, Some results about $\Delta I-$statistically pre-Cauchy sequences with an orcilz function, Journal of Computational Analysis and Applications 28 (1) (2020) 180–188.

[16] M. Mursaleen, A. Alotaibi, On I-convergence in random 2-normed spaces, Math. Slovaca 61 (6) (2011) 933–940.

[17] M. Mursaleen, S.A. Mohiuddine, O.H.H. Edely, On the ideal convergence of double sequences in intuitionistic fuzzy normed spaces, Computers and Mathematics with Applications 59 (2) (2010) 603–611.

[18] M. Mursaleen and S.A. Mohiuddine, On ideal convergence in probabilistic normed spaces, Math. Slovaca 62 (2012) 49–62.

[19] M. Mursaleen, S. Debnath, D. Rakshit, I-statistical limit superior and I-statistical limit inferior, Filomat 31 (7) (2017) 2103–2108.

[20] T.Šalát, B.C. Tripathy, M. Ziman, On I-convergence field. Ital. J. Pure Appl. Math. 17 (5) (2005) 1–8.

[21] E. Savas, P. Das, A generalized statistical convergence via ideals, Applied Mathematics Letters 24 (6) (2011) 826–830.

[22] E. Savas, P. Das, S. Dutta, A note on some generalized summability methods, Acta Math. Univ. Comenianae 82 (2) (2013) 297-304.

[23] O.H.H. Edely, M. Mursaleen, On statistical A-summability, Mathematical and Computer Modelling 49 (3) (2009) 672–680.

[24] O.H.H. Edely, On some properties of A^I summability and A^{I^*} summability, Azerbaijan Journal of Mathematics 11 (1) (2021) 189–200.

[25] F. Moricz, Tauberian conditions under which statistical convergence follows from statistical summability (C, 1), Journal of Mathematical Analysis and Applications 275 (1) (2002) 277–287.

[26] J. Connor, Two valued measures and summability, Analysis 10 (4) (1990) 373–386.

[27] R.G. Bartle, J.I. Joichi, The preservation of convergence of measurable functions under composition, Proceedings of the American Mathematical Society 12 (1) (1961) 122–126.