Some New Formulas for Horn's Hypergeometric Functions H_1, H_2, H_3, H_4, H_5, H_6 and H_7

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Abstract The aim of this work is to demonstrate various an interesting recursion formulas, differential and integral operators, integration formulas, and infinite summation for each of Horn's hypergeometric functions H_1, H_2, H_3, H_4, H_5, H_6 and H_7 by the contiguous relations of Horn's hypergeometric series. Some interesting different cases of our main consequences are additionally constructed.

1. Introduction and Definitions

In [1], the $\Gamma(\mu)$ Gamma function is defined as follows

$$\Gamma(\mu) = \int_0^\infty e^{-t} t^{\mu-1} dt, \Re(\mu) > 0. \tag{1.1}$$

Let N and $\mathbb{C}$ be the set of natural and complex numbers, the Pochhammer symbol $(\mu)_k$ is given by

$$(\mu)_k = \frac{\Gamma(\mu+n)}{\Gamma(\mu)} = \begin{cases} \mu(\mu+1)(\mu+2)\dots(\mu+k-1), & \mu \in \mathbb{C}\backslash\{0\}, k \in \mathrm{N}; \\ 1, & \mu \in \mathbb{C}\backslash\{0\}, k = 0, \end{cases} \tag{1.2}$$

and

$$(\mu)_{-k} = \frac{(-1)^n}{(1-\mu)_k}, (\mu \neq 0, \pm 1, \pm 2, \pm 3, \dots, \forall \mu > k; k \in \mathrm{N}). \tag{1.3}$$

Following abbreviated notations to recall the investigation are needed in [1, 2]. We write

$$
\begin{aligned}
(\mu)_{k+1} &= \mu(\mu+1)_k = (\mu+k)(\mu)_k, \\
(\mu)_{k-1} &= \frac{1}{\mu-1}(\mu-1)_k; \mu \neq 1, \\
(\mu+1)_k &= \left(1 + \frac{k}{\mu}\right)(\mu)_k; \mu \neq 0, \\
\frac{1}{(\mu-1)_k} &= \left[1 + \frac{k}{\mu-1}\right]\frac{1}{(\mu)_k}; \mu \neq 1, 0, -1, -2, \ldots.
\end{aligned}
\tag{1.4}
$$

For $n, k \in \mathrm{N}$ and ν satisfy the condition in (1.3), we have

$$
\begin{aligned}
(\mu)_{n+k} &= (\mu)_n(\mu+n)_k = (\mu)_k(\mu+k)_n, \\
(\nu)_{n-k} &= \frac{(-1)^k(\nu)_n}{(1-\nu-n)_k}, 0 \leq k \leq n.
\end{aligned}
\tag{1.5}
$$

Definition 1.1. The Horn hypergeometric functions H_1, H_2, H_3, H_4, H_5, H_6 and H_7 are defined as [[1], page 56, Eqs. (25)-(27), page 57, Eqs. (28)-(31), [2], page 225, Eqs. (13)-(19), [3]]

$$
H_1(\alpha, \beta, \gamma; \delta; x, y) = \sum_{m,n=0}^{\infty} \frac{(\alpha)_{m-n}(\beta)_{m+n}(\gamma)_n}{(\delta)_m m! n!} x^m y^n,
\tag{1.6}
$$

$(|x| < r, |y| < s, 4rs = (s-1)^2, \delta \neq 0, -1, -2, \ldots, \alpha$ satisfies condition (1.5)),

$$
H_2(\alpha, \beta, \gamma, \delta; \epsilon; x, y) = \sum_{m,n=0}^{\infty} \frac{(\alpha)_{m-n}(\beta)_m(\gamma)_n(\delta)_n}{(\epsilon)_m m! n!} x^m y^n,
\tag{1.7}
$$

$(|x| < r, |y| < s, -r + \frac{1}{s} = 1, e \neq 0, -1, -2, \ldots, \alpha$ satisfies condition (1.5)),

$$
H_3(\alpha, \beta; \gamma; x, y) = \sum_{m,n=0}^{\infty} \frac{(\alpha)_{2m+n}(\beta)_n}{(\gamma)_{m+n} m! n!} x^m y^n,
\tag{1.8}
$$

$(|x| < r, |y| < s, r + \left(s - \frac{1}{2}\right)^2 = \frac{1}{4}, \gamma \neq 0, -1, -2, \ldots),$

$$
H_4(\alpha, \beta; \gamma; \delta; x, y) = \sum_{m,n=0}^{\infty} \frac{(\alpha)_{2m+n}(\beta)_n}{(\gamma)_m(\delta)_n m! n!} x^m y^n,
\tag{1.9}
$$

$(|x| < r, |y| < s, 4r = (s-1)^2, \gamma, \delta \neq 0, -1, -2, \ldots),$

$$
H_5(\alpha, \beta; \gamma; x, y) = \sum_{m,n=0}^{\infty} \frac{(\alpha)_{2m+n}(\beta)_{n-m}}{(\gamma)_n m! n!} x^m y^n,
\tag{1.10}
$$

$(|x| < r, |y| < s, 1 + 16r^2 - 36rs \pm (8r - s + 27rs^2) = 0,$
$\gamma \neq 0, -1, -2, \ldots, \beta$ satisfies condition (1.5)),

$$H_6(\alpha, \beta, \gamma; x, y) = \sum_{m,n=0}^{\infty} \frac{(\alpha)_{2m-n}(\beta)_{n-m}(\gamma)_n}{n!m!} x^m y^n,$$

$$(|x| < r, |y| < s, s^2 r + s = 1, \alpha, \beta \text{ satisfies condition } (1.5)) \tag{1.11}$$

and

$$H_7(\alpha, \beta, \gamma; \delta; x, y) = \sum_{m,n=0}^{\infty} \frac{(\alpha)_{2m-n}(\beta)_n(\gamma)_n}{(\delta)_m m! n!} x^m y^n,$$

$$(|x| < r, |y| < s, 4r = (\frac{1}{s} - 1)^2, \delta \neq 0, -1, -2, \ldots, \alpha \text{ satisfies condition } (1.5)). \tag{1.12}$$

An integral operator is defined by [4, 5]

$$\hat{\mathfrak{I}} = \frac{1}{x} \int_0^x dx + \frac{1}{y} \int_0^y dy. \tag{1.13}$$

Recently, various recursion formulas involving some well known special functions of mathematical physics have been developed by way of a quantity of authors. Hypergeometric series of two variables have been investigated significantly from their mathematical analysis point of view. Horn essentially identified 34 distinct convergent series. Amongst them, we have chosen seven Horn series that appear more frequently in an extensive variety of problems in theoretical physics, applied mathematics, chemistry, statistics and engineering sciences. These connections of Horn hypergeometric functions with various other research areas have led many researchers to the field of special functions.

Motivated by the work of [6–17], the authors organized the results of the research. The paper is organized as follows. In Section 2, we discuss the various recursion formulas of Horn hypergeometric functions with all the parameters. In Section 3, we establish many differential formulas and differential operators for Horn's hypergeometric functions. In Section 4, we obtain some integral formulas of Horn hypergeometric functions. In Section 5, we derive the infinite summation formulas for the functions H_1, H_2, H_3, H_4, H_5, H_6 and H_7.

2. Recursion Formulas for Horn Hypergeometric Functions H_1, H_2, H_5, H_6 and H_7

Here, we establish an interesting recursion formulas for the Horn hypergeometric functions H_1, H_2, H_5, H_6 and H_7.

Theorem 2.1. *For $k \in \mathrm{N}$, the Horn hypergeometric function H_1 satisfies the recursion formulas:*

$$H_1(\alpha + k, \beta, \gamma; \delta; x, y) = H_1(\alpha, \beta, \gamma; \delta; x, y) + \frac{\beta x}{\delta} \sum_{r=1}^{k} H_1(\alpha + r, \beta + 1, \gamma; \delta + 1; x, y)$$

$$- \beta \gamma y \sum_{r=1}^{k} \frac{1}{(\alpha + r - 1)(\alpha + r - 2)} H_1(\alpha + r - 2, \beta + 1, \gamma + 1; \delta; x, y), \tag{2.1}$$

$$(\delta \neq 0, \alpha \neq 1 - r, \alpha \neq 2 - r, r \in \mathrm{N}),$$

$$H_1(\alpha, \beta + k, \gamma; \delta; x, y) = H_1(\alpha, \beta, \gamma; \delta; x, y) + \frac{\alpha x}{\delta} \sum_{r=1}^{k} H_1(\alpha + 1, \beta + r, \gamma; \delta + 1; x, y)$$

$$+ \frac{\gamma y}{\alpha - 1} \sum_{r=1}^{k} H_1(\alpha - 1, \beta + r, \gamma + 1; \delta; x, y), \alpha \neq 1, \delta \neq 0, \tag{2.2}$$

$$H_1(\alpha, \beta, \gamma + k; \delta; x, y) = H_1(\alpha, \beta, \gamma; \delta; x, y)$$

$$+ \frac{\beta y}{\alpha - 1} \sum_{r=1}^{k} H_1(\alpha - 1, \beta + 1, \gamma + r; \delta; x, y), \alpha \neq 1 \tag{2.3}$$

and

$$H_1(\alpha, \beta, \gamma; \delta - k; x, y) = H_1(\alpha, \beta, \gamma; \delta; x, y)$$

$$+ \alpha \beta x \sum_{r=1}^{k} \frac{H_1(\alpha + 1, \beta + 1, \gamma; \delta - r + 2; x, y)}{(\delta - r)(\delta - r + 1)}, \delta \neq r, \delta \neq r - 1, r \in \mathbb{N}. \tag{2.4}$$

Proof. Referring to the definition H_1 (1.6) and transformation

$$(\alpha + 1)_{m-n} = (\alpha)_{m-n} \left(1 + \frac{m - n}{\alpha} \right), \alpha \neq 0, \tag{2.5}$$

we get the contiguous relation:

$$H_1(\alpha + 1, \beta, \gamma; \delta; x, y) = H_1(\alpha, \beta, \gamma; \delta; x, y) + \frac{\beta x}{\delta} H_1(\alpha + 1, \beta + 1, \gamma; \delta + 1; x, y)$$

$$- \frac{\beta \gamma y}{\alpha(\alpha - 1)} H_1(\alpha - 1, \beta + 1, \gamma + 1; \delta; x, y), \delta \neq 0, \alpha \neq 0, \alpha \neq 1. \tag{2.6}$$

Again, we get the function H_1 with the parameter $\alpha + 2$ by adding this contiguous relation

$$H_1(\alpha + 2, \beta, \gamma; \delta; x, y) = H_1(\alpha, \beta, \gamma; \delta; x, y) + \frac{\beta x}{\delta} \left[H_1(\alpha + 1, \beta + 1, \gamma; \delta + 1; x, y) \right.$$

$$+ H_1(\alpha + 2, \beta + 1, \gamma; \delta + 1; x, y) \bigg] - \beta \gamma y \left[\frac{1}{\alpha(\alpha - 1)} H_1(\alpha - 1, \beta + 1, \gamma + 1; \delta; x, y) \right. \tag{2.7}$$

$$+ \frac{1}{(\alpha + 1)\alpha} H_1(\alpha, \beta + 1, \gamma + 1; \delta; x, y) \bigg], \alpha \neq -1, 0, 1.$$

Computing the Horn function H_1 with the numerator parameter $\alpha + k$ for k times, we obtain (2.1). Using (1.6) and the relation

$$(\beta + 1)_{m+n} = (\beta)_{m+n} \left(1 + \frac{m + n}{\beta} \right), \beta \neq 0,$$

we obtain the contiguous function

$$H_1(\alpha, \beta + 1, \gamma; \delta; x, y) = H_1(\alpha, \beta, \gamma; \delta; x, y) + \frac{\alpha x}{\delta} H_1(\alpha + 1, \beta + 1, \gamma; \delta + 1; x, y)$$

$$+ \frac{\gamma y}{\alpha - 1} H_1(\alpha - 1, \beta + 1, \gamma + 1; \delta; x, y), \alpha \neq 1, \delta \neq 0. \tag{2.8}$$

Repeating the previous step when $\beta = \beta + 1$, we get

$$H_1(\alpha, \beta + 2, \gamma; \delta; x, y) = H_1(\alpha, \beta + 1, \gamma; \delta; x, y) + \frac{\alpha x}{\delta} H_1(\alpha + 1, \beta + 2, \gamma; \delta + 1; x, y)$$

$$+ \frac{\gamma y}{\alpha - 1} H_1(\alpha - 1, \beta + 2, \gamma + 1; \delta; x, y) = H_1(\alpha, \beta, \gamma; \delta; x, y)$$

$$+ \frac{\alpha x}{\delta} \left[H_1(\alpha + 1, \beta + 1, \gamma; \delta + 1; x, y) + H_1(\alpha + 1, \beta + 2, \gamma; \delta + 1; x, y) \right]$$

$$+ \frac{\gamma y}{\alpha - 1} \left[H_1(\alpha - 1, \beta + 1, \gamma + 1; \delta; x, y) + H_1(\alpha - 1, \beta + 2, \gamma + 1; \delta; x, y) \right].$$

By iterating this method on H_1 with $\beta + k$ for k times, we find (2.2).

Using the definition H_1 (1.6) and the relation

$$(\gamma + 1)_n = (\gamma)_n \left(1 + \frac{n}{\gamma} \right), \gamma \neq 0,$$

we get the contiguous function

$$H_1(\alpha, \beta, \gamma + 1; \delta; x, y) = H_1(\alpha, \beta, \gamma; \delta; x, y)$$

$$+ \frac{\beta y}{\alpha - 1} H_1(\alpha - 1, \beta + 1, \gamma + 1; \delta; x, y), \alpha \neq 1. \tag{2.9}$$

Repeating the previous relation when $\gamma = \gamma + 1$, we get

$$H_1(\alpha, \beta, \gamma + 2; \delta; x, y) = H_1(\alpha, \beta, \gamma; \delta; x, y)$$

$$+ \frac{\beta y}{\alpha - 1} \left[H_1(\alpha - 1, \beta + 1, \gamma + 1; \delta; x, y) + H_1(\alpha - 1, \beta + 1, \gamma + 2; \delta; x, y) \right].$$

By iterating this method on H_1 with $\gamma + k$ for k times, we get (2.3).

Referring to the definition H_1 (1.6) and the relation

$$\frac{1}{(\delta - 1)_m} = \frac{1}{(\delta)_m} + \frac{m}{(\delta - 1)(\delta)_m}, \delta \neq 1, 0, -1, -2, -3, ...,$$

we obtain the contiguous function

$$H_1(\alpha, \beta, \gamma; \delta - 1; x, y) = H_1(\alpha, \beta, \gamma; \delta; x, y)$$

$$+ \frac{\alpha \beta x}{(\delta - 1)\delta} H_1(\alpha + 1, \beta + 1, \gamma; \delta + 1; x, y), \delta \neq 0, 1. \tag{2.10}$$

Repeating the above relation when $\delta = \delta - 1$, we get

$$H_1(\alpha, \beta, \gamma; \delta - 2; x, y) = H_1(\alpha, \beta, \gamma; \delta; x, y) + \alpha \beta x \left[\frac{H_1(\alpha + 1, \beta + 1, \gamma; \delta + 1; x, y)}{(\delta - 1)\delta} \right.$$

$$+ \left. \frac{1}{(\delta - 2)(\delta - 1)} H_1(\alpha + 1, \beta + 1, \gamma; \delta; x, y) \right], \delta \neq, 0, 1, 2.$$

If we apply this contiguous relation for k times for Horn function H_1 with the denominator parameter with $\delta - k$ for k times, we get (2.4). ∎

Here, we establish several recursion formulas for Horn functions H_2, H_5, H_6 and H_7.

Theorem 2.2. *The Horn function* H_2 *satisfy the recursion formulas*

$$H_2(\alpha + k, \beta, \gamma, \delta; \epsilon; x, y) = H_2(\alpha, \beta, \gamma, \delta; \epsilon; x, y)$$

$$+ \frac{\alpha x}{\epsilon} \sum_{r=1}^{k} H_2(\alpha + r, \beta + 1, \gamma, \delta; \epsilon + 1; x, y)$$

$$- \gamma \delta y \sum_{r=1}^{k} \frac{H_2(\alpha + r - 2, \beta, \gamma + 1, \delta + 1; \epsilon; x, y)}{(\alpha + r - 1)(\alpha + r - 2)};$$

$$\epsilon \neq 0, \alpha \neq 1 - r, \alpha \neq 2 - r, r \in \mathrm{N},$$

$$\tag{2.11}$$

$$H_2(\alpha, \beta + k, \gamma, \delta; \epsilon; x, y) = H_2(\alpha, \beta, \gamma, \delta; \epsilon; x, y)$$

$$+ \frac{\alpha x}{\epsilon} \sum_{r=1}^{k} H_2(\alpha + 1, \beta + r, \gamma, \delta; \epsilon + 1; x, y), \epsilon \neq 0,$$

$$\tag{2.12}$$

$$H_2(\alpha, \beta, \gamma + k, \delta; \epsilon; x, y) = H_2(\alpha, \beta, \gamma, \delta; \epsilon; x, y)$$

$$+ \frac{\delta y}{\alpha - 1} \sum_{r=1}^{k} H_2(\alpha - 1, \beta, \gamma + r, \delta + 1; \epsilon; x, y), \alpha \neq 1,$$

$$\tag{2.13}$$

$$H_2(\alpha, \beta, \gamma, \delta + k; \epsilon; x, y) = H_2(\alpha, \beta, \gamma, \delta; \epsilon; x, y)$$

$$+ \frac{\gamma y}{\alpha - 1} \sum_{r=1}^{k} H_2(\alpha - 1, \beta, \gamma + 1, \delta + r; \epsilon; x, y), \alpha \neq 1$$

$$\tag{2.14}$$

and

$$H_2(\alpha, \beta, \gamma, \delta; \epsilon - k; x, y) = H_2(\alpha, \beta, \gamma, \delta; \epsilon; x, y)$$

$$+ \alpha \beta x \sum_{r=1}^{k} \frac{H_2(\alpha + 1, \beta + 1, \gamma, \delta; \epsilon - r + 2; x, y)}{(\epsilon - r)(\epsilon - r + 1)}, \epsilon \neq r - 1, r \in \mathrm{N}$$

$$\tag{2.15}$$

Theorem 2.3. *The Horn hypergeometric function* H_5 *satisfy the identity:*

$$H_5(\alpha + k, \beta; \gamma; x, y) = H_5(\alpha, \beta; \gamma; x, y) + \frac{2x}{\beta - 1} \sum_{r=1}^{k} (\alpha + r) H_5(\alpha + r + 1, \beta - 1; \gamma; x, y)$$

$$\tag{2.16}$$

$$+ \frac{\beta y}{\gamma} \sum_{r=1}^{k} H_5(\alpha + r, \beta + 1; \gamma + 1; x, y), \beta \neq 1, \gamma \neq 0,$$

$$H_5(\alpha, \beta + k; \gamma; x, y) = H_5(\alpha, \beta; \gamma; x, y) + \frac{\alpha y}{\gamma} \sum_{r=1}^{k} H_5(\alpha + 1, \beta + r; \gamma + 1; x, y)$$

$$- \alpha(\alpha + 1) x \sum_{r=1}^{k} \frac{H_5(\alpha + 2, \beta + r - 2; \gamma + 1; x, y)}{(\beta + r - 1)(\beta + r - 2)}; \gamma \neq 0, \beta \neq 1 - r, 2 - r, r \in \mathrm{N}$$

$$\tag{2.17}$$

and

$$H_5(\alpha, \beta; \gamma - k; x, y) = H_5(\alpha, \beta; \gamma; x, y)$$

$$+ \alpha \beta y \sum_{r=1}^{k} \frac{H_5(\alpha + 1, \beta + 1; \gamma - r + 2; x, y)}{(\gamma - r)(\gamma - r + 1)}; \gamma \neq r, \gamma \neq r - 1, r \in \mathrm{N}.$$

$$\tag{2.18}$$

Theorem 2.4. *The Horn hypergeometric function* H_6 *satisfy the recursion formulas:*

$$H_6(\alpha + k, \beta, \gamma; x, y) = H_6(\alpha, \beta, \gamma; x, y) + \frac{2x}{\beta - 1}\sum_{r=1}^{k}(\alpha + r)H_6(\alpha + r + 1, \beta - 1, \gamma; x, y)$$

$$(2.19)$$

$$- \beta\gamma y\sum_{r=1}^{k}\frac{H_6(\alpha + r - 2, \beta + 1, \gamma + 1; x, y)}{(\alpha + r - 1)(\alpha + r - 2)}; \beta \neq 1, \alpha \neq 1 - r, \alpha \neq 2 - r, r \in \mathrm{N},$$

$$H_6(\alpha, \beta + k, \gamma; x, y) = H_6(\alpha, \beta, \gamma; x, y) + \frac{\gamma y}{\alpha - 1}\sum_{r=1}^{k}H_6(\alpha - 1, \beta + r, \gamma + 1; x, y)$$

$$(2.20)$$

$$- \alpha(\alpha + 1)x\sum_{r=1}^{k}\frac{H_6(\alpha + 2, \beta + r - 2, \gamma; x, y)}{(\beta + r - 1)(\beta + r - 2)}; \alpha \neq 1, \beta \neq 1 - r, \beta \neq 2 - r, r \in \mathrm{N}$$

and

$$H_6(\alpha, \beta; \gamma + k; x, y) = H_6(\alpha, \beta, \gamma; x, y)$$

$$+ \frac{\beta y}{\alpha - 1}\sum_{r=1}^{k}H_6(\alpha - 1, \beta + 1, \gamma + r; x, y), \alpha \neq 1.$$

$$(2.21)$$

Theorem 2.5. *The Horn hypergeometric function* H_7 *satisfy the identity:*

$$H_7(\alpha + k, \beta, \gamma; \delta; x, y) = H_7(\alpha, \beta, \gamma; \delta; x, y)$$

$$+ \frac{2x}{\delta}\sum_{r=1}^{k}(\alpha + r)H_7(\alpha + r + 1, \beta, \gamma; \delta + 1; x, y)$$

$$(2.22)$$

$$- \beta\gamma y\sum_{r=1}^{k}\frac{H_7(\alpha + r - 2, \beta + 1, \gamma + 1; \delta; x, y)}{(\alpha + r - 1)(\alpha + r - 2)}; \delta \neq 0, \alpha \neq 1 - r, \alpha \neq 2 - r, r \in \mathrm{N},$$

$$H_7(\alpha, \beta + k, \gamma; \delta; x, y) = H_7(\alpha, \beta, \gamma; \delta; x, y)$$

$$+ \frac{\gamma y}{\alpha - 1}\sum_{r=1}^{k}H_7(\alpha - 1, \beta + r, \gamma + 1; x, y), \alpha \neq 1,$$

$$(2.23)$$

$$H_7(\alpha, \beta, \gamma + k; \delta; x, y) = H_7(\alpha, \beta, \gamma; \delta; x, y)$$

$$+ \frac{\beta y}{\alpha - 1}\sum_{r=1}^{k}H_7(\alpha - 1, \beta + 1, \gamma + r; x, y), \alpha \neq 1$$

$$(2.24)$$

and

$$H_7(\alpha, \beta, \gamma; \delta - k; x, y) = H_7(\alpha, \beta, \gamma; \delta; x, y)$$

$$+ \alpha(\alpha + 1)x\sum_{r=1}^{k}\frac{H_7(\alpha + 2, \beta, \gamma; \delta - r + 2; x, y)}{(\delta - r)(\delta - r + 1)}; \delta \neq r, r - 1, r \in \mathrm{N}.$$

$$(2.25)$$

3. Differential Recursion Formulas and Differential Operators for Horn's Hypergeometric Functions

Here, We derive many differential relationships and differential operators for functions H_1, H_2, H_3, H_4, H_5, H_6 and H_7.

Theorem 3.1. *Differential recursion formulas for Horn function* H_1 *are as follows*

$$H_1(\alpha + 1, \beta, \gamma; \delta; x, y) = \left(1 + \frac{\vartheta_x - \vartheta_y}{\alpha}\right) H_1(\alpha, \beta, \gamma; \delta; x, y), \alpha \neq 0, \tag{3.1}$$

$$H_1(\alpha, \beta + 1, \gamma; \delta; x, y) = \left(1 + \frac{\vartheta_y + \vartheta_x}{\beta}\right) H_1(\alpha, \beta, \gamma; \delta; x, y), \beta \neq 0, \tag{3.2}$$

$$H_1(\alpha, \beta, \gamma + 1; \delta; x, y) = \left(1 + \frac{\vartheta_y}{\gamma}\right) H_1(\alpha, \beta, \gamma; \delta; x, y), \gamma \neq 0 \tag{3.3}$$

and

$$H_1(\alpha, \beta, \gamma; \delta - 1; x, y) = \left(1 + \frac{\vartheta_x}{\delta - 1}\right) H_1(\alpha, \beta, \gamma; \delta; x, y), \delta \neq 1. \tag{3.4}$$

Proof. Starting from the differential operators

$$\vartheta_x x^m = x \frac{\partial}{\partial x} x^m = m x^m,$$

$$\vartheta_y y^n = y \frac{\partial}{\partial y} y^n = n y^n,$$

and using (2.5), we get the differential formula for H_1

$$H_1(\alpha + 1, \beta, \gamma; \delta; x, y) = \sum_{m,n=0}^{\infty} \left(1 + \frac{m - n}{\alpha}\right) (\alpha)_{m-n} \frac{(\beta)_{m+n}(\gamma)_n}{(\delta)_m m! n!} x^m y^n$$

$$= \sum_{m,n=0}^{\infty} \frac{(\alpha)_{m-n}(\beta)_{m+n}(\gamma)_n}{(\delta)_m m! n!} x^m y^n + \frac{1}{\alpha} \sum_{m,n=0}^{\infty} \frac{m(\alpha)_{m-n}(\beta)_{m+n}(\gamma)_n}{(\delta)_m m! n!} x^m y^n$$

$$- \frac{1}{\alpha} \sum_{m,n=0}^{\infty} \frac{n(\alpha)_{m-n}(\beta)_{m+n}(\gamma)_n}{(\delta)_m m! n!} x^m y^n$$

$$= H_1(\alpha, \beta, \gamma; \delta; x, y) + \frac{\vartheta_x - \vartheta_y}{\alpha} H_1(\alpha, \beta, \gamma; \delta; x, y), \alpha \neq 0.$$

A similar way, we obtain (3.2)-(3.4).   ∎

Theorem 3.2. *For Horn hypergeometric function* H$_2$, H$_3$, H$_4$, H$_5$, H$_6$ *and* H$_7$

$$H_2(\alpha+1,\beta,\gamma,\delta;\epsilon;x,y) = \left(1 + \frac{\vartheta_x - \vartheta_y}{\alpha}\right) H_2(\alpha,\beta,\gamma,\delta;\epsilon;x,y), \alpha \neq 0,$$

$$H_2(\alpha,\beta+1,\gamma,\delta;\epsilon;x,y) = \left(1 + \frac{\vartheta_x}{\beta}\right) H_2(\alpha,\beta,\gamma,\delta;\epsilon;x,y), \beta \neq 0,$$

$$H_2(\alpha,\beta,\gamma+1,\delta;\epsilon;x,y) = \left(1 + \frac{\vartheta_y}{\gamma}\right) H_2(\alpha,\beta,\gamma,\delta;\epsilon;x,y), \gamma \neq 0, \tag{3.5}$$

$$H_2(\alpha,\beta,\gamma,\delta+1;\epsilon;x,y) = \left(1 + \frac{\vartheta_y}{\delta}\right) H_2(\alpha,\beta,\gamma,\delta;\epsilon;x,y), \delta \neq 0,$$

$$H_2(\alpha,\beta,\gamma,\delta;\epsilon-1;x,y) = \left(1 + \frac{\vartheta_x}{\epsilon-1}\right) H_2(\alpha,\beta,\gamma,\delta;\epsilon;x,y), \epsilon \neq 1,$$

$$H_3(\alpha+1,\beta;\gamma;x,y) = \left(1 + \frac{2\vartheta_x + \vartheta_y}{\alpha}\right) H_3(\alpha,\beta;\gamma;x,y), \alpha \neq 0,$$

$$H_3(\alpha,\beta+1;\gamma;x,y) = \left(1 + \frac{\vartheta_y}{\beta}\right) H_3(\alpha,\beta;\gamma;x,y), \beta \neq 0, \tag{3.6}$$

$$H_3(\alpha,\beta;\gamma-1;x,y) = \left(1 + \frac{\vartheta_x + \vartheta_y}{\gamma-1}\right) H_2(\alpha,\beta;\gamma;x,y), \gamma \neq 1$$

$$H_4(\alpha+1,\beta;\gamma,\delta;x,y) = \left(1 + \frac{2\vartheta_x + \vartheta_y}{\alpha}\right) H_4(\alpha,\beta;\gamma,\delta;x,y), \alpha \neq 0,$$

$$H_4(\alpha,\beta+1;\gamma,\delta;x,y) = \left(1 + \frac{\vartheta_y}{\beta}\right) H_4(\alpha,\beta;\gamma,\delta;x,y), \beta \neq 0,$$

$$H_4(\alpha,\beta;\gamma-1,\delta;x,y) = \left(1 + \frac{\vartheta_x}{\gamma-1}\right) H_4(\alpha,\beta;\gamma,\delta;x,y), \gamma \neq 1, \tag{3.7}$$

$$H_4(\alpha,\beta;\gamma,\delta-1;x,y) = \left(1 + \frac{\vartheta_y}{\delta-1}\right) H_4(\alpha,\beta;\gamma,\delta;x,y), \delta \neq 1,$$

$$H_5(\alpha+1,\beta;\gamma;x,y) = \left(1 + \frac{2\vartheta_x + \vartheta_y}{\alpha}\right) H_5(\alpha,\beta;\gamma;x,y), \alpha \neq 0,$$

$$H_5(\alpha,\beta+1;\gamma;x,y) = \left(1 + \frac{\vartheta_y - \vartheta_x}{\beta}\right) H_5(\alpha,\beta;\gamma;x,y), \beta \neq 0, \tag{3.8}$$

$$H_5(\alpha,\beta;\gamma-1;x,y) = \left(1 + \frac{\vartheta_y}{\gamma-1}\right) H_5(\alpha,\beta;\gamma;x,y), \gamma \neq 1,$$

$$H_6(\alpha+1,\beta,\gamma;x,y) = \left(1 + \frac{2\vartheta_x - \vartheta_y}{\alpha}\right) H_6(\alpha,\beta,\gamma;x,y), \alpha \neq 0,$$

$$H_6(a,\beta+1,\gamma;x,y) = \left(1 + \frac{\vartheta_y - \vartheta_x}{\beta}\right) H_6(\alpha,\beta,\gamma;x,y), \beta \neq 0, \tag{3.9}$$

$$H_6(\alpha,\beta,\gamma+1;x,y) = \left(1 + \frac{\vartheta_y}{\gamma}\right) H_6(\alpha,\beta,\gamma;x,y), \gamma \neq 0$$

and

$$H_7(\alpha + 1, \beta, \gamma; \delta; x, y) = \left(1 + \frac{2\vartheta_x - \vartheta_y}{\alpha}\right) H_7(\alpha, \beta, \gamma; \delta; x, y), \alpha \neq 0,$$

$$H_7(\alpha, \beta + 1, \gamma; \delta; x, y) = \left(1 + \frac{\vartheta_y}{\beta}\right) H_7(\alpha, \beta, \gamma; \delta; x, y), \beta \neq 0,$$

$$H_7(\alpha, \beta, \gamma + 1; \delta; x, y) = \left(1 + \frac{\vartheta_y}{\gamma}\right) H_7(\alpha, \beta, \gamma; \delta; x, y), \gamma \neq 0,$$

$$H_7(\alpha, \beta, \gamma; \delta - 1; x, y) = \left(1 + \frac{\vartheta_x}{\delta - 1}\right) H_7(\alpha, \beta, \gamma; \delta; x, y), \delta \neq 1$$

(3.10)

hold true.

Theorem 3.3. *The derivative formulas hold true for function* H_1

$$\frac{\partial^s}{\partial x^s} H_1(\alpha, \beta, \gamma; \delta; x, y) = \frac{(\alpha)_s (\beta)_s}{(\delta)_s} H_1(\alpha + s, \beta + s, \gamma; \delta + s; x, y) \tag{3.11}$$

and

$$\frac{\partial^s}{\partial y^s} H_1(\alpha, \beta, \gamma; \delta; x, y) = \frac{(-1)^s (\beta)_s (\gamma)_s}{(1 - \alpha)_s} H_1(\alpha - s, \beta + s, \gamma + s; \delta; x, y), \alpha \neq 1, 2, 3, \ldots \tag{3.12}$$

Proof. The derivative of H_1 with respect to x yields

$$\frac{\partial}{\partial x} H_1(\alpha, \beta, \gamma; \delta; x, y) = \frac{\alpha\beta}{\delta} H_1(\alpha + 1, \beta + 1, \gamma; \delta + 1; x, y). \tag{3.13}$$

Again, taking the derivative (3.13) with respect to x, we get

$$\frac{\partial^2}{\partial x^2} H_1(\alpha, \beta, \gamma; \delta; x, y) = \frac{\alpha\beta}{\delta} \frac{\partial}{\partial x} H_1(\alpha + 1, \beta + 1, \gamma; \delta + 1; x, y)$$
$$= \frac{\alpha(\alpha + 1)\beta(\beta + 1)}{\delta(\delta + 1)} H_1(\alpha + 2, \beta + 2, \gamma; \delta + 2; x, y). \tag{3.14}$$

Iteration the above process, we arrive at

$$\frac{\partial^s}{\partial x^s} H_1(\alpha, \beta, \gamma; \delta; x, y) = \frac{(\alpha)_s (\beta)_s}{(\delta)_s} H_1(\alpha + s, \beta + s, \gamma; \delta + r; x, y).$$

A similar way, we obtain (3.12).   ∎

Theorem 3.4. *For Horn hypergeometric functions* H_2, H_3, H_4, H_5, H_6 *and* H_7,

$$\frac{\partial^s}{\partial x^s} H_2(\alpha, \beta, \gamma, \delta; \epsilon; x, y) = \frac{(\alpha)_s (\beta)_s}{(e)_s} H_2(\alpha + s, \beta + s, \gamma, \delta; \epsilon + s; x, y),$$

$$\frac{\partial^s}{\partial y^s} H_2(\alpha, \beta, \gamma, \delta; \epsilon; x, y) = \frac{(-1)^s (\gamma)_s (\delta)_s}{(1 - \alpha)_s} H_2(\alpha - s, \beta, \gamma + s, \delta + s; \epsilon; x, y), \alpha \neq 1, 2, 3, \ldots,$$

(3.15)

$$\frac{\partial^s}{\partial x^s} H_3(\alpha, \beta; \gamma; x, y) = \frac{(\alpha)_{2s}}{(\gamma)_s} H_3(\alpha + 2s, \beta; \gamma + s; x, y),$$

$$\frac{\partial^s}{\partial y^s} H_3(\alpha, \beta; \gamma; x, y) = \frac{(\alpha)_s (\beta)_s}{(\gamma)_s} H_3(\alpha + s, \beta + s; \gamma + s; x, y),$$

(3.16)

$$\frac{\partial^s}{\partial x^s} H_4(\alpha, \beta; \gamma, \delta; x, y) = \frac{(\alpha)_{2s}}{(\gamma)_s} H_4(\alpha + 2s, \beta; \gamma + s, \delta; x, y),$$

$$\frac{\partial^s}{\partial y^s} H_4(\alpha, \beta; \gamma, \delta; x, y) = \frac{(\alpha)_s(\beta)_s}{(\delta)_s} H_4(\alpha + s, \beta + s; \gamma, \delta + s; x, y),$$

(3.17)

$$\frac{\partial^s}{\partial x^s} H_5(\alpha, \beta; \gamma; x, y) = \frac{(-1)^s(\alpha)_{2s}}{(1 - \beta)_s} H_5(\alpha + 2s, \beta - s; \gamma; x, y),$$

$$\frac{\partial^s}{\partial y^s} H_5(\alpha, \beta; \gamma; x, y) = \frac{(\alpha)_s(\beta)_s}{(\gamma)_s} H_5(\alpha + s, \beta + s; \gamma + s; x, y),$$

(3.18)

$$\frac{\partial^s}{\partial x^s} H_6(\alpha, \beta, \gamma; x, y) = \frac{(-1)^s(\alpha)_{2s}}{(1 - \beta)_s} H_6(\alpha + 2s, \beta - s, \gamma; x, y), \beta \neq 1, 2, 3, \ldots,$$

$$\frac{\partial^s}{\partial y^s} H_6(\alpha, \beta, \gamma; x, y) = \frac{(-1)^s(\beta)_s(\gamma)_s}{(1 - \alpha)_s} H_6(\alpha - s, \beta + s, \gamma + s; x, y), \alpha \neq 1, 2, 3, \ldots$$

(3.19)

and

$$\frac{\partial^s}{\partial x^s} H_7(\alpha, \beta, \gamma; \delta; x, y) = \frac{(\alpha)_{2s}}{(\delta)_s} H_7(\alpha + 2r, \beta, \gamma; \delta + r; x, y),$$

$$\frac{\partial^s}{\partial y^s} H_7(\alpha, \beta, \gamma; \delta; x, y) = \frac{(-1)^s(\beta)_s(\gamma)_s}{(1 - \alpha)_s} H_7(\alpha - s, \beta + s, \gamma + s; \delta; x, y), \alpha \neq 1, 2, 3, \ldots$$

(3.20)

are satisfied.

4. INTEGRAL OPERATORS FOR HORN HYPERGEOMETRIC FUNCTIONS

Here, we present the integral operators for the functions H_1, H_2, H_3, H_4, H_5, H_6 and H_7 about numerator and denominator parameters by different expressions.

Theorem 4.1. *For Horn hypergeometric function* H_1, *we have the integral operators* $\hat{\mathfrak{I}}_x^s$ *and* $\hat{\mathfrak{I}}_y^s$:

$$\hat{\mathfrak{I}}_x^s\, H_1(\alpha, \beta, \gamma; \delta; x, y) = \frac{(-1)^s(1 - \delta)_s}{x^s(1 - \alpha)_s(1 - \beta)_s} H_1(\alpha - s, \beta - s, \gamma; \delta - s; x, y),$$
$$(\alpha, \beta \neq 1, 2, 3, \ldots, x \neq 0),$$

(4.1)

$$\hat{\mathfrak{I}}_y^s\, H_1(\alpha, \beta, \gamma; \delta; x, y) = \frac{(\alpha)_s}{y^s(1 - \beta)_s(1 - \gamma)_s} H_1(\alpha + s, \beta - s, \gamma - s; \delta; x, y),$$
$$(\beta, \gamma \neq 1, 2, 3, \ldots, y \neq 0)$$

(4.2)

and

$$\left(\hat{\mathfrak{I}}_x\hat{\mathfrak{I}}_y\right)^s H_1(\alpha, \beta, \gamma; \delta; x, y) = \frac{(1 - \delta)_s H_1(\alpha, \beta - 2s, \gamma - s; \delta - s; x, y)}{x^s y^s(1 - \beta)_{2s}(1 - \gamma)_s},$$
$$(\beta, \gamma \neq 1, 2, 3, \ldots, x, y \neq 0).$$

(4.3)

Proof. From (1.13), we write $\hat{\mathfrak{I}} = \hat{\mathfrak{I}}_x + \hat{\mathfrak{I}}_y$ where $\hat{\mathfrak{I}}_x = \frac{1}{x}\int_0^x dx$ and $\hat{\mathfrak{I}}_y = \frac{1}{y}\int_0^y dy$. Also, we get relations concerning the integral operators $\hat{\mathfrak{I}}_x$ and $\hat{\mathfrak{I}}_y$

$$
\begin{aligned}
\hat{\mathfrak{I}}_x \, \mathrm{H}_1(\alpha,\beta,\gamma;\delta;x,y) &= \sum_{m,n=0}^{\infty} \frac{(\alpha)_{m-n}(\beta)_{m+n}(\gamma)_n}{(\delta)_m m! n!}\left[\frac{1}{x}\int_0^x \mathrm{x}^m \mathrm{y}^n dx\right]\\
&= \sum_{m,n=0}^{\infty} \frac{(\alpha)_{m-n}(\beta)_{m+n}(\gamma)_n}{(\delta)_m m! n!}\frac{\mathrm{x}^m \mathrm{y}^n}{m+1}\\
&= \sum_{m,n=0}^{\infty} \frac{(\alpha)_{m-n-1}(\beta)_{m+n-1}(\gamma)_n}{(\delta)_{m-1} m! n!}\mathrm{x}^{m-1}\mathrm{y}^n\\
&= \frac{\delta-1}{x(\alpha-1)(\beta-1)}\mathrm{H}_1(\alpha-1,\beta-1,\gamma;\delta-1;x,y), \alpha,\beta\neq 1, x\neq 0.
\end{aligned}
$$

Repeating the above relation, we get

$$
\begin{aligned}
\hat{\mathfrak{I}}_x^2 \, \mathrm{H}_1(\alpha,\beta,\gamma;\delta;x,y) &= \frac{\delta-1}{x(\alpha-1)(\beta-1)}\hat{\mathfrak{I}}_x \mathrm{H}_1(\alpha-1,\beta-1,\gamma;\delta-1;x,y)\\
&= \frac{(\delta-1)(\delta-2)}{x^2(\alpha-1)(\alpha-2)(\beta-1)(\beta-2)}\mathrm{H}_1(\alpha-2,\beta-2,\gamma;\delta-2;x,y), \alpha,\beta\neq 1,2, x\neq 0.
\end{aligned}
$$

Iterating for s times, we obtain (4.1). Similarly, for $\mathfrak{I}_y$, we get

$$
\begin{aligned}
\hat{\mathfrak{I}}_y \, \mathrm{H}_1(\alpha,\beta,\gamma;\delta;x,y) &= \sum_{m,n=0}^{\infty} \frac{(\alpha)_{m-n}(\beta)_{m+n}(\gamma)_n}{(\delta)_m m! n!}\left[\frac{1}{y}\int_0^y \mathrm{x}^m \mathrm{y}^n dy\right]\\
&= \sum_{m,n=0}^{\infty} \frac{(\alpha)_{m-n}(\beta)_{m+n}(\gamma)_n}{(\delta)_m m! n!}\left[\frac{\mathrm{x}^m \mathrm{y}^n}{n+1}\right]\\
&= \sum_{m,n=0}^{\infty} \frac{(\alpha)_{m-n}(\beta)_{m+n}(\gamma)_n}{(\delta)_m m! (n+1)!}\mathrm{x}^m \mathrm{y}^n\\
&= \sum_{m,n=0}^{\infty} \frac{(\alpha)_{m-n+1}(\beta)_{m+n-1}(\gamma)_{n-1}}{(\delta)_m m! n!}\mathrm{x}^m \mathrm{y}^{n-1}\\
&= \frac{\alpha}{y(\beta-1)(\gamma-1)}\mathrm{H}_1(\alpha+1,\beta-1,\gamma-1;\delta;x,y), \beta,\gamma\neq 1, y\neq 0.
\end{aligned}
$$

Repeating the previous relation, we get

$$
\begin{aligned}
\hat{\mathfrak{I}}_y^2 \, \mathrm{H}_1(\alpha,\beta,\gamma;\delta;x,y) &= \frac{\alpha}{y(\beta-1)(\gamma-1)}\hat{\mathfrak{I}}_y \mathrm{H}_1(\alpha+1,\beta-1,\gamma-1;\delta;x,y)\\
&= \frac{\alpha(\alpha-1)}{y^2(\beta-1)(\beta-2)(\gamma-1)(\gamma-2)}\mathrm{H}_1(\alpha+2,\beta-2,\gamma-2;\delta;x,y), \beta,\gamma\neq 1,2, y\neq 0.
\end{aligned}
$$

By iterating the above relation for s times, we get (4.2). Using the operator $\hat{\mathfrak{I}}_x\hat{\mathfrak{I}}_y$, we have

$$
\hat{\mathfrak{I}}_x\hat{\mathfrak{I}}_y \, \mathrm{H}_1(\alpha,\beta,\gamma;\delta;x,y) = \frac{(\delta-1)\mathrm{H}_1(\alpha,\beta-2,\gamma-1;\delta-1;x,y)}{xy(\beta-1)(\beta-2)(\gamma-1)}, \beta,\gamma\neq 1, x,y\neq 0.
$$

Iterating the above relation for s times, we obtain (4.5). ∎

Theorem 4.2. *For $\alpha, \beta, \gamma \neq 1, 2$ and $x, y \neq 0$. The integration formulas for the function* H$_1$ *holds true:*

$$\hat{\mathfrak{J}}^2 H_1(\alpha, \beta, \gamma; \delta; x, y) = \frac{(\delta - 1)(\delta - 2)}{x^2(\alpha - 1)(\alpha - 2)(\beta - 1)(\beta - 2)} H_1(\alpha - 2, \beta - 2, \gamma; \delta - 2; x, y)$$

$$+ \frac{2(\delta - 1)}{xy(\beta - 1)(\beta - 2)(\gamma - 1)} H_1(\alpha, \beta - 2, \gamma - 1; \delta - 1; x, y) \qquad (4.4)$$

$$+ \frac{\alpha(\alpha + 1)}{y^2(\beta - 1)(\beta - 2)(\gamma - 1)(\gamma - 2)} H_1(\alpha + 2, \beta - 2, \gamma - 2; \delta; x, y).$$

Proof. Consider $\hat{\mathfrak{J}}$ acts on the function H$_1$, we get

$$\hat{\mathfrak{J}} H_1(\alpha, \beta, \gamma; \delta; x, y) = \sum_{m,n=0}^{\infty} \frac{(\alpha)_{m-n}(\beta)_{m+n}(\gamma)_n}{(\delta)_m m! n!} \left[\frac{1}{x} \int_0^x x^m y^n dx + \frac{1}{y} \int_0^y x^m y^n dy \right]$$

$$= \sum_{m,n=0}^{\infty} \frac{(\alpha)_{m-n}(\beta)_{m+n}(\gamma)_n}{(\delta)_m m! n!} \left[\frac{x^m y^n}{m + 1} + \frac{x^m y^n}{n + 1} \right]$$

$$= \sum_{m,n=0}^{\infty} \frac{(\alpha)_{m-n}(\beta)_{m+n}(\gamma)_n}{(\delta)_m (m + 1)! n!} x^m y^n + \sum_{m,n=0}^{\infty} \frac{(\alpha)_{m-n}(\beta)_{m+n}(\gamma)_n}{(\delta)_m m! (n + 1)!} x^m y^n$$

$$= \frac{\delta - 1}{x(\alpha - 1)(\beta - 1)} H_1(\alpha - 1, \beta - 1, \gamma; \delta - 1; x, y)$$

$$+ \frac{\alpha}{y(\gamma - 1)(\beta - 1)} H_1(\alpha + 1, \beta - 1, \gamma - 1; \delta; x, y).$$

The operator $\hat{\mathfrak{J}}^2$ is such that

$$\hat{\mathfrak{J}}^2 = \hat{\mathfrak{J}}\hat{\mathfrak{J}} = (\hat{\mathfrak{J}}_x)^2 + 2\hat{\mathfrak{J}}_x\hat{\mathfrak{J}}_y + (\hat{\mathfrak{J}}_x)^2 = \frac{1}{x^2} \int_0^x \int_0^x dx dx + \frac{2}{xy} \int_0^y \int_0^x dx dy + \frac{1}{y^2} \int_0^y \int_0^y dy dy.$$

We see that

$$\hat{\mathfrak{J}}^2 H_1(\alpha, \beta, \gamma; \delta; x, y) = \frac{(\delta - 1)(\delta - 2)}{x^2(\alpha - 1)(\alpha - 2)(\beta - 1)(\beta - 2)} H_1(\alpha - 2, \beta - 2, \gamma; \delta - 2; x, y)$$

$$+ \frac{2(\delta - 1)}{xy(\beta - 1)(\beta - 2)(\gamma - 1)} H_1(\alpha, \beta - 2, \gamma - 1; \delta - 1; x, y)$$

$$+ \frac{\alpha(\alpha + 1)}{y^2(\beta - 1)(\beta - 2)(\gamma - 1)(\gamma - 2)} H_1(\alpha + 2, \beta - 2, \gamma - 2; \delta; x, y). \qquad \blacksquare$$

Theorem 4.3. *The following integration formulas for the function* **H**$_1$ *holds true:*

$$\hat{\mathfrak{J}}^s H_1(\alpha, \beta, \gamma; \delta; x, y) = \frac{(1 - \delta)_s \prod_{k=1}^s (\vartheta_x + \vartheta_y - k + 1) H_1(\alpha, \beta - 2s, \gamma - s; \delta - s; x, y)}{x^s y^s (1 - \beta)_{2s}(1 - \gamma)_s}, \quad (4.5)$$

$$(\beta, \gamma \neq 1, 2, \ldots, x, y \neq 0).$$

Proof. By using (1.13) and (1.6), we have

$$\hat{\mathfrak{J}} H_1(\alpha, \beta, \gamma; \delta; x, y) = \frac{(\delta - 1)(\vartheta_x + \vartheta_y) H_1(\alpha, \beta - 2, \gamma - 1; \delta - 1; x, y)}{xy(\beta - 1)(\beta - 2)(\gamma - 1)},$$

$$(\beta \neq 1, 2, \gamma \neq 1, x, y \neq 0).$$

By iterating for s times, we get (4.5). $\blacksquare$

Theorem 4.4. *For the functions* H_2, H_3, H_4, H_5, H_6 *and* H_7, *we have the results*

$$\hat{\mathfrak{J}}_x^s \, H_2(\alpha, \beta, \gamma, \delta; \epsilon; x, y) = \frac{(-1)^s (1-\epsilon)_s}{x^s (1-\alpha)_s (1-\beta)_s} H_2(\alpha - s, \beta - s, \gamma, \delta; \epsilon - s; x, y),$$

$$(\alpha, \beta \neq 1, 2, \ldots, x \neq 0),$$

$$\hat{\mathfrak{J}}_y^s \, H_2(\alpha, \beta, \gamma, \delta; \epsilon; x, y) = \frac{(\alpha)_s}{y^s (1-\gamma)_s (1-\delta)_s} H_2(\alpha + s, \beta, \gamma - s, \delta - s; \epsilon; x, y),$$

$$(\gamma, \delta \neq 1, 2, \ldots, y \neq 0),$$

$$\left(\hat{\mathfrak{J}}_x \hat{\mathfrak{J}}_y\right)^s H_2(\alpha, \beta, \gamma, \delta; \epsilon; x, y) = \frac{(1-\epsilon)_s H_2(\alpha, \beta - s, \gamma - s, \delta - s; \epsilon - s; x, y)}{x^s y^s (1-\beta)_s (1-\gamma)_s (1-\delta)_s},$$

$$(\beta, \gamma, \delta \neq 1, 2, \ldots, x, y \neq 0),$$

$$\hat{\mathfrak{J}}^2 H_2(\alpha, \beta, \gamma, \delta; \epsilon; x, y) = \frac{(\epsilon - 1)(\epsilon - 2) H_2(\alpha - 2, \beta - 2, \gamma, \delta; \epsilon - 2; x, y)}{x^2 (\alpha - 1)(\alpha - 2)(\beta - 1)(\beta - 2)} \quad (4.6)$$

$$+ \frac{2(\epsilon - 1)}{xy(\beta - 1)(\gamma - 1)(\delta - 1)} H_1(\alpha, \beta - 1, \gamma - 1, \delta - 1; \epsilon - 1; x, y)$$

$$+ \frac{\alpha(\alpha + 1)}{y^2 (\gamma - 1)(\gamma - 2)(\delta - 1)(\delta - 2)} H_2(\alpha + 2, \beta, \gamma - 2, \delta - 2; \epsilon; x, y),$$

$$(\alpha, \beta, \gamma, \delta \neq 1, 2, x, y \neq 0),$$

$$\hat{\mathfrak{J}}^s H_2(\alpha, \beta, \gamma; \delta; x, y) = \frac{(1-\epsilon)_s \prod_{k=1}^s (\vartheta_x + \vartheta_y - k + 1) H_2(\alpha, \beta - s, \gamma - s, \delta - s; \epsilon - s; x, y)}{x^s y^s (1-\beta)_s (1-\gamma)_s (1-\delta)_s},$$

$$(\beta, \gamma, \delta \neq 1, 2, \ldots, x, y \neq 0).$$

$$\hat{\mathfrak{J}}_x^s \, H_3(\alpha, \beta; \gamma; x, y) = \frac{(-1)^s (1-\gamma)_s}{x^s (1-\alpha)_{2s}} H_3(\alpha - 2s, \beta; \gamma - s; x, y),$$

$$(\alpha \neq 1, 2, \ldots, x \neq 0),$$

$$\hat{\mathfrak{J}}_y^s \, H_3(\alpha, \beta; \gamma; x, y) = \frac{(-1)^s (1-\delta)_s}{y^s (1-\alpha)_s (1-\beta)_s} H_3(\alpha - s, \beta - s; \gamma - s; x, y), \quad (4.7)$$

$$(\alpha, \beta \neq 1, 2, \ldots, y \neq 0),$$

$$\left(\hat{\mathfrak{J}}_x \hat{\mathfrak{J}}_y\right)^s H_3(\alpha, \beta; \gamma; x, y) = \frac{(1-\gamma)_{2s}}{x^s y^s (1-\alpha)_{3s} (1-\beta)_s} H_3(\alpha - 3s, \beta - s; \gamma - 2s; x, y)$$

$$(\alpha, \beta \neq 1, 2, \ldots, x, y \neq 0),$$

$$\hat{\mathfrak{J}}^2 H_3(\alpha, \beta; \gamma; x, y) = \frac{(\gamma - 1)(\gamma - 2)}{x^2 (\alpha - 1)(\alpha - 2)(\alpha - 3)(\alpha - 4)} H_3(\alpha - 4, \beta; \gamma - 2; x, y)$$

$$+ \frac{2(\gamma - 1)(\gamma - 2)}{xy(\alpha - 1)(\alpha - 2)(\alpha - 3)(\beta - 1)} H_3(\alpha - 3, \beta - 1; \gamma - 2; x, y)$$

$$+ \frac{(\gamma - 1)(\gamma - 2)}{y^2 (\alpha - 1)(\alpha - 2)(\beta - 1)(\beta - 2)} H_3(\alpha - 2, \beta - 2; \gamma - 2; x, y), \quad (4.8)$$

$$(\alpha \neq 1, 2, 3, 4, \beta \neq 1, 2, x, y \neq 0),$$

$$\hat{\mathfrak{J}}^s H_3(\alpha, \beta; \gamma; x, y) = \frac{(1-\gamma)_{2r} \prod_{k=1}^s (\vartheta_x + \vartheta_y - k + 1) H_1(\alpha - 3s, \beta - s; \gamma - 2s; x, y)}{x^s y^s (1-\alpha)_{3s} (1-\beta)_s},$$

$$(\alpha\beta \neq 1, 2, \ldots, x, y \neq 0),$$

$$\hat{\mathfrak{J}}_x^s\,H_4(\alpha,\beta;\gamma,\delta;x,y) = \frac{(-1)^s(1-\gamma)_s}{x^s(1-\alpha)_{2s}}H_4(\alpha-2s,\beta;\gamma-s,\delta;x,y),$$
$$(\alpha \neq 1,2,\ldots,x \neq 0),$$

$$\hat{\mathfrak{J}}_y^s\,H_4(\alpha,\beta;\gamma,\delta;x,y) = \frac{(-1)^s(1-\gamma)_s}{y^s(1-\alpha)_s(1-\beta)_s}H_4(\alpha-s,\beta-s;\gamma,\delta-s;x,y),$$
$$(\alpha,\beta \neq 1,2,\ldots,y \neq 0),$$
$$\left(\hat{\mathfrak{J}}_x\hat{\mathfrak{J}}_y\right)^s H_4(\alpha,\beta;\gamma,\delta;x,y) = \frac{(1-\gamma)_s(1-\delta)_s}{x^sy^s(1-\alpha)_{3r}(1-\beta)_s}H_4(\alpha-3s,\beta-s;\gamma-s,\delta-s;x,y)$$
$$(\alpha,\beta \neq 1,2,\ldots,x,y \neq 0),$$

$$\hat{\mathfrak{J}}^2H_4(\alpha,\beta;\gamma,\delta;x,y) = \frac{(\gamma-1)(\gamma-2)}{x^2(\alpha-1)(\alpha-2)(\alpha-3)(\alpha-4)}H_4(\alpha-4,\beta;\gamma-2,\delta;x,y)$$
$$+\frac{2(\gamma-1)(\delta-1)}{xy(\alpha-1)(\alpha-2)(\alpha-3)(\beta-1)}H_4(\alpha-3,\beta-1;\gamma-1,\delta-1;x,y)$$
$$+\frac{(\delta-1)(\delta-2)}{y^2(\alpha-1)(\alpha-2)(\beta-1)(\beta-2)}H_4(\alpha-2,\beta-2;\gamma,\delta-2;x,y),$$
$$(\alpha,\beta \neq 1,2,x,y \neq 0),$$

$$\hat{\mathfrak{J}}^sH_4(\alpha,\beta;\gamma,\delta;x,y) = \frac{(1-\gamma)_s(1-\delta)_s}{x^sy^s(1-\alpha)_{3s}(1-\beta)_s}$$
$$\times \prod_{k=1}^{s}(\vartheta_x+\vartheta_y-k+1)H_4(\alpha-3s,\beta-s;\gamma-s,\delta-r;x,y),\alpha,\beta \neq 1,2,\ldots,x,y \neq 0,$$

(4.9)

(4.10)

$$\hat{\mathfrak{J}}_x^s\,H_5(\alpha,\beta;\gamma;x,y) = \frac{(\beta)_s}{x^s(1-\alpha)_{2s}}H_5(\alpha-2s,\beta+s;\gamma;x,y),$$
$$(\alpha \neq 1,2,\ldots,x \neq 0),$$

$$\hat{\mathfrak{J}}_y^s\,H_5(\alpha,\beta;\gamma;x,y) = \frac{(-1)^s(1-\gamma)_s}{y^s(1-\alpha)_s(1-\beta)_s}H_5(\alpha-s,\beta-s;\gamma-s;x,y),$$
$$(\alpha,\beta \neq 1,2,\ldots,y \neq 0),$$
$$\left(\hat{\mathfrak{J}}_x\hat{\mathfrak{J}}_y\right)^s H_5(\alpha,\beta;\gamma;x,y) = \frac{(1-\gamma)_s}{x^sy^s(1-\alpha)_{3s}}H_5(\alpha-3s,\beta;\gamma-s;x,y)$$
$$(\alpha \neq 1,2,\ldots,x,y \neq 0),$$

(4.11)

$$\hat{\mathfrak{J}}^2H_5(\alpha,\beta;\gamma;x,y) = \frac{b(\beta+1)}{x^2(\alpha-1)(\alpha-2)(\alpha-3)(\alpha-4)}H_5(\alpha-4,\beta+2;\gamma;x,y)$$
$$+\frac{2(\gamma-1)}{xy(\alpha-1)(\alpha-2)(\alpha-3)}H_5(\alpha-3,b;\gamma-1;x,y)$$
$$+\frac{(\gamma-1)(\gamma-2)}{y^2(\alpha-1)(\alpha-2)(\beta-1)(\beta-2)}H_5(\alpha-2,\beta-2;\gamma-2;x,y),$$
$$(\alpha \neq 1,2,3,4,\beta \neq 1,2,x,y \neq 0),$$

$$\hat{\mathfrak{J}}^s \mathrm{H}_5(\alpha, \beta; \gamma; x, y) = \frac{(1-\gamma)_s}{x^s y^s (1-\alpha)_{3s}} \prod_{k=1}^{s} (\vartheta_x + \vartheta_y - k + 1) \mathrm{H}_5(\alpha - 3s, \beta; \gamma - r; x, y),$$

$$(\alpha \neq 1, 2, \ldots, x, y \neq 0). \tag{4.12}$$

$$\hat{\mathfrak{J}}_x^s \, \mathrm{H}_6(\alpha, \beta, \gamma; x, y) = \frac{(\beta)_s}{x^s (1-\alpha)_{2s}} \mathrm{H}_6(\alpha - 2s, \beta + s, \gamma; x, y),$$

$$(\alpha \neq 1, 2, \ldots, x \neq 0),$$

$$\hat{\mathfrak{J}}_y^s \, \mathrm{H}_6(\alpha, \beta, \gamma; x, y) = \frac{(\alpha)_s}{y^s (1-\beta)_s (1-\gamma)_s} \mathrm{H}_6(\alpha + s, \beta - s, \gamma - s; x, y),$$

$$(\beta, \gamma \neq 1, 2, \ldots, y \neq 0), \tag{4.13}$$

$$\left(\hat{\mathfrak{J}}_x \hat{\mathfrak{J}}_y \right)^s \mathrm{H}_6(\alpha, \beta, \gamma; x, y) = \frac{1}{x^s y^s (1-\alpha)_s (1-\gamma)_s} \mathrm{H}_6(\alpha - s, \beta, \gamma - s; x, y),$$

$$(\alpha, \gamma \neq 1, 2, \ldots, x, y \neq 0),$$

$$\hat{\mathfrak{J}}^2 \mathrm{H}_6(\alpha, \beta, \gamma; x, y) = \frac{\beta(\beta+1)}{x^2 (\alpha-1)(\alpha-2)(\alpha-3)(\alpha-4)} \mathrm{H}_6(\alpha - 4, \beta + 2, \gamma; x, y)$$

$$+ \frac{2}{xy(\alpha-1)(\gamma-1)} \mathrm{H}_6(\alpha - 1, \beta, \gamma - 1; x, y)$$

$$+ \frac{\alpha(\alpha+1)}{y^2 (\beta-1)(\beta-2)(\gamma-1)(\gamma-2)} \mathrm{H}_6(\alpha + 2, \beta - 2, \gamma - 2; x, y), \tag{4.14}$$

$$(\alpha \neq 1, 2, 3, 4, \beta, \gamma \neq 1, 2, x, y \neq 0),$$

$$\hat{\mathfrak{J}}^s \mathrm{H}_6(\alpha, \beta, \gamma; x, y) = \frac{1}{x^s y^s (1-\alpha)_s (1-\gamma)_s} \prod_{k=1}^{s} (\vartheta_x + \vartheta_y - k + 1) \mathrm{H}_6(\alpha - s, \beta, \gamma - s; x, y),$$

$$(\alpha \neq 1, 2, \ldots, x, y \neq 0),$$

$$\hat{\mathfrak{J}}_x^s \, \mathrm{H}_7(\alpha, \beta; \gamma; x, y) = \frac{(-1)^s (1-\delta)_s}{x^s (1-\alpha)_{2s}} \mathrm{H}_7(\alpha - 2s, \beta, \gamma; \delta - s; x, y),$$

$$(\alpha \neq 1, 2, \ldots, x \neq 0),$$

$$\hat{\mathfrak{J}}_y^s \, \mathrm{H}_7(\alpha, \beta, \gamma; \delta; x, y) = \frac{(\alpha)_s}{y^s (1-\beta)_s (1-\gamma)_s} \mathrm{H}_7(\alpha + s, \beta - s, \gamma - s; \delta; x, y),$$

$$(\beta, \gamma \neq 1, 2, \ldots, y \neq 0), \tag{4.15}$$

$$\left(\hat{\mathfrak{J}}_x \hat{\mathfrak{J}}_y \right)^s \mathrm{H}_7(\alpha, \beta, \gamma; \delta; x, y) = \frac{(1-\delta)_s \mathrm{H}_7(\alpha - s, \beta - s, \gamma - s; \delta - r; x, y)}{x^s y^s (1-\alpha)_s (1-\beta)_s (1-\gamma)_s},$$

$$(\alpha, \beta, \gamma \neq 1, 2, \ldots, x, y \neq 0)$$

and

$$\hat{\mathfrak{J}}^2 H_7(\alpha, \beta, \gamma; \delta; x, y) = \frac{(\delta - 1)(\delta - 2)}{x^2(\alpha - 1)(\alpha - 2)(\alpha - 3)(\alpha - 4)} H_7(\alpha - 4, \beta, \gamma; \delta - 2; x, y)$$

$$+ \frac{2(\delta - 1)}{xy(\alpha - 1)(\beta - 1)(\gamma - 1)} H_7(\alpha - 1, \beta - 1, \gamma - 1; \delta - 1; x, y)$$

$$+ \frac{\alpha(\alpha + 1)}{y^2(\beta - 1)(\beta - 2)(\gamma - 1)(\gamma - 2)} H_7(\alpha + 2, \beta - 2, \gamma - 2; \delta; x, y) \quad (4.16)$$

$$(\alpha \neq 1, 2, 3, 4, \beta, \gamma \neq 1, 2, x, y \neq 0),$$

$$\hat{\mathfrak{J}}^s H_7(\alpha, \beta, \gamma; \delta; x, y) = \frac{(1 - \delta)_s \prod_{k=1}^{s}(\vartheta_x + \vartheta_y - k + 1) H_7(\alpha - s, \beta - s, \gamma - s; \delta - s; x, y)}{x^s y^s (1 - \alpha)_s (1 - \beta)_s (1 - \gamma)_s}$$

$$(\alpha, \beta, \gamma \neq 1, 2, \ldots, x, y \neq 0).$$

5. Infinite Summations for Horn's Hypergeometric Functions

Here, we derive some infinite summations for the functions H₁, H₂, H₃, H₄, H₅, H₆ and H₇.

Theorem 5.1. *The infinite summations for the function H₁ hold true:*

$$\sum_{r=0}^{\infty} \frac{(\alpha)_r}{r!} H_1(\alpha + r, \beta, \gamma; \delta; x, y) t^r = (1 - t)^{-\alpha} H_1\left(\alpha, \beta, \gamma; \delta; \frac{x}{1 - t}, y(1 - t)\right), |t| < 1, \quad (5.1)$$

$$\sum_{r=0}^{\infty} \frac{(\beta)_r}{r!} H_1(\alpha, \beta + r, \gamma; \delta; x, y) t^r = (1 - t)^{-\beta} H_1\left(\alpha, \beta, \gamma; \delta; \frac{x}{1 - t}, \frac{y}{1 - t}\right), |t| < 1, \quad (5.2)$$

and

$$\sum_{r=0}^{\infty} \frac{(\gamma)_r}{r!} H_1(\alpha, \beta, \gamma + r; \delta; x, y) t^r = (1 - t)^{-\gamma} H_1\left(\alpha, \beta, \gamma; \delta; x, \frac{y}{1 - t}\right), |t| < 1. \quad (5.3)$$

Proof. Using the fact that [1]

$$(1 - t)^{-\alpha} = \sum_{r=0}^{\infty} \frac{(\alpha)_r}{r!} t^r, |t| < 1.$$

Using (1.5) and (1.6), we obtain

$$(1 - t)^{-\alpha} H_1\left(\alpha, \beta, \gamma; \delta; \frac{x}{1 - t}, y(1 - t)\right) = \sum_{m,n=0}^{\infty} \frac{(\alpha)_{m-n}(\beta)_{m+n}(\gamma)_n}{(\delta)_m m! n!} x^m y^n (1 - t)^{-\alpha + n - m}$$

$$= \sum_{r,m,n=0}^{\infty} \frac{(\alpha)_{m-n}(\alpha + m - n)_r (\beta)_{m+n}(\gamma)_n}{(\delta)_m r! m! n!} x^m y^n t^r = \sum_{r,m,n=0}^{\infty} \frac{(\alpha)_{m-n+r}(\beta)_{m+n}(\gamma)_n}{(\delta)_m r! m! n!} x^m y^n t^r$$

$$= \sum_{r,m,n=0}^{\infty} \frac{(\alpha)_r (\alpha + r)_{m-n}(\beta)_{m+n}(\gamma)_n}{(\delta)_m r! m! n!} x^m y^n t^r = \sum_{r=0}^{\infty} \frac{(\alpha)_r}{r!} t^r H_1\left(\alpha + r, \beta, \gamma; \delta; x, y\right).$$

Similarly, we obtain (5.2) and (5.3).   ∎

Theorem 5.2. *The infinite summations for the functions* H_2, H_3, H_4, H_5, H_6 *and* H_7 *hold true:*

$$\sum_{r=0}^{\infty} \frac{(\gamma)_r}{r!} H_2(\alpha, \beta, \gamma + r, \delta; \epsilon; x, y) t^r = (1 - t)^{-\gamma} H_2\left(\alpha, \beta, \gamma, \delta; \epsilon; x, \frac{y}{1-t}\right), |t| < 1,$$

$$\sum_{r=0}^{\infty} \frac{(\delta)_r}{r!} H_2(\alpha, \beta, \gamma, \delta + r; x, y) t^r = (1 - t)^{-\delta} H_1\left(\alpha, \beta, \gamma; \delta; x, \frac{y}{1-t}\right), |t| < 1,$$
(5.4)

$$\sum_{r=0}^{\infty} \frac{(\alpha)_r}{r!} H_3(\alpha + r, \beta; \gamma; x, y) t^r = (1 - t)^{-\alpha} H_3\left(\alpha, \beta; \gamma; \frac{x}{(1-t)^2}, \frac{y}{1-t}\right), |t| < 1,$$

$$\sum_{r=0}^{\infty} \frac{(\beta)_r}{r!} H_3(\alpha, \beta + r; \gamma; x, y) t^r = (1 - t)^{-\beta} H_3\left(\alpha, \beta; \gamma; x, \frac{y}{1-t}\right), |t| < 1,$$
(5.5)

$$\sum_{r=0}^{\infty} \frac{(\alpha)_r}{r!} H_4(\alpha + r, \beta; \gamma, \delta; x, y) t^r = (1 - t)^{-\alpha} H_4\left(\alpha, \beta; \gamma, \delta; \frac{x}{(1-t)^2}, \frac{y}{1-t}\right), |t| < 1,$$

$$\sum_{r=0}^{\infty} \frac{(\beta)_r}{r!} H_4(\alpha, \beta + r; \gamma, \delta; x, y) t^r = (1 - t)^{-\beta} H_4\left(\alpha, \beta; \gamma, \delta; x, \frac{y}{1-t}\right), |t| < 1,$$
(5.6)

$$\sum_{r=0}^{\infty} \frac{(\alpha)_r}{r!} H_5(\alpha + r, \beta; \gamma; x, y) t^r = (1 - t)^{-\alpha} H_5\left(\alpha, \beta; \gamma; \frac{x}{(1-t)^2}, \frac{y}{1-t}\right), |t| < 1,$$

$$\sum_{r=0}^{\infty} \frac{(\beta)_r}{r!} H_5(\alpha, \beta + r; \gamma; x, y) t^r = (1 - t)^{-\beta} H_5\left(\alpha, \beta; \gamma; x(1-t), \frac{y}{1-t}\right), |t| < 1,$$
(5.7)

$$\sum_{r=0}^{\infty} \frac{(\alpha)_r}{r!} H_6(\alpha + r, \beta, \gamma; x, y) t^r = (1 - t)^{-\alpha} H_6\left(\alpha, \beta, \gamma; \frac{x}{(1-t)^2}, y(1-t)\right), |t| < 1,$$

$$\sum_{r=0}^{\infty} \frac{(\beta)_r}{r!} H_6(\alpha, \beta + r, \gamma; x, y) t^r = (1 - t)^{-\beta} H_6\left(\alpha, \beta, \gamma; x(1-t), \frac{y}{1-t}\right), |t| < 1,$$
(5.8)

$$\sum_{r=0}^{\infty} \frac{(\gamma)_r}{r!} H_6(\alpha, \beta, \gamma + r; x, y) t^r = (1 - t)^{-\gamma} H_6\left(\alpha, \beta, \gamma; x, \frac{y}{1-t}\right), |t| < 1,$$

and

$$\sum_{r=0}^{\infty} \frac{(\alpha)_r}{r!} H_7(\alpha + r, \beta, \gamma; \delta; x, y) t^r = (1 - t)^{-\alpha} H_7\left(\alpha, \beta, \gamma; \delta; \frac{x}{(1-t)^2}, y(1-t)\right), |t| < 1,$$

$$\sum_{r=0}^{\infty} \frac{(\beta)_r}{r!} H_7(\alpha, \beta + r, \gamma; \delta; x, y) t^r = (1 - t)^{-\beta} H_7\left(\alpha, \beta, \gamma; \delta; x, \frac{y}{1-t}\right), |t| < 1,$$
(5.9)

$$\sum_{r=0}^{\infty} \frac{(\gamma)_r}{r!} H_7(\alpha, \beta, \gamma + r; \delta; x, y) t^r = (1 - t)^{-\gamma} H_7\left(\alpha, \beta, \gamma; \delta; x, \frac{y}{1-t}\right), |t| < 1.$$

6. Concluding Remarks

As a direct outcome of the numerous new recursion, differential recursion formulas, differential and integral operators, infinite summations and interesting results of Horn hypergeometric functions H$_1$, H$_2$, H$_3$, H$_4$, H$_5$, H$_6$ and H$_7$ which we have established, an analytical approach to computation some of the many comprehensive results has been discussed. Our analytical expressions serve as a benchmark for the accuracy of various approximation techniques designed to study radiation field problems. The results of this study are general in nature and likely to find specific applications in the theory of special functions.

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